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EDITORIAL CORRESPONDENCE AND BOOKS FOR REVIEW should be addressed to the EDITOR-IN-CHIEF, R. C. ARCHIBALD, Brown University, Providence, R. I.

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THE AMERICAN MATHEMATICAL MONTHLY

The endless torch race first began
  Who knoweth where? Who knoweth when?
. . . .
The runners give from hand to hand,—
In lines of flame from land to land;
  And still again, and yet again,
They follow straight the soul's command,—
Though but in part they understand,—
And still pass on the deathless brand
  From man to man—and we are men!

WILLIAM ADAMS SLADE.

THE APRIL MEETING OF THE IOWA SECTION.

The Iowa Section of the Mathematical Association of America met for its fourth regular meeting April 26, 1919, in connection with the Iowa Academy of Science, at the Iowa State Teachers College, Cedar Falls, Iowa. There were present as members of the Association: L. M. Coffin, Coe College; I. S. Condit, Iowa State Teachers College; F. M. McGaw, Cornell College; J. F. Reilly, University of Iowa; H. L. Rietz, University of Iowa; Maria M. Roberts, Iowa State College; F. M. Weida, University of Iowa; and C. W. Wester, Iowa State Teachers College.

The following papers were read and discussed:

"The teaching of a first course in mathematics." By Professor H. L. Rietz.

"A course in arithmetic." By Professor I. S. Condit.

"Effect of delaying algebra until the tenth grade." By Miss Lida Pittman, Fort Dodge (by invitation).

"An example of curve fitting." By G. W. Snedecor, Ames (by invitation).

"Outlines of a course in trigonometry." By Professor J. F. Reilly.

"Some analogies between algebraic equations and linear differential equations." By Peter Luteyn, State Teachers College (by invitation).

The following were elected officers for the ensuing year: Chairman, I. S. CONDIT; Vice Chairman, F. M. MCGAW; Secretary-Treasurer, L. M. COFFIN.

C. W. WESTER, *Secretary-Treasurer*.

SOME EXTENSIONS OF THE WORK OF PAPPUS AND STEINER ON TANGENT CIRCLES.

By J. H. WEAVER, Ohio State University.

Introduction. The figure of three mutually tangent semicircles with their centers in the same straight line was known among the Greeks as the "Shoemaker's Knife" (ἄρβηλος). A few of the properties of the figure are found in the works of Archimedes.[1] Others occur in the Collection of Pappus.[2] After the Greek period we find no work done on the problem until Steiner generalized the results of Pappus and added several others dealing with the perspective properties of the figure.[3] Later Sir Thomas Muir added a theorem giving formulæ for various sets of radii involved.[4] Habicht has discussed some of the properties of elliptic functions connected with the figure[5] while M. G. Fontené has generalized certain formulæ arising from sets of tangent circles.[6]

In the present paper formulæ for the radii of certain sets of circles are developed and used to build up several types of infinite series which may be summed geometrically. Then some general properties of tangents and normals to conics associated with three mutually tangent circles are set forth.[7] These properties lead to a quadrangular-quadrilateral configuration and incidentally furnish some methods for constructing conics. And finally some theorems connected with centers of perspectivity of the various sets of circles are proved.

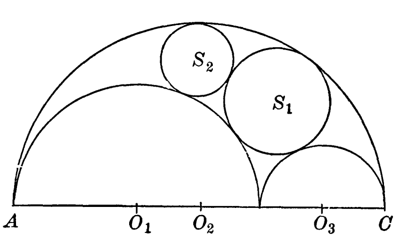


FIG. 1.

1. General Considerations. Let there be two circles tangent internally, with centers O_1 and O_2 and let a circle with center S_n ($n = 1, 2, \cdots$) (Fig. 1) be tangent to these. Then S_n lies on an ellipse with foci O_1 and O_2. If we take the midpoint of O_1O_2 as origin and O_1O_2 as the x-axis, the equation of the ellipse will be

$$\frac{4x^2}{(r_1 + r_2)^2} + \frac{y^2}{r_1r_2} = 1, \tag{1}$$

where r_1 and r_2 ($r_2 > r_1$) are the radii of the circles (O_1) and (O_2) respectively.

[1] *Works of Archimedes*, ed. Heath, Cambridge, 1897, Lemmas, 4–6.

[2] *Collectio*, ed. Hultsch., Berlin, 1876–8, Vol. I, pp. 209 and ff.

[3] Steiner, *Gesammelte Werke*, Berlin, 1881, Vol. 1, pp. 47–76.

[4] *Proceedings of Edinburgh Math. Soc.*, Vol. 3, p. 119. In the same volume, pages 2–11, J. S. Mackay has collected some of the simpler theorems connected with the problem.

[5] Konrad Habicht, *Die Steinerschen Kreisreihen*, Berne, 1904, 35 pp. In this work are found extensive references bearing on the subject.

[6] "Sur les cercles de Pappus," *Nouvelles Annales de Mathématiques* (4), tome 1918, pp. 383–90.

[7] The center of a circle tangent to two given circles lies on a conic having the centers of the two given circles as foci. This is, of course, equivalent to the definition that the sum or difference of the focal radii is constant. I have called such conics "associated" conics.

[8] In what follows circles will be designated by their centers in brackets.

Let ρ_n be the radius of (S_n) and let the coördinates of the point S_n be x_n and y_n. From a fundamental property of the ellipse we have

$$r_1 + \rho_n = a + ex_n, \tag{2}$$

where $2a = r_1 + r_2$, and

$$e = \frac{r_2 - r_1}{r_2 + r_1}.$$

Pappus has shown that if another circle (S_{n+1}) with radius ρ_{n+1}, center at point x_{n+1}, y_{n+1} and coming after (S_n) in the positive direction around the circles, is tangent to (S_n), the following relation holds

$$\frac{y_n + 2\rho_n}{\rho_n} = \frac{y_{n+1}}{\rho_{n+1}}$$

or

$$\frac{y_n}{\rho_n} = \frac{y_{n-1}}{\rho_{n-1}} + 2 = \frac{y_1}{\rho_1} + 2(n-1). \qquad (3)^1$$

2. **Formulæ arising from the figure of three mutually tangent circles with their centers in the same straight line.** Let there be three mutually tangent circles (O_1, O_2) and (O_3) having their centers in the same straight line and radii r_1, r_2, and r_3 respectively (Fig. 1). Then let a series of circles (S_n) be drawn tangent to (O_1) and (O_2), the first circle in the series being also tangent to (O_3) and each of the others tangent to the one preceding it in the series. There are two other such sets of tangent circles. The set tangent to (O_2) and (O_3) we will designate as (S_n'), and the set tangent to (O_1) and (O_3) as (S_n''). Let the radii of the various sets be ρ_n, ρ_n' and ρ_n'' respectively, and the coördinates of the centers be x_n, y_n, x_n', y_n' and x_n'', y_n'' respectively. We will now consider the set (S_n).

By means of equations (1), (2) and (3) and the use of induction we have in this particular case, since the y-coördinate of $O_3 = 0$

$$\rho_n = \frac{r_1 r_2 r_3}{n^2 r_3^2 + r_1 r_3 + r_1^2}. \tag{4}$$

This result is arrived at by Muir and Fontené by different methods.[2] Also from equations (2) and (4)

$$x_{n-1} - x_n = \frac{(2n-1) r_3^2 r_1 r_2 (r_1 + r_2)}{[(n-1)^2 r_3^2 + r_1 r_3 + r_1^2][n^2 r_3^2 + r_1 r_3 + r_1^2]} = i_n, \text{ say.} \tag{5}$$

From the geometric properties of the figure

$$\sum_{n=1}^{\infty} \rho_n \tag{6}$$

is a convergent series, and if r_3 approaches the limit 0, then (6) approaches the value $\pi r_2/2$ but is 0 at the limit.

[1] Pappus, *Collectio*, p. 224.
[2] See Introduction.

Also

$$\sum_{n=1}^{\infty} i_n = 2r_1 + r_3.$$

If we define i_n as the nth intercept of the series (S_n) (i_1, projection of O_3S_1 on AC), then (4) and (5) are the formulæ for the nth radius and intercept in the series (S_n). An interchange of r_1 and r_3 will give the corresponding formulæ for the series (S_n'), while an interchange of r_2 and r_3 with r_2 considered negative will give the corresponding formulæ for the set (S_n'').

If $r_2 = 2r_1$ we have the special case

$$\sum_{n=1}^{\infty} \rho_n'' = r_2/2.$$

We will now establish the following theorem.

THEOREM: If two circles (O_1) and (O_2) are tangent internally, and a circle (S) is drawn tangent to these two, such that SO_i ($i = 1, 2$) is perpendicular to O_1O_2 then it is possible to draw a circle (S') tangent to (O_1), (O_2) and (S) such that the four centers S, S', O_1 and O_2 determine a rectangle.

Proof: Let SO_2 be perpendicular to O_1O_2, and let the coördinates of S and S' be x, y and x', y' respectively and let the radii be ρ and ρ'. Then

$$x = \frac{r_2 - r_1}{2}, \qquad y = \frac{2r_1r_2}{r_1 + r_2}, \qquad \rho = \frac{r_2(r_2 - r_1)}{r_1 + r_2}, \tag{6}$$

and by virtue of equations (1), (2) and (3)

$$y' = y \text{ and } x' = -x,$$

which proves the theorem.

THEOREM: If in the series (S_n), the points S_n, S_{n+1}, O_1 and O_2 determine a rectangle, then $r_1 = nr_3$.

Proof: Equate the values of ρ given in equations (4) and (6).

Let the foot of the perpendicular from S_n to O_1O_2 be P_n.

Let angle $P_nS_nS_{n+1} = \angle B_n$.

Then if $r_1 = kr_3$ (k an integer or a rational fraction)

$$\tan B_n = \frac{(2k + 1)(2n + 1)}{2(n^2 + n - k - k^2)} \tag{7}$$

and the slopes of the lines of successive centers of the series (S_n) are all rational. Moreover if in (7) k is an integer, $B_k = \pi/2$, and

$$\sum_{n=k+1}^{\infty} (B_{n-1} - B_n) = B_k - \lim_{n=\infty} B_n = \pi/2.$$

From the identity $(\rho_n + \rho_{n+1})^2 = i_{n+1}^2 + (y_n - y_{n+1})^2$ we get by using equa-

tions (3), (4) and (5) the equation

$$[(2n^2 + 2n + 1)r_3^2 + 2r_1r_3 + 2r_1^2]^2 = [(2n^2 + 2n)r_3^2 - 2r_1r_3 - 2r_1^2]^2 + [(2n + 1)r_3(r_2 + r_1)]^2$$

giving a triply infinite set of rational right triangles.

3. Formulæ arising from three mutually tangent circles, one of which is tangent to the line of centers of the other two. Let there be two circles (O_1) and (O_2) (Fig. 2) tangent internally at A and let a series of circles be drawn tangent to these, the first one in the series being tangent to the line O_1O_2 and each of the others tangent to the one preceding it in the series. Let the radius of (O_1) be r_1 and of (O_2) be r_2, and let the centers of the circles be S_n. Then from equations (1), (2) and (3), since $y_1 = \rho_1$ we get by induction

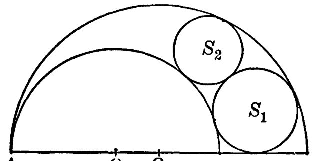
$$\rho_n = \frac{4r_1r_2(r_2 - r_1)}{4(n^2 - n)(r_2 - r_1)^2 + (r_2 + r_1)^2} \tag{8}$$

$$i_n = \frac{32(n - 1)r_1r_2(r_2 - r_1)^2(r_2 + r_1)}{[4(n^2 - n)(r_2 - r_1)^2 + (r_2 + r_1)^2][(4n^2 - 12n + 8)(r_2 - r_1)^2 + (r_2 + r_1)^2]} \tag{9}$$

where for i_n $n \geqq 2$.

Here

$$\sum_{n=2}^{\infty} i_n = 2r_2 - i_1{}^1$$



A O_1 O_2

FIG. 2.

and if r_1 approaches r_2

$$\sum_{n=2}^{\infty} \rho_n$$

approaches $\pi r_2/2$.

THEOREM: If in this series the points S_n, S_{n+1}, O_1 and O_2 determine a rectangle then

$$r_2 = \frac{2n + 1}{2n - 1} \cdot r_1.$$

Let

$$r_2 = \frac{2k + 1}{2k - 1} \cdot r_1.$$

Then

$$\tan B_n = \frac{2nk}{n^2 - k^2}.$$

Moreover,

$$\sum_{n=k+1}^{\infty} (B_{n-1} - B_n) = B_k - \lim_{n=\infty} B_n = \pi/2.$$

Also the equation $(\rho_n + \rho_{n+1})^2 = (i_{n+1})^2 + (y_n - y_{n+1})^2$ gives the triply infinite set of rational right triangles[2]

$$[4n^2(r_2 - r_1)^2 + (r_2 + r_1)^2]^2 = [4n^2(r_2 - r_1)^2 - (r_2 + r_1)^2]^2 + [4n(r_2^2 - r_1^2)]^2.$$

[1] The distance from the point of contact of S_1 with the diameter AO_2 to the end of this diameter, on the side opposite from the point O_2, is taken as i_1.—Editor

[2] This result is but a special case of a rational right triangle with sides $u^2 + v^2$, $u^2 - v^2$, and $2uv$.—Editor.

4. Formulæ arising from a series of tangent circles, tangent to two given circles, the first circle in the series being tangent to a line tangent to the smaller circle and perpendicular to the line of centers. Let there be two circles (O_1) and (O_2) tangent internally at A and let $(O_1) < (O_2)$ and let the tangent to (O_1) perpendicular to O_1O_2 be drawn and let a series of tangent circles be drawn tangent to (O_1) and (O_2), the first circle in the series being also tangent to the perpendicular just drawn (Fig. 3).

Then from Pappus, Book IV., lemma XIX, we have

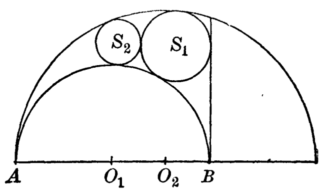


FIG. 3.

$$\frac{r_2 - r_1}{r_2} = \frac{4\rho_1^2}{y_1^2}.$$

Let

$$\frac{r_2 - r_1}{r_2} = m^2,$$

then

$$2\rho_1 = my_1. \tag{10}$$

Using equations (1), (2), (3) and (10) we have by induction

$$\rho_n = \frac{r_1 m^2}{(n-1)^2 m^4 + 2(n-1)m^3 + 1},$$

$$i_n = \frac{r_1 m^3(2 - m^2)[(2n-3)m + 2]}{[(n-2)^2 m^4 + 2(n-2)m^3 + 1][(n-1)^2 m^4 + 2(n-1)m^3 + 1]}.$$

If r_1 approaches r_2, the sum $\sum_{n=1}^{\infty} \rho_n$ approaches $r_2 \cdot \pi/2$ but is zero at the limit. Also $\sum_{n=1}^{\infty} i_n = 2r_1$, if $i_1 = \rho_1$. Let $\angle B_n = \pi/2$. Then since

$$\sin B_n = \frac{i_{n+1}}{\rho_n + \rho_{n+1}}$$

we have the relation

$$n = \frac{1 - m}{m^2},$$

and if $n = 1$, $r_1 : r_2 =$ side of decagon inscribed in a circle of unit radius. Here also as in sections (2) and (3) we may obtain an equation

$$[(2n^2 - 2n + 1)m^4 + (4n - 2)m^3 + 2]^2$$
$$= [2n^2 - 2n)m^4 + (4n - 2)m^3 + 4m^2 - 2]^2 + [m(2 - m^2)((2n - 1)m + 2)]^2,$$

which gives a doubly infinite set of rational right triangles.

5. Formulæ arising from a series of tangent circles tangent to a given circle and a given straight line. Let (Cf. figure 3) the series of circles (S_n) be tangent to the perpendicular at B and to O_2, (S_1) being also tangent to O_1.

The centers S_n lie on a parabola with vertex O_1 and focus O_2. Using O_1 as origin the equation of the parabola is

$$y^2 = 4(r_2 - r_1)x. \tag{11}$$

Moreover

$$\rho_n = r_1 - x_n \tag{12}$$

and

$$(y_n - y_{n-1})^2 = (\rho_n + \rho_{n-1})^2 - (x_n - x_{n-1})^2. \tag{13}$$

By a substitution from (12) equation (13) reduces to

$$(y_n - y_{n-1})^2 = 4\rho_n\rho_{n-1}. \tag{14}$$

Let $r_1 = \lambda r_2$. Let D_n denote the sum of the odd-numbered terms in the expansion of $(1 + \sqrt{\lambda})^n$: and N_n the sum of the even-numbered terms, that is

$$D_n + N_n = (1 + \sqrt{\lambda})^n,$$

$$D_n - N_n = (1 - \sqrt{\lambda})^n.$$

Then using equations (11), (12) and (14) and induction

$$\rho_n = \left(1 - \frac{N_n^2}{D_n^2}\right) r_1,$$

$$y_n = \frac{2N_n}{D_n}\sqrt{r_1(r_2 - r_1)}. \tag{15}$$

From (15)

$$\frac{2\sqrt{(1-\lambda)r_1r_2} - y_n}{2\sqrt{(1-\lambda)r_1r_2} + y_n} = \frac{(\sqrt{r_2} - \sqrt{r_1})^n}{(\sqrt{r_2} + \sqrt{r_1})^n}.$$

Therefore

$$\sum_{n=1}^{\infty} \frac{2\sqrt{(1-\lambda)r_1r_2} - y_n}{2\sqrt{(1-\lambda)r_1r_2} + y_n} = \frac{\sqrt{r_2} - \sqrt{r_1}}{2\sqrt{r_1}}.$$

Also we have

$$\sum_{n=1}^{\infty} (y_n - y_{n-1}) \mp 2\sqrt{r_1(r_2 - r_1)}.$$

6. Some properties of conics associated with three mutually tangent circles. Let there be two circles (O_1) and (O_2) tangent internally at A (Fig. 4) and let the radii of these circles be r_1 and r_2 respectively, and let the radius of a circle tangent to these two and having its center on the line O_1O_2 be r_3 and let its center be O_3. Let O be the center of any circle tangent to (O_1) and (O_2) and let r be its radius.

The conic associated with (O) and (O_2) is an ellipse with the points O and O_2 as foci, and passing through O_1: the conic associated with (O) and (O_1) is a hyperbola passing through O_2 and having O and O_1 as foci. Likewise we will have an ellipse passing through O and having O_1 and O_2 as foci. Draw from A the line AT tangent to the circles (O_1) and (O_2). Then with the three circles $(O_1,)$ (O_2) and (O) the straight line AT there will be associated four parabolas[1] two of which pass through A, one through O_1 and the fourth through O_2.

[1] See article "Some Properties of a Straight Line and Circle and their Associated Parabolas," *Annals of Math.*, second series, Vol. 19, pp. 174–5. Also "Some properties of circles and related conics," *Annals of Math*, second series, vol. 20, pp. 279–280.

Call the conic through O_2, H_2, the one through O_1, E_1 and the one through O and O_3, E_3, and the parabolas through O_1 and O_2, P_1 and P_2 respectively.

With this notation the following may be readily proved analytically:

THEOREM: The normals to E_1 and H_2 at the points O_1 and O_2 intersect on a line through O_3 perpendicular to O_1O_2.

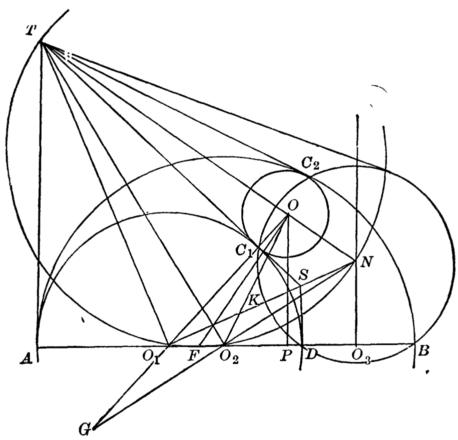


FIG. 4.

THEOREM: The tangents to H_2 and E_1 at the points O_2 and O_1 intersect on the line AT.

THEOREM: If two circles are tangent internally, and any circle is tangent to these two, there are associated with these circles three non-degenerate conics, six points of which are the three points of contact of the three circles and the three centers of the three circles, and the six tangents to the conics at these six points pass through a common point.

Proof: Let the normal to E_1 at O_1 and to H_2 at O_2 intersect at N, and let the intersection of the tangents at these points be T. Let the point of contact of (O) and (O_1) be C_1 and of (O) and (O_2) be C_2. Then the angle AO_1C_1 is bisected by O_1T. Therefore a line from T to C_1 will be tangent to (O_1) and (O). Moreover since T and N are ex-centers of the triangle OO_1O_2, T, N, and O are collinear. It is also evident that the tangent to (O_2) at C_2 will pass through T. We have therefore the six lines TA, TO, TO_1, TC_2 TO_2, and TC_1, and these lines are the six tangents to E_3, E_1 and H_2 at the points A, O, O_1, C_2, O_2, C_1.

THEOREM: E_1 and P_1 have the same normal at O_1.

Proof: Since O is the focus of P_1 and the axis is parallel to O_1O_2, then the bisector of the angle OO_1O_2 will be normal to P_1. But this is also normal to E_1 because O and O_2 are the foci of E_1.

THEOREM: The three axes of the three non-degenerate conics associated with three tangent circles, and the three normals at the centers of the circles, meet in points that are collinear.

Proof: Let the normal and axis of E_3 intersect in F, the normal and axis of H_2 intersect in G and the normal and axis of E_1 in K. Then since we have the triangle O_1O_2O and the two bisectors of two interior angles and the bisector of the opposite exterior angle, the points F, K and G are collinear.

The right angles formed by the tangents and normals at O_1 and O_2 are inscribed in a semicircle with TN as diameter. Call this circle C_t. Steiner has pointed out the fact that D, B, C_1 and C_2 are points of a circle C_n with center N.[1] It is then evident that the tangents to C_n at its points of intersection with C_t pass through T. We then have two sets of coaxial circles C_n and C_t, the centers

[1] See reference to Steiner in Introduction.

of one being on a line through O_3 perpendicular to O_1O_2 and the diameters of the other being segments of tangents to E_3 cut off by the tangents at A and O_3.[1] It should also be noted in this connection that the point T is the pole of the line drawn from the point of tangency of any two of the circles to the center of the third circle with respect to the conic passing through that center.

Also if there is drawn at D a line perpendicular to O_1O_2 and TC_1 is produced intersecting this line in S, then N, S and O_1 are collinear.

THEOREM: If three circles are mutually tangent and tangents and normals be drawn to the three associated conics at the points of contact and the centers of the three circles, and if the normals of two of the conics be chosen, these will intersect by twos on a tangent to the third conic.

Proof: Consider the lines O_1N, O_2N, OO_1, OO_2. These intersect in the points O and N which are on the tangent to E_3.

THEOREM: The axes and normals to two of the conics, together with the tangents to these two conics drawn at the centers of the circles determine two perspective triangles whose center of perspectivity is the intersection of the axis and normal to the third conic.

Proof: Consider the lines O_1N, O_2N, OO_1, OO_2, O_1T, O_2T. These intersect by twos on the line TN. They may therefore be considered as the sides of two perspective triangles. Let the corresponding sides be

$$\begin{array}{ll} O_1T, & O_2T, \\ O_2N, & O_1N, \\ OO_1, & OO_2. \end{array}$$

These determine the perspective triangles $A_{12}A_{13}A_{23}$ and $B_{12}B_{13}B_{23}$ and the center of perspectivity is the point F on O_1O_2 (see Fig. 4 where F is marked). But this point is also on the normal OB_{12}.

Corollary: There will be three such sets of perspective triangles and the centers of perspective will be collinear (second theorem before the last).

THEOREM: N and T are double points of an involution, of which O and the point (Y), where TN intersects O_1O_2, are a conjugate pair, and therefore O and Y are inverse points with respect to the circle C_t.

The proof of this theorem follows immediately from a consideration of the quadrangle $O_1A_{12}O_2B_{12}$.

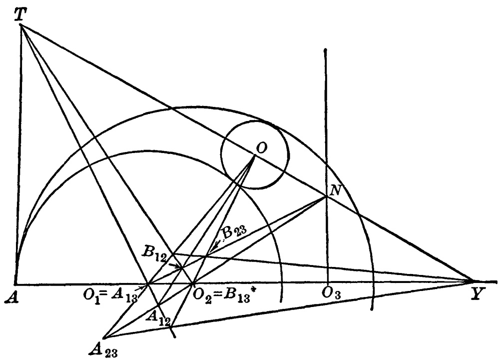


FIG. 5.

The following theorems are also evident.

THEOREM: If the three axes and the three normals to the three associated

[1] See *Conics* of Apollonius, Book III, prop. 45 (Heath's ed., p. 114).

conics be drawn the axis and normal to each conic being taken as corresponding sides, they form two perspective triangles with T as center of perspective.

THEOREM: The four axes of perspective of the three circles and the six lines, three of which are normals and the other three are the tangents to the three conics at the three centers of the three circles, determine a quadrangular-quadrilateral configuration, whose diagonal triangle is the triangle determined by the three centers of the three circles.

In connection with the above discussion it should be noted that it furnishes a method for constructing points on a conic. For let O_1, O_2 and O_3 be any three points on a line, and let the perpendicular be drawn at O_3 and let N be any point in the perpendicular. Let O_1 and O_2 be points such that we have the order $O_1O_2O_3$ or $O_2O_1O_3$. Then from N draw lines to O_1 and O_2, making the angles NO_1O_3 and NO_2O_3, and draw from O_1 and O_2 the lines O_1O and O_2O such that

$$\angle NO_1O_3 = \angle OO_1N,$$

$$\angle NO_2O_3 = \angle OO_2N.$$

Then the point O is on an ellipse. If we have the order $O_1O_3O_2$, O will be on a hyperbola, and if O_1 or O_2 is at infinity we have a parabola. And in each instance O_1 and O_2 are foci of the conic. This also gives a method for establishing a (1, 1) correspondence between the points of a conic and the points of a straight line.

7. Some Projective Properties of the Figure in Section 2. Let A and C be the ends of the diameter O_1O_2 of the circle (O_2) (Fig. 1), and let there be drawn from A lines to S_n and from C lines to S_n', and let C and C' be the angles that these lines make with AC. Then

$$\tan C = \frac{2nr_3}{r_1 + r_2}, \tag{16}$$

$$\tan C' = \frac{2nr_1}{r_2 + r_3}. \tag{17}$$

By means of equations (16) and (17) we may find the equations of the lines AS_n and CS_n', a solution of which reveals the fact that the line AS_n and the line CS_n' intersect on a line through S_1 perpendicular to AC in points whose ordinates are $2\rho_n$.[1]

By very simple analytical considerations we may prove the following

THEOREM: The triangles $S_{n+1}S_nS_{n-1}$ and $S_{n+1}'S_n'S_{n-1}'$ are perspective and their center of perspective is the external center of perspective of the circles O_1 and O_3.

THEOREM: The locus of the point of contact of two tangent circles which are tangent to two given tangent circles (internally tangent) is a circle whose center is the center of perspective of the two given circles and whose radius is the harmonic mean between the radii of the two given circles.

[1] In this connection see Steiner, p. 69 and ff.

Proof: The center of perspective, P_3, of the two given circles, O_1 and O_2, has the same power with respect to all circles tangent to these two in a given way. Therefore, the locus of the point of contact of any two such circles which are tangent to each other is a circle orthogonal to them all.

THEOREM: The circle with P_3 as center and P_3A as radius cuts every C_n orthogonally.

Proof: Let there be drawn with T as center and TA as radius a circle. This will pass through C_1 and C_2. Therefore $C_1C_2P_3$ will be the radical axis of the circle just drawn and C_n. The circle with P_3 as center and P_3A as radius is orthogonal to the circle with center T. It is therefore orthogonal to every C_n.

SOME VANISHING AGGREGATES CONNECTED WITH CIRCULANTS.

By W. H. METZLER, Syracuse University.

In the course of certain investigations on Lagrange's Equation for circulants[1] by Dr. Muir[2] in 1912 attention was called to the vanishing aggregate:

$$\begin{vmatrix} 1 & b & c & d \\ 1 & c & d & e \\ 1 & d & e & a \\ 1 & e & a & b \end{vmatrix} + \begin{vmatrix} 1 & b & c & e \\ 1 & c & d & a \\ 1 & d & e & b \\ 1 & a & b & d \end{vmatrix} + \begin{vmatrix} 1 & b & d & e \\ 1 & c & e & a \\ 1 & e & b & c \\ 1 & a & c & d \end{vmatrix} + \begin{vmatrix} 1 & c & d & e \\ 1 & e & a & b \\ 1 & a & b & c \\ 1 & b & c & d \end{vmatrix} = 0,$$

where a, b, c, d, e are the elements of a circulant of order five. He obtained it as the coefficient of the first power of x in Lagrange's equations, which power (as well as all the odd powers) was proven not to exist, and next enunciates the following general theorem: *If the elements of the first column of any odd-ordered circulant, axisymmetric with respect to the principal diagonal, be replaced by units, the sum of the complementary minors of the elements in the places* (2, 2), (3, 3), $\cdots$, (n, n) *vanishes.*

He next points out that in the case $n = 7$ we may substitute for

$$[2, 2]_1 + [3, 3]_1 + [4, 4]_1 + [5, 5]_1 + [6, 6]_1 + [7, 7]_1 = 0,$$

the two relations

$$[2, 2]_1 + [3, 3]_1 + [5, 5]_1 = 0,$$

$$[4, 4]_1 + [6, 6]_1 + [7, 7]_1 = 0,$$

where $[p, q]_r$ denotes the complementary minor of the element in the pth row and qth column after the rth column of the circulant has been replaced by units.

[1] For the purposes of this paper the following definition will be assumed: If each row of a determinant may be derived from the preceding row by passing the first element over all the others to the last place, the determinant is called a circulant.

[2] "Lagrange's determinantal equation in the case of a circulant," *Messenger of Mathematics*, New Series, vol. 41, March, 1912.

The object of this note is to investigate these aggregates a little more closely and from a somewhat different viewpoint, and to determine those that are fundamental.

By equating the coefficients of odd powers of x to zero we would get various vanishing aggregates connecting minors of even order but these aggregates would not be fundamental.

Starting with the determinant

$$\begin{vmatrix} b & c+a & d+e & 1 \\ c & d+b & e+a & 1 \\ d & e+c & a+b & 1 \\ e & a+d & b+c & 1 \end{vmatrix},$$

which obviously vanishes since two columns may be made identical, we have

$$\begin{vmatrix} b & c & d & 1 \\ c & d & e & 1 \\ d & e & a & 1 \\ e & a & b & 1 \end{vmatrix} + \begin{vmatrix} b & c & e & 1 \\ c & d & a & 1 \\ d & e & b & 1 \\ e & a & c & 1 \end{vmatrix} + \begin{vmatrix} b & a & d & 1 \\ c & b & e & 1 \\ d & c & a & 1 \\ e & d & b & 1 \end{vmatrix} + \begin{vmatrix} b & a & e & 1 \\ c & b & a & 1 \\ d & c & b & 1 \\ e & d & c & 1 \end{vmatrix} = 0,$$

or

$$[1, 4]_5 - [1, 5]_3 + [1, 2]_4 - [1, 3]_2 = 0. \tag{1}$$

From the properties of circulants of odd order we have

1. When $1 + r$ is even

$$[1, r]_s = \left[\frac{1+r}{2}, \frac{1+r}{2}\right]_{s-[(r-1)/2]} \quad \text{if } s > \frac{r-1}{2},$$

$$= \left[\frac{1+r}{2}, \frac{1+r}{2}\right]_{n+s-[(r-1)/2]} \quad \text{if } s \overline{\gtrless} \frac{r-1}{2}.$$

2. When $1 + r$ is odd

$$[1, r]_s = -\left[\frac{n+r+1}{2}, \frac{n+r+1}{2}\right]_{s-[(n+r-1)/2]} \quad \text{if } s > \frac{n+r-1}{2},$$

$$= -\left[\frac{n+r+1}{2}, \frac{n+r+1}{2}\right]_{s+[(n-r+1)/2]} \quad \text{if } s \overline{\gtrless} \frac{n+r-1}{2}.$$

We also have

1. When $r + s$ is even

$$[r, r]_s = -\left[\frac{r+s}{2}, \frac{r+s}{2}\right]_{n+[(3r-s)/2]} \quad \text{if } 3r \overline{\gtrless} s,$$

$$= -\left[\frac{r+s}{2}, \frac{r+s}{2}\right]_{(3r-s)/2} \quad \text{if } 3r > s;$$

2. When $r + s$ is odd

$$[r, r]_s = -\left[\frac{n+r+s}{2}, \frac{n+r+s}{2}\right]_{n+[(3r-n-s)/2]} \quad \text{if } 3r \lesseqgtr s+n,$$

$$= -\left[\frac{n+r+s}{2}, \frac{n+r+s}{2}\right]_{(3r-n-s)/2} \quad \text{if } 3r > s+n.$$

Making use of these relations (1) takes the form

$$[2, 2]_1 + [3, 3]_1 + [4, 4]_1 + [5, 5]_1 = 0,$$

which is as given by Muir.

If we had started with

$$\begin{vmatrix} b & c+e+a & d & 1 \\ c & d+a+b & e & 1 \\ d & e+b+c & a & 1 \\ e & a+c+d & b & 1 \end{vmatrix},$$

which is also equal to zero, we would have

$$[1, 4]_5 - [1, 5]_2 + [1, 2]_4 = 0. \tag{2}$$

Similarly

$$[1, 3]_5 + [1, 5]_2 - [1, 2]_3 = 0. \tag{3}$$

From (2) and (3) we get

$$[1, 3]_5 - [1, 2]_3 + [1, 4]_5 + [1, 2]_4 = 0,$$

or

$$[3, 3]_1 + [2, 2]_1 + [5, 5]_1 + [4, 4]_1 = 0,$$

which is (1).

Again

$$[1, 1]_2 + [3, 3]_2 + [4, 4]_2 + [5, 5]_2 = 0;$$

and since $[5, 5]_1 = -[3, 3]_2$, we have

$$[1, 1]_2 + [4, 4]_2 + [5, 5]_2 + [2, 2]_1 + [3, 3]_1 + [4, 4]_1 = 0. \tag{4}$$

The general law for (2) is, in the case of circulants of odd order,

$$(-1)^{r-1}[1, r]_s + (-1)^{s-1}[1, s]_t + (-1)^{t-1}[1, t]_r = 0. \tag{5}$$

The general law for (1) is

$$(-1)^{r-1}[1, r]_s + (-1)^{s-1}[1, s]_t + (-1)^{t-1}[1, t]_u + (-1)^{u-1}[1, u]_r = 0, \tag{6}$$

but, as has been seen, (6) is made up of two of (5).

The method here used will give vanishing aggregates of minors of any order. Thus from

$$\begin{vmatrix} 1 & a+b+c+d & e \\ 1 & b+c+d+e & a \\ 1 & c+d+e+a & b \end{vmatrix} = 0,$$

where as before, a, b, c, d, e are the elements of a circulant of order five, we have

$$\begin{vmatrix} 1 & a & e \\ 1 & b & a \\ 1 & c & b \end{vmatrix} + \begin{vmatrix} 1 & b & e \\ 1 & c & a \\ 1 & d & b \end{vmatrix} + \begin{vmatrix} 1 & c & e \\ 1 & d & a \\ 1 & e & b \end{vmatrix} + \begin{vmatrix} 1 & d & e \\ 1 & e & a \\ 1 & a & b \end{vmatrix} = 0,$$

or

$$-\begin{bmatrix} 45 \\ 23 \end{bmatrix}_4 + \begin{bmatrix} 45 \\ 34 \end{bmatrix}_1 - \begin{bmatrix} 45 \\ 41 \end{bmatrix}_2 + \begin{bmatrix} 45 \\ 12 \end{bmatrix}_3 = 0,$$

or

$$-\begin{bmatrix} 12 \\ 12 \end{bmatrix}_5 + \begin{bmatrix} 12 \\ 12 \end{bmatrix}_4 - \begin{bmatrix} 15 \\ 15 \end{bmatrix}_3 + \begin{bmatrix} 15 \\ 15 \end{bmatrix}_2 = 0,$$

where $\begin{bmatrix} r\,s \\ t\,u \end{bmatrix}_v$, is the complementary of the minor of the elements in the intersection of the rth and sth rows, and the tth and uth columns, after the elements in the vth column of the circulant are replaced by units.

Using this same method we may also obtain vanishing aggregates of minors of circulants of even order. Thus for $n = 4$ we have

$$\begin{vmatrix} 1 & a+b+c & d \\ 1 & b+c+d & a \\ 1 & c+d+a & b \end{vmatrix} = 0,$$

and

$$\begin{vmatrix} 1 & a & d \\ 1 & b & a \\ 1 & c & b \end{vmatrix} + \begin{vmatrix} 1 & b & d \\ 1 & c & a \\ 1 & d & b \end{vmatrix} + \begin{vmatrix} 1 & c & d \\ 1 & d & a \\ 1 & a & b \end{vmatrix} = 0,$$

or

$$-[1, 1]_2 + [1, 2]_4 + [1, 4]_1 = 0,$$

which is in accord with (5).

Again

$$\begin{vmatrix} 1 & a+c & b+d \\ 1 & b+d & c+a \\ 1 & c+a & d+b \end{vmatrix} = 0,$$

and

$$[1, 2]_3 - [1, 3]_4 + [1, 4]_1 - [1, 1]_2 = 0,$$

which is in accord with (6). It is otherwise evident that we may remove from (5) and (6) the restriction that n must be odd, and have them true for any order of circulant.

November, 1918.

QUESTIONS AND DISCUSSIONS.

EDITED BY W. A. HURWITZ, Cornell University, Ithaca, N. Y.

DISCUSSIONS.

We present this month only one discussion, by Professor Florence P. Lewis, dealing with Euclid's parallel postulate. Professor Lewis sketches the history of the controversy over this postulate, and indicates its bearing on the problems of teaching. The subject is of great interest and importance. While it would be rash to assert that a knowledge of the history of non-Euclidean geometry and an acquaintance with the modern views regarding the nature of a postulate are indispensable prerequisites to successful teaching of elementary geometry, nevertheless it can not be denied that such equipment must notably enrich and vivify a teacher's own appreciation of the subject and thus add materially to the effectiveness of teaching.

The subject is an extensive one, not readily amenable to adequate treatment in a short article. It is unlikely that any two persons, in giving a short account of the *history* of the parallel postulate and the development of non-Euclidean geometry, would make exactly the same selection of names to be mentioned or the same interpretation of historical facts. Still less likely would be the agreement of independent writers on the *pedagogic* aspects of the question. Thus our readers will probably occasionally disagree with parts of Professor Lewis's treatment or dissent from some of her conclusions. But it is believed that the article as a whole represents a consistent and just account, which should be of value to many readers, especially those concerned with the teaching of geometry.

Professor Lewis's contention that we should cease to fear redundancy in our list of assumptions seems irrefutable. A course in geometry for adolescents should not be planned in the same way as a course for graduate students in the university. Why "prove" to a high school student that circles with equal radii are congruent? To the poor student the proof brings no added conviction; while the good student wonders why, if this proof is logically necessary, it is not even more essential to show that no arc of a circle is a segment of a straight line. Both statements would be welcome as assumptions.

Perhaps it is not easy to draw the line between what is to be assumed and what is to be proved. Probably every child would accept as assumptions the propositions on congruence of triangles; probably almost none would be satisfied to accept the angle sum of a triangle without proof. It may require some discrimination to decide just which theorems arouse in the student that *demand for an answer* which is the kernel of all successful teaching; but there can be little doubt that such decision should form the basis of our future treatment of elementary geometry.

History of the Parallel Postulate.[1]

By Florence P. Lewis, Goucher College.

Like the famous problems of construction, Euclid's postulate concerning parallels is a thought that links the ages. Its history is a long story with dramatic climax and far-reaching influence on modern mathematical and general scientific thought. I wish to recall briefly the salient features of the story, and to state what seem to me its suggestions in regard to the teaching of elementary geometry.

Euclid's fifth postulate (called also the eleventh or twelfth axiom) states: "If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines if produced indefinitely meet on that side on which are the angles less than two right angles." The earliest commentators found fault with this statement as being not self-evident. Concerning the meaning of *axiom,* Aristotle says: "That which it is necessary for anyone to hold who is to learn anything at all is an axioms;" and "It is ignorance alone that could lead anyone to try to prove the axiom." Without going into the difficult question of the precise distinction to the Greek mind between axiom and postulate, we may take it that the character of being indisputable pertained to each. Postulates stating that a straight line joining any two points can be drawn, that a circle can be drawn with given center and radius, or that all right angles are equal, were accepted, while the postulate of parallels was scrutinized and admitted at best with reluctance.

Proclus, writing in the fifth century A. D., gives some of the reasons for this attitude, and we may surmise others. The postulate makes a positive statement about a region beyond the reach of possible observation or geometrical intuition. Proclus insinuates that those who "suppose they have ground for instantaneous belief" are "yielding to mere plausible imaginings"; the conclusion is "plausible but not necessary."[2] The converse is proved in Proposition 27, book I of Euclid's *Elements,* and there seems to be no reason why this proposition should be more or less self-evident than its converse. The fact that the two lines continually approach each other[3] was not a convincing argument to the Greek geometer who was acquainted with the relation of the hyperbola to its asymptote. The form of statement of the postulate is long and awkward compared with that of the others, and its obviousness thereby lessened. There is evidence that Euclid himself endeavored to prove the statement before putting it down as a postulate; for in some manuscripts it appears not with the others but only just before Proposition 29, where it is indispensable to the proof. If the order is significant, it indicates that the author did not at first intend to include this among the postulates, and that he finally did so only when he found that he could neither prove it nor proceed without it.

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[1] Read before the Association of Teachers of Mathematics in New England, May 3, 1919.

[2] Cf. Heath's Euclid, Vol. I, and Bonola's *Non-Euclidean Geometry,* Chicago, Open Court, 1912, to which reference is made throughout this paper.

[3] Even the meaning of this phrase requires further elucidation.

Most of the early geometers appear to have attacked the problem. Proclus quotes and criticizes several proofs, and gives one of his own. He instances one writer who even attempted to prove the falsity of the statement, the argument being similar to those used in Zeno's paradoxes. The common opinion, however, seems to have been that the postulate stated a truth, but that it ought to be proved. Euclid had proved two sides of a triangle greater than the third, which is far more obvious than this. If the statement was true it should be proved in order to convince the doubters; if false, it should be removed. In no case should it be retained among the fundamental presuppositions. Sir Henry Savile (1621) and the Italian Saccheri (1733) refer to it as a blot or blemish on a work that is otherwise perfect, and this expresses the common attitude of mathematicians until the first quarter of the nineteenth century.

Early attempts at proof usually took the form of a change in the definition of parallels, or the substitution, conscious or unconscious, of a new assumption. Neither of these methods resulted in satisfaction to any but their inventors; for the definitions usually concealed an assumption, and the new postulates were no more obvious than the old. Posidonius, quoted by Proclus, defines parallels as lines everywhere equidistant. This begs the question; surely such parallels do not meet, but may there not be in the same plane other lines, not equidistant,which also do not meet? The definition involves also the assumption, that the locus of points in a plane at a given distance from a straight line is a straight line, and this was not self-evident.[1] Ptolemy says that two lines on one side of a transversal are *no more parallel* than their extensions on the other side; hence if the two angles on one side are together less than two right angles, so also are the two angles on the other side, which is impossible since the sum of the four angles is four right angles. This is another way of saying that through a point but one parallel to a given line can be drawn, which is exactly Euclid's postulate. Proclus himself assumes (with some concealment) that if a line cuts one of two parallels it cuts the other, which is again postulate 5. Even as late as the close of the eighteenth century we find this argument advanced by one Thibault, and attributed also to Playfair: Let a line segment with one end A at a vertex of a triangle be rotated through the exterior angle. Translate it along the side until A comes to the next vertex and repeat the process. We finally arrive at the original position and must therefore have rotated through 360°. Hence the sum of the interior angles of a triangle is 180°; and, since Legendre had satisfactorily proved that this proposition entails Euclid's postulate of parallels, the latter is at last demonstrated. The fact that the same process could equally well be carried out with a spherical triangle, in which the angle-sum is not 180°, might have given him pause. The assumption that translation and rotation are independent operations is in fact equivalent to Euclid's postulate.

[1] It should be noted that even the meaning of the criterion suggests several questions of logic. If two lines are so placed that perpendiculars to one of them from points on the other are equal, will the same statement hold when the rôles of the two lines are reversed? Will a perpendicular to one of two non-intersecting lines necessarily be perpendicular to the other? Of course the answers to these questions are closely bound up with the very postulate under discussion.—EDITOR.

Heath gives (l. c.) a long and instructive list of these substitutes. In the course of centuries the minds of those interested became clear on one point: they did not wish merely to know whether it was possible to substitute some other assumption for Euclid's, though this question has its interest; they wished to know primarily whether exactly his form of the postulate was logically deducible from his other postulates and established theorems. To change the postulate was merely to re-state the problem.

After certain Arabs and Persians had had their say in their day, the curtain rises on the Italian Renaissance of the sixteenth century, where the problem was attacked with great vigor. French and British assailants were not lacking. The first modern work devoted entirely to the subject was published by Cataldi in 1603. When the eighteenth century took up the unfinished business of proving the parallel postulate, we find most of the giants of those days attacking the enemy of geometers with an even keener sense that without victory there could be no peace. Yet d'Alembert toward the close of the century could still refer to the state of the theory of parallels as "the scandal of elementary geometry." Klügel in 1763 examined thirty demonstrations of the postulate. He was perhaps the first to express doubt of its demonstrability. Lagrange, according to De Morgan, in about 1800, when in the act of presenting to the French Academy a prepared memoir on parallels, interrupted his reading with the exclamation, "Il faut que j'y songe encore," and withdrew his manuscript.

While the results of these investigations were on the whole negative, certain positive and valuable results were nevertheless obtained. The relation between the parallel postulate and the angle sum of a triangle was clearly brought out. Legendre proved that if in a single triangle the angle sum is two right angles, the postulate holds. Other equivalents are of interest. John Wallis and Laplace wished to assume: There exists a figure of arbitrary size similar to any given figure. Gauss could proceed rigorously provided he could prove the existence of a rectilinear triangle whose area is greater than any previously assigned area. W. Bolyai could have succeeded with the assumption that a circle can be passed through any three points not in a straight line. It must be borne in mind, moreover, that few mathematical questions have served so well as whetstones on which to sharpen the critical powers of mankind.

The work of the Italian priest Saccheri deserves notice because his method is that which finally brought the discussion to a close. Though published in 1733 his results did not become well known until after 1880, and therefore had little influence on other investigations. Legendre's *Réflexions*, published a hundred years later, covered much of the same ground without advancing quite so far. The title of Saccheri's work is *Euclides ab omni Naevo Vindicatus*, Euclid Vindicated of every Flaw. His plan was to prove the postulate by assuming its contradictory and showing that an inconsistency followed. He succeeded in proving that, according as in one triangle the angle sum is greater than, equal to, or less than two right angles, the same holds in every triangle, and that accordingly Euclid's postulate or one of its contradictories will hold.

He makes three hypotheses which were recognized later to correspond to the elliptic, Euclidean and hyperbolic geometries. But at the end of his work, in order to exhibit a contradiction when Euclid's postulate is denied, he is forced to make use of a somewhat vague and unacceptable assumption about "the nature of a straight line."

Gauss's activity in connection with the parallel postulate is of especial interest because of its psychologic aspect. It is difficult for us to picture a mathematician hesitating to publish a discovery for fear of the outcry that its publication might produce—perhaps not many would be displeased to awaken an echo; yet this is believed by some to have been the attitude of Gauss. Though he was keenly interested and thought deeply on the subject of parallels, he published nothing; he feared, as he said, "the clamor of the Bœotians." When forced to write a letter on the subject, he begs his correspondent to keep silence as to the information imparted. In 1831 he writes in a letter: "In the last few weeks I have begun to put down a few of my Meditations [on parallels] which are already to some extent forty years old. These I had never put in writing, so that I have been compelled three or four times to go over the whole matter afresh in my head. Also I wished that it should not perish with me." It is only when we call to mind the unrivalled place of honor held by Euclidean geometry among branches of human knowledge—a respect no doubt enhanced by the prominence given it in Kant's *Critique of Pure Reason*—that we realize the uncomfortable position of one who even appeared to attack its validity. Gauss's meditations were leading him through tedious and painstaking labors to the conclusion that Euclid's fifth postulate was not deducible from his other postulates. The minds of those not conversant with the intricacies of the problem might easily rush to the conclusion that Euclid's geometry was therefore untrue, and feel the whole structure of human learning crashing about their ears.

Between 1820 and 1830, the conclusion toward which Gauss tended was finally made sure by the invention of the hyperbolic non-Euclidean geometry by Lobachevsky and Johann Bolyai, working simultaneously and independently. The question, Is Euclid's fifth postulate logically deducible from his other postulates? is answered by showing that the denial of this postulate while all the others are retained leads to a geometry as consistent as Euclid's own. The method, we recall, was that used by Saccheri, whose intellectual conservatism alone prevented his reaching the same result. The famous postulate is only one of three mutually exclusive hypotheses which are logically on the same footing. Thus was Euclid "vindicated" in an unexpected manner. Knowingly or not, the wise Greek had stated the case correctly, and only his followers had been at fault in their efforts for improvement. To quote Heath: "We cannot but admire the genius of the man who concluded that such an hypothesis, which he found necessary to the validity of his whole system, was really indemonstrable."

Thus in some sense the problem of the parallel postulate was laid to rest, but its spirit marches on. If the fifth postulate could without logical error be replaced by its contradictory, could the other postulates be similarly treated? What is

the nature of a postulate or axiom? What requirements should a satisfactory system of axioms fulfill? Are we sure that accepted proofs will bear as keen scrutiny as that to which proofs of the postulate have been subjected? The facing of these questions has brought us to the modern critical study of the foundations of geometry. It has been realized that if geometry is to continue to enjoy its reputation for logical perfection, it should at least try to deserve it. The edge of criticism, sharpened on the parallel postulate, is turned against the whole structure. Out of this movement has grown the critical examination of the foundations of algebra, of projective geometry, of mechanics, of logic itself; and the end is not in sight.

One obvious result of this critical study is that geometrical axioms are not necessary truths, but merely presuppositions: they are the hypotheses on which the whole body of theorems rests. It is essential that a system of axioms should be consistent with each other, and desirable that they be non-redundant, and complete. No one has found Euclid's system inconsistent,[1] and redundancy would be a crime against elegance rather than against logic. But on the score of completeness Euclid is far from giving satisfaction. He not infrequently states conclusions which could be arrived at only by looking at a figure, *i.e.*, by space intuition; but we are all familiar with cases where space intuition misleads (for example in the fallacious proof that all triangles are isosceles), and if we accept it as a guide how can we be sure that our intuitions will always agree? In constructing an equilateral triangle Euclid says, "From point C where the two circles meet, draw $\cdots$." Perhaps they do meet—but not on the basis of anything previously stated. In dropping a perpendicular from a point to a line, he says a certain circle will meet the line twice. Why should it not cut thrice or not at all? In another proof he says that a certain line will lie within a certain angle. I *see* that it does, but I do not see it proved. We are told in the midst of a proof to bisect an angle of a triangle and produce the bisector to meet the opposite side. How do we know it will meet? Because it is not a parallel. And probably it is not parallel because it is inside the triangle. How do we know it is inside; or, being inside, that it must get outside? When have these terms been defined? You may answer: It is not necessary to define them because everyone with common sense knows inside from outside without being told. "Who is so dull as not to perceive $\cdots$?" says Simson, one of Euclid's apologists. This may be granted. But it must be pointed out that common sense knows that two straight lines cannot enclose space, yet this is given prominence as an axiom; or that a straight line is the shortest distance between two points, yet this is proved as a proposition. To state in words what distinguishes the inside from the outside of a polygon is not easy. The word "between" is likewise difficult of definition. But the modern geometer imbued with the critical spirit feels it necessary to define such terms, and what is more, he finds a way of doing it.

[1] It is in fact possible to show that the system is consistent, provided we agree to accept the axioms of arithmetic as consistent. While this is only a transformation of the problem, it is logically important to recognize the possibility of such a transformation.—EDITOR.

Hilbert's *Grundlagen der Geometrie*, published in 1899, is a classic product of this movement. It presupposes no space concepts at all, but only such general logical terms as "corresponds to," "associated with," "determined by." Contrary to tradition, it does not begin by defining terms. The first sentence is: "I think of three systems of things which I call points, lines and planes." Note the unadorned simplicity of the concept *things*. The axioms serve as definitions. They state, in non-spatial terms, relations between these "things"; that is to say, the points, lines and planes are such things as have such and such relations. "That is all ye know on earth, and all ye need to know." Twenty-one axioms are found necessary, as against Euclid's meager five. The whole work could be read and comprehended by a being with no space intuitions whatever. We could substitute the names of colors or sounds for points, lines and planes, and get on equally well. The ideal of making a thing "so plain that a blind man could see it" is literally realized. And the age-old ideal of a body of proved propositions, close-knit together by unassailable logic, is immeasurably nearer realization.

Although to the best of my knowledge no one has yet had the hardihood to invite a child of fourteen to consider "three systems of things,"[1] the modern critical movement is not without bearing on problems of teaching. I wish, with proper humility, to put forward a few ideas on this subject.

If it is true that our traditional formal geometry, taken directly or indirectly from Euclid, is not the logically perfect thing we had imagined, and if its modern perfected descendant is so abstract that not even the most rationalistic of us would venture to force it on beginners, why not acknowledge these facts and bravely face anew the question of how we can best make the study of elementary geometry serve its proclaimed purpose of training the mind? I would suggest two lines along which progress might be made. First, by sacrificing the ideal of non-redundancy in our underlying assumptions we could save time and stimulate interest by arriving more quickly at propositions whose truth is not immediately evident and which could be presented as subjects for investigation. Must we, because Euclid did it, prove that the base angles of an isosceles triangle are equal? A child that has cut the triangle out of paper and folded it over knows as much as any proof can teach him. If to treat the proposition in this way is repugnant to the teacher's logical conscience, let him privately label it "axiom" or "postulate," and proceed, even though this proposition could have been proved. *The place to begin producing arguments is the place where the truth of the proposition is even momentarily in doubt.* One statement which presents itself with a question mark and is found after investigation to be true or to be false is worth ten obviously true statements proved with all the paraphernalia of hypothesis, conclusion, step one and step two, with references. The only apparent reason for proving in the traditional way the theorems on the isosceles triangle and congruent triangles is in order to familiarize the student with the above-mentioned paraphernalia. This brings me to my second suggestion.

Formal geometry has been looked upon as a complete and perfect thing to

[1] It is possible that Halsted's *Rational Geometry* has a somewhat similar purpose.—EDITOR.

which the learner can with profit play the sedulous ape. Yet I sometimes think that by emphasizing too early the traditional form of presentation of geometrical argument, and paying too little attention to the psychology of the learner, we may have corrupted some very good minds. "I wish to prove . . .," says the student; meaning, I wish to prove something stated and accepted as true in advance of argument. Should we not prefer to have our students say, "I wish to examine . . ., to understand, to find out whether . . ., to discover a relation between . . ., to invent a means of doing . . ."? What better slogan could prejudice desire than "I wish to prove"? The conscienceless way in which college debaters collect and enumerate arguments regardless of the issues involved is another aspect of the same evil. A student said not long ago, "The study of mathematics would be good fun if we did not have to learn proofs." It had never been brought home to her that mathematical reasoning is not a thing to be acquired, like a knowledge of Latin verbs, but a thing to be participated in like any other form of exercise. Another said, "I cannot apply my geometry because all we did in school was to learn the proofs and pass the examinations."

In the midst of a proof the student hesitates and says, "I am sure this is the next step, but I cannot recall the reason for it." The step and its reason would occur simultaneously to a mind that had faced the proposition as a problem and thought it through. I should like to see in every text an occasional page of exercises to prove or disprove. And if formal proofs must be printed in full, by all means let some of the proofs be wrong.

When the student has thus halted with one foot in the air in this progress from step to step down the printed page, on what does the ability to proceed depend? On the ability to quote something: to quote, usually, a single statement—compact, authoritative, triumphantly produced. Surely it is bad training that leads the mind to *expect* to find support for its surmises in a form so simple; and the temptation to substitute ability to quote in place of the labor of finding out the truth may be a real danger. What wonder if the mind so trained quotes Washington's Farewell Address or the Monroe Doctrine and feels that its work is done?

I do not mean that formal proof should never be given. It has its place as an exercise in literary composition; for it deals with the form in which thought is expressed. We should, however, take every possible precaution to see that the thought is first there to be expressed, lest the form be mistaken for the substance. Just how this is to be brought about I am not prepared to discuss, although I suspect that drawing and measurement in the early stages of study, problems of construction and investigation, and the total absence of complete proofs from the printed page, would help. I wish merely to state my belief that only in so far as we succeed in these aims shall we succeed in making geometry really train the mind.

> It can be done, said the butcher, I *think;*
> It must be done, I am sure.

One point further. Perhaps we are a little too modest about the importance

of having our students retain something of the subject-matter of the courses we teach. Evidently it is here that memory, based on understanding, should rightly be used. I sometimes think we might in some way collectively take out insurance against a student's arriving at the junior year in college in the belief that two triangles are similar whenever they have a side in common.

RECENT PUBLICATIONS.

REVIEW.

Introductory Mathematical Analysis. By W. Paul Webber and Louis C. Plant. New York, Wiley, 1919. 13 + 304 pages. Price $2.00.

This combination book for freshmen covers plane trigonometry and topics from advanced algebra, analytic geometry, and the calculus, which are treated in the order named.

The first seven chapters (70 pages) may be said to form an introduction containing a review of algebra and geometry, and such general topics as computations, including logarithms and the slide rule, rectangular coördinates with graphical representations of statistics and equations, classes of numbers, variables, functions, and limits. Interpolation in tables involving two independent variables is given, and simple problems to determine the form of an equation representing a given empirical table are solved. In a later chapter some problems of the latter sort are solved by using logarithmic and semi-logarithmic paper.

Chapter VIII (50 pages) gives the usual course in plane trigonometry in condensed form. In defining the six functions a tabular form is used in which a row is given to each function and a column to an angle in each of the four quadrants, a point $P_1(x_1,y_1)$ being chosen on the terminal line of an angle α_1 in the first quadrant, a point $P_2(x_2, y_2)$ on the terminal line of an angle α_2 in the second quadrant, etc. The table includes, in parentheses, the sign of each function for each quadrant. A radian "is defined by the equation Arc $AB = r \cdot \theta$, where r is the radius of the arc AB and θ is the angle in radians." Portions of a traverse table and of a table of haversines, and some problems in indirect fire, appear among the applications.

The chapter on the circular functions is followed by one on polar coördinates, complex numbers, and vectors, which connects trigonometry with algebra. Vector and scalar products are given considerable attention, with application to equilibrium of particles and rigid bodies. A short chapter on equations gives a few of the theorems from the theory of equations, but the opportunity to tie it to the preceding chapter by the theorem that complex roots occur in conjugate pairs was not improved. The reviewer holds no brief for this theorem, especially as he thinks other material more suitable for and valuable to freshmen than complex numbers, but the omission is noticeable in a text which considers the derivative of u^n for a complex value of n, which treats series with complex terms, and which expresses the sine and cosine in the well-known exponential forms in order to

obtain their derivatives. Real roots of equations of higher than the second degree are approximated by synthetic division without shifting the y-axis.

The chapters on analytic geometry (41 pages) are devoted chiefly to the straight line and conic sections. These are treated very compactly, but with the completeness of many texts. Space is saved in the chapter on conics by leaving much of the theory to be worked out as exercises. For example, the equation of an ellipse is derived in the form

$$(1 - e^2)x^2 - 2epx + y^2 = 0,$$

and the following exercises given.

"2. Move the origin to the point $(ep/(1 - e^2), 0)$ to remove the term in the first power of x."

"3. Call $ep/(1 - e^2) = a$ and $(1 - e^2)a^2 = b^2$ in the equation obtained in example (2) and show that the equation reduces to

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1."$$

Nearly a third of the book (88 pages) is given to the calculus. The ϵ, δ form of proof is used for the preliminary theorems on limits, the derivative introduced in a purely formal way, and the theorems necessary for differentiating algebraic functions, implicit as well as explicit, are obtained before it comes to light that the derivative has any concrete significance, either as the slope of a tangent line or as a rate. Series receive an extended treatment. Maclaurin's series is obtained by differentiating an assumed expansion and determining the coefficients. The function e^x is defined as a series, and the expansion of $e^{\Delta u}$ is the basis for obtaining the derivative of e^u. Integration is considered as the inverse of differentiation, and also as the limit of a sum. The applications of the calculus most emphasized are tangent lines, maxima and minima, velocity and acceleration, plane areas, volume of a solid of revolution, work, and centroids.

Much of the material is presented in a very compact form, so that, as stated in the preface, "The teacher will find an opportunity for originality in developing the text and at times a necessity for more details." The authors' style is sometimes abrupt, as in the sentence "Let student solve and verify," and a few sentences are open to more serious criticisms. For example: "Let x be a variable and a some constant. Then if $a - x$ assumes its sequence of values in order, there comes a stage, such that, for all subsequent values of x, the numerical value of $a - x$ becomes and remains less than any assigned small value ϵ, then a is called the **limit** of x" (page 64). "If, at a given instant of time, the speed of a body becomes uniform, then the distance Δs, passed over in the time Δt, is the instantaneous speed at the given instant" (page 214). The treatment of instantaneous speed which follows the latter quotation is unsatisfactory in that it confuses two different meanings of Δs.

ARTHUR SULLIVAN GALE.

BRIER HILL, N. Y.,
August, 1919.

NOTES.

"In 1880 the Cambridge University Press began the republication in collected form of Stokes's *Mathematical and Physical Papers*. In this publication he introduced for the first time the *solidus* notation for division, originally introduced by De Morgan in his article on the Calculus of Functions in the *Encyclopaedia Metropolitana*. If a fraction like $\frac{a}{b}$ or a differential coefficient such as $\frac{dy}{dx}$, is mentioned in the text, the printing of such expressions requires a good deal of "justification" on the part of the compositor. To avoid this expense and the loss of space Stokes introduced the linear notation a/b and dy/dx." [A. Macfarlane's lecture on "Sir George Gabriel Stokes" in *Lectures on Ten British Physicists of the Nineteenth Century* (New York, 1919), pp. 100–101.]

The Mathematical Association of Japan for Secondary Education issued the first number of Volume 1 of its *Journal* under date of April, 1919. It is published by M. Kaba, Tokyo, as chief editor, assisted by M. Kuroda, M. Watanabe, S. Nagao and H. Furnkawa. While all papers in this issue were printed in Japanese, the announcement is made that contributions in English are accepted. The contents of the initial issue are as follows:

Treatise
Presidential Address by T. Hayashi.
Miscellanies

On the graph of log x by M. Kaba; list of market prices and some technical terms in business circles by M. Kaba; on some syllabi of mathematics in the European and American secondary schools by M. Kuroda.

Problems for Consideration

On entrance examination by M. Kaba; on the equipment for teaching of mathematics by H. Furnkawa; on the teaching of arithmetic in the middle school course by M. Kuniyeda; on the teaching of graphs in the middle school course by M. Kuroda.

Mathematical Problems by M. Watanabe; Book Reviews; Questions; Miscellaneous Reports; Social Column; Special Reports.

ARTICLES IN CURRENT PERIODICALS.

Bulletin des sciences Mathematiques, volume 54, March, 1919: Review by R. Garnier of E. Turrière's *Sur le calcul des objectifs astronomiques de Fraunhofer* (1917), 49–50; "Sur un théorème relatif a l'extension du théorème de Rolle aux fonctions de plusieurs variables" by E. Gau, 50–51; "Sur le calcul des perturbations" by H. Vergne, 51–72; Revue des publications, 17–24.—April: Review by A. Boulanger of P. Duhem's *Etudes sur Léonard de Vinci*. Troisième série (Paris, 1913), 73–77; "Sur le calcul des perturbations" (suite et fin) by H. Vergne, 78–79; "La série $\frac{1}{5}+\frac{1}{7}+\frac{1}{11}+\frac{1}{13}+\frac{1}{17}+\frac{1}{19}+\frac{1}{29}+\frac{1}{31}+\frac{1}{41}+\frac{1}{43}+\frac{1}{59}+\frac{1}{61}+\cdots$ où les dénominateurs sont 'nobres premiers jumeaux' est convergente ou finie" by V. Brun, 100–104 (à suivre).

Bulletin of the American Mathematical Society, volume 26, no. 1, October, 1919: "In memory of Gabriel Marcus Green" by E. J. Wilczynski, 1–13 [Quotation: "In the six short years of his mathematical career, from 1913–1919, he enriched geometry with so many new ideas and important results as would suffice to excite our admiration if they had been spread over all of a normal life time. In his death we have suffered a heavy loss, but his life and work will continue to be, for many of us, an everlasting source of strength and inspiration"]; "Reduction of the elliptic element to the Weierstrass form" by F. H. Safford, 13–16; "A note on 'continuous mathematical induction'" by Y. R. Chao, 17–18; "On the number of representations of $2n$ as a sum of $2r$ squares" by E. T. Bell, 19–25; "Some functional equations in the theory of relativity" by A. C. Lunn, 26–34; "Formulas for constructing abridged mortality tables for decennial ages" by C. H. Forsyth, 34–38; Review by F. W. Owens of Dowling's *Projective Geometry* (New York, 1917), 39–40; Review by C. F. Craig of McClenon's *Introduction to the Elementary Functions* (Boston, 1918), 40–41; "Notes" and "New Publications," 41–48.

Bulletin of the First District Normal School, Kirksville, Mo., volume 15, no. 9, September, 1915; Mathematics Series no. 1: "The use of the graph in geography teaching" by B. Cosby, 7–8; "A plan for the study of stocks and bonds" by W. H. Zeigel, 9–14; "The history of mathematics as an incentive to mathematical study" by G. H. Jamison, 15–20; "The value of a life setting to algebra problems" by B. Cosby, 21–25; "Analytic geometry in our high schools" by W. H. Zeigel, 26–32; "Sets of orthogonal functions and their oscillation properties" by C. A. Epperson, 33–38; "A teacher's library" by G. H. Jamison, 38–39; "Henry Ward Beecher and mathematics," 40—Volume 16, no. 10, October, 1916; Mathematics Series no. 2: "Building and loan associations explained" by W. H. Zeigel, 5–10; "An experiment in teaching algebra" by C. A. Epperson, 11–13; "Some suggestions and observations on the teaching of high-school mathematics" by G. H. Jamison, 14–20; "The purpose and content of high-school arithmetic" by B. Cosby, 21–30; "The cube root of a binomial surd" by W. H. Zeigel, 31–38—Volume 18, no. 12, December, 1918 (published August, 1919). Mathematics Series No. 3[1]: "A geometrical problem —proof and application" by W. H. Zeigel, 5–10; "Keeping abreast of the times" by G. H. Jamison, 11–13; "Investments" by Byron Cosby, 14–20; "A suggested problem for classes in analytic geometry and surveying" by C. A. Epperson, 21–23; "The rôle of assumptions and definitions in high school mathematics" by G. H. Jamison, 24–30.

Journal of the United States Artillery, Volume 50., June 1919: "A method of computing differential corrections for a trajectory" by G. A. Bliss, 455–460 [reprinted from corrected proof in volume 51 (October), 445–449]—Volume 51, August: "Equations of differential variations in exterior ballistics" by W. E. Milne, 154–159—September: "The new ballistics" by R. S. Hoar, 285–295; "The use of adjoint systems in the problem of differential corrections for trajectories" by G. A. Bliss, 296–311; "Effect of the earth's rotation upon the point of fall" by P. Field, 328–329—October: "Rotating bands" by O. Veblen and P. L. Alger, 355–390.

Nature, volume 104, September 25, 1919: "Mathematics at the University of Stressburg" by H. B. Heywood, 74; The "algebraic cube," 79 [model in eight pieces illustrating the formula $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$. "The blocks are supplied in a neat cubical box, 10 cm. to the edge, by Messrs. Barnes and Morris, Ltd., scientific instrument makers, Audrey House, Ely Place, London, E. C. 1]; The University of Edinburgh Mathematical Institute, 87—October 9: Review by S. Brodetsky of T. R. Running's *Empirical Formulas* (New York, 1907), H. B. Phillips's *Differential and Integral Calculus* (New York, 1916–17), W. P. Milne and G. S. B. Westcott's *First Course in the Calculus*, Part 1 (London, 1918), R. C. Fawdry's *Dynamics*, Part 2 (London, 1919), and R. S. Heath's *Solid Geometry* (London, 1919), 109–110; P. E. B. Jourdain, 117 ["With the mathematician Philip Edward Bertrand Jourdain there died on October 1 a truly remarkable man. Jourdain lived only thirty-nine years, but the amount and value of the work that he accomplished, considering the disability under which he labored, are almost incredible. . . . He went up to Cambridge in 1898, then already a cripple. During his course at Cambridge he spent some time in Germany and became a fluent and scholarly linguist, speaking and reading several European languages. In 1904, though now physically quite incapacitated, he was awarded the Allen mathematical scholarship for research, and throughout the remainder of his short career his main activities were directed to the prosecution of mathematical investigations. His most important work was the discovery of certain series of infinite numbers. Working with Russell and Whitehead, he showed that certain arithmetical processes could be applied to them, and thus he obtained new and interesting results. He continued on this line of research, and even a few days before his death, of the imminence of which he was fully aware, he succeeded in demonstrating the existence of a previously unsuspected series of infinities. . . . Jourdain contributed extensive mathematical articles to the last edition of the *Encyclopaedia Brittanica.* He founded and edited the *International Journal of Ethics.* He was for some years the English editor, and since the death of Carus in 1918, the chief editor of the *Monist.* He also made a number of translations for the Open Court Publishing Co. Jourdain took the liveliest interest in the movement for encouraging the history of science. He was a contributor to *Isis*, and at the time of his death he had in preparation an article for the *Studies in the History and Method of Science* which it is hoped he may have left in a state ready for publication"]; "Hindu Spherical Astronomy," 119 ["Mr. G. R. Kaye has published a paper on 'Ancient Hindu spherical astronomy' in the *Journal and Proceeding of the Asiatic Society of Bengal* (Vol. 15). In this he summarises, with the aid of modern mathematical formulae, the fundemantal portions of the principal classical astronomical texts, which date from between A.D. 498 (the Aryabhatiya) and about A.D. 1000, when the redaction of the Surya Sidd-

[1] For this number the *Bulletin* was called *Bulletin of the State Teachers College.*

hanta now extant was written. Indian trigonometry is, like Indian astronomy, of Greek origin, but the Indians developed the methods received from the Greeks in various ways. There seems to be no doubt that the Indians were the first to introduce the use of sines instead of chords, and to compute the table of sines. But they never went further, and did not make use of the tangent function. They never gave a proof of any rule they enunciated. . . ."]

Open Court, volume 33, September, 1919: [a Paul Carus number with portrait] "The ideals of the life and work of Paul Carus" by P. E. B. Jourdain, 521–523 [Quotation: "At the Gymnasium of Stettin he came under the influence of the great mathematician Hermann Grassmann, of whom he always spoke with affectionate respect. Later he studied at the Universities of Strassburg and Tübingen. Owing to the need he felt so strongly for keeping his independence of thought, he resigned a teaching post in Germany and came first to England and then to America"].

Revue de Métaphysique et de Morale, 26. année, July–August, 1919: "A propos de la démonstration géométrique: Réponse à M. Goblot" by L. Rougier, 517–521.

Revue générale des Sciences pures et appliquées, 30. année, nos. 15–16, August, 1919: "La vie et l'oeuvre de Léonard de Vinci. A propos du quatrième centenaire de sa mort" by F. Bottazzi, 465–477; Review by R. D'Adhémar of P. Boutroux's *Les Principes de l'analyse mathématique. Exposé historique et critique,* volume 2 (Paris, 1919), 492.

Revue Scientifique, August 16–23, 1919: "Le temps et sa mesure" by M. Moulin, 486–494.

School Science and Mathematics, volume 19, no. 7, October, 1919: "What graphical and statistical material should be included in the ninth-grade mathematics course?" by L. E. Mensenkamp, 595–598; "Developing ability to solve the verbal problem; the basic aim of the ninth grade course" by Elsie G. Parker, 599–604; Problems and solutions, 655–658; "The theorem of Nichomachus" by U. P. Davis, 663 ["Cubical numbers are sums of consecutive odd integers"]; "Central Association of Science and Mathematics Teachers" by C. S. Winslow, 664–665.

Science, new series, volume 50, August 30, 1919: "New activities in the history of science" by L. C. Karpinski, 213–214—September 12: "Not ten but twelve!" by W. B. Smith, 239–242 [advocating duodenary scale of notation].

Scientia, volume 26, no. 3, September, 1919: "La teoria di relatività nel suo sviluppo storico. Porte I^a: La relatività della prima maniera" by A. Palatini, 195–207, supplément, 59–72; "Le danger de l'application du calcul des probabilités aux sciences de la nature et en particulier à l'astronomie" by E. Belot, 242–246; Review by G. Scorza of Montessus de Ballore's *Leçons sur les fonctions elliptiques en vue de leurs applications* (Paris, 1917), and *Exercices et leçons de mécanique analytique* (Paris, 1915), 247–248; Review by U. Ricci of G. H. Knibbs's The Mathematical theory of population (Melbourne, 1917), 260 [The review: "L'éminent directeur de la statistique officielle de l'Australie a voulu donner une preuve de sa bravoure en réunissant en un gros volume une collection très étendue de formules mathématiques permettant de mesurer les phénomènes démographiques. Les instruments analytiques propres à définir et à décrire la consistance et les variations d'une population dans ses divers aspects (distribution d'une population par sexe et par âge, masculinité, c'est-à-dire proportion entre hommes et femmes, natalité, nuptialité, fécondité, mortalité, migrations) sont forgés, adaptés et appliqués avec grande richesse, nous dirons presque avec luxe. De très nombreuses courbes et surfaces sont dessinées dans le volume, et côte-à-côte avec la théorie s'avancent les exemples tirés du recensement australien de 1911 et d'autres statistiques australiennes, sans parler de comparaisons établies avec les données empruntées à d'autres pays. Tous ceux qui s'occupent de statistique générale et de démographie consulteront avec profit ce volume qui est une mine de précieuses et minutieuses études"].

Scientific American Supplement, volume 88, September 20, 1919: "Derivation of new magic squares" by "Weg," 191.

Scientific Monthly, volume 9, no. 4, October, 1919: "Linkages" by F. V. Morley, 366–378.

Transactions of the American Mathematical Society, volume 20, no. 3, July, 1919: "Certain types of involutorial space transformations" by F. R. Sharpe and V. Snyder, 185–202; "On a new treatment of theorems of finiteness" by O. E. Glenn, 203–212; "On the theory of developments of an abstract class in relation to the calcul fonctionnel" by E. W. Chittenden and A. D. Pitcher, 213–233; "On the influence of keyways on the stress distribution in cylindrical shafts" by T. H. Gronwall, 234–244; "Some convergent developments associated with

irregular boundary conditions" by J. W. Hopkins and D. Jackson, 245–259; "Groups possessing a small number of sets of conjugate operators" by G. A. Miller, 260–270.

Zeitschrift für mathematischen und naturwissenschaftlichen Unterricht, volume 50, no. 4–5, April, 1919: "Ein neues elementares Verfahren zur Lösung von Extremaufgaben" by H. Dörrie, 153–177; "Ueber die Konstruction der Ellipse" by E. Wiedemann, 177–181; "Aufgaben-Repertorium," 189–192.

AMERICAN DOCTORAL DISSERTATION

J. H. Weaver, "Some extensions of the work of Pappus and Steiner on tangent circles." American Mathematical Monthly, January, 1920, volume 27, pp. 2–11; also reprinted. (Pennsylvania, 1916.)

UNDERGRADUATE MATHEMATICS CLUBS.

Edited by U. G. Mitchell, University of Kansas, Lawrence.

CLUB ACTIVITIES.

The Mathematics Club of Brown University, Providence, R. I. [*1918*, 33–34, 459; *1919*, 167].

April 18, 1919: "The mathematical theory of investments" by Professor Clinton H. Currier.

May 15: "Mathematics in chemistry" by Esther A. Brintzenhoff '19; "Isosceles trigonometry" by Chauncey D. Wentworth '20; "History of calculating machines" by Mr. W. L. Morden, New England manager of the Monroe Calculating Machine Company. Election of officers. Taking of club photograph.

June 4: Picnic.

The average attendance at the meetings for the year 1918–19 was 47. The officers elected for the year 1919–20 were:

Chairman of Club, Professor Roland G. D. Richardson;

Committee on Program, Professor Raymond C. Archibald, Alice F. Hildreth Gr., Pauline A. Barrows '21, Chauncey D. Wentworth '20, Daniel E. Whitford '20;

Committee on Arrangements, Professor Ray E. Gilman, Frances M. Merriam '20, Constance W. Haley '21, Raymond L. Wilder '20, Marshall H. Cannell '22, Bruce H. McCurdy '22.

The Mathematics Club of Connecticut College, New London, Conn. [*1918*, 270, 460].

Active membership in this club is limited to students pursuing courses in mathematics beyond the regular freshman requirement. There were ten members of the club in 1918–19 and due to influenza and diphtheria quarantines as well as to war conditions only four formal meetings were held.

The officers for 1918–19 were: President, Margaret Maher '19; secretary, Justine McGowan '20; treasurer, Louise Avery '21. These officers constitute

the committee on program and arrangements. At the meeting of May 16, 1919, Justine McGowan '20 was elected president for the year 1919–20.

Programs for the four meetings are given below.

November 23, 1918: "Arithmetic in the educational program of earlier days" by Professor David D. Leib.

February 4, 1919: "Flatland" by Dorothea E. Peck '19.

April 15: "Calculating machines" by Marie Munger '20.

May 16: "Mathematical puzzles and paper cutting" by Dorothy Pryde '21.

Mathematics Club of Iowa State University, Iowa City.

This club was organized and held its first meeting April 3, 1919. During the year 1918–19 it was not primarily an undergraduate club since its officers were all graduate students or members of the faculty and students had no part in the programs presented. However, junior and senior students were asked to attend the meetings, were allowed to vote and an effort was made to keep the matter presented within the range of their mathematical advancement. It is intended that undergraduate students shall take part in the programs to be given in 1919–20.

The officers for the remainder of the year 1918–19 were: President, E. M. Berry Gr.; secretary-treasurer, Helen J. Williams Gr.; program committee, E. M. Berry Gr., Professor Richard P. Baker and Professor Edward W. Chittenden.

The programs for the remainder of the year 1918–19 are given below.

April 3, 1919: "On functional relations for which the correlation coefficient is zero" by Professor Henry L. Rietz.

April 17: "Graphical solutions for the imaginary roots of an algebraic equation" by Rutherford E. Gleason, Instructor in Mathematics.

May 1: "Computing the mean and standard deviations of a frequency distribution" by Professor John F. Reilly.

May 15: "Some applications of differential equations" by Professor Raymond B. McClenon, Grinnell College, Grinnell, Iowa.

May 28: "Some properties of cubic curves" by Frank M. Weida, Instructor in Mathematics.

Mathematics Club of Iowa State Teachers College, Cedar Falls [*1918*, 311–312, 459].

The officers for the summer term 1918 were: President, Professor Peter Luteyn; secretary, Lorena F. French '18; executive committee, Professor Ira S. Condit, Professor Charles W. Wester and Margery Kinne '21.

The officers for the fall, winter and spring terms were: President, Professor Emma F. Lambert; secretary, Mary A. Peters '19; executive committee, Professor Charles W. Wester, Professor Robert D. Daugherty and Garnet Maulsby '20. The executive committee prepares the programs and decides on the dates of meetings.

Programs between May, 1918, and May, 1919, were as follows:

May 8, 1918: "Group recitations in geometry" by Phoebe Cowan '18; "Teaching of algebra in secondary schools" by Garnet Maulsby '20; "The number concept" by Verna Zarr '18.

May 29: "The correlation of mathematics in secondary schools" by Lorena F. French '18; "Calculus applied to physics" by Laura Huber '19.

June 27: "Tests of efficiency in teaching" by Mr. Charles W. Kline, superintendent of schools, East Waterloo, Iowa.

July 10: "Supervised study in secondary schools" by Miss Jessie Cunning, Instructor in Mathematics, Ft. Dodge High School, Ft. Dodge, Iowa; "The equation as an interpretation of the problem" by Principal W. E. Beck, Iowa City High School.

July 31: "Important points to be emphasized in the teaching of algebra" by Principal W. E. Beck, Iowa City High School.

August 14: "Standard forms in teaching arithmetic" by Miss Olive Tilton, Supervisor of Mathematics, Iowa State Teachers College Training School.

November 20: "An introduction to modern geometry" by Professor Wester; "Pascal's theorem" by Eleanor Sweeney '19; "Brianchon's theorem" by Garnet Maulsby '20; "Harmonic sets" by Dora Hospers '20; "Poles and Polars and applications to metrical relations" by Mary A. Peters '19.

January 29, 1919: "Rhythm in number land" by Professor Daugherty.

February 19: "The development of number" by Garnet Maulsby '20.

March 19: "Chicago meeting of the Department of Superintendence, N. E. A." by Professor Condit; "The fourth dimension" by Eleanor Sweeney '19.

April 16: "The metric system" by Professor Lambert.

April 30: "Fractions" by Professor Wester; "Irrational number" by Bernice Wilcox '19.

Mathematics Club of Mount Holyoke College, South Hadley, Mass. [*1918*, 312–313, 458].

October 19, 1918: Social meeting.

November 16: "The abacus and other forms of counting machines" by Dorothy C. Smith '19.

January 18, 1919: "Fundamental principles of flight" (illustrated with models of monoplanes) by Elizabeth R. Laird, Professor of Physics.

February 15: "History and applications of logarithms and the slide rule" by Thelma Bridge '20, Margaret F. Wilcox '19 and Mildred Allen, Instructor in Physics.

Mathematics Club of the University of Washington, Seattle [*1919*, 170].

An account of this club was published in the April, 1919, number of the Monthly. The supplementary items of information given below are found in the Monthly for December, 1905.

The club was organized in the autumn of 1905. The first officers were: President, Professor Robert E. Moritz; secretary, Professor Frank M. Morrison. The monthly meetings of the club were devoted to reviews, current mathematical literature and reports on original work.

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL AND OTTO DUNKEL.

Send all communications about problems and solutions to B. F. FINKEL, Springfield, Missouri.

PROBLEMS FOR SOLUTION.

2799. Proposed by H. C. BRADLEY, Massachusetts Institute of Technology.

A newspaper recently gave this problem: Cut a regular six-pointed star into the fewest number of pieces which will fit together and make a square. The newspaper gave a solution in seven pieces. First cut off two opposite points of the star. Divide each into two parts, and fit to the remaining portion of the star so as to make a reetangle. Find the mean proportional between the length and breadth of this rectangle (construction not shown); this is the side of the required square. Using this dimension on the two long sides of the rectangle, divide the latter into three pieces, which make the square. Total seven pieces.

How may the square be formed with not more than five pieces?

2800. Proposed by A. M. HARDING, University of Arkansas.

If $x + y + z = xyz$, show that

$$\frac{2x}{1-x^2} + \frac{2y}{1-y^2} + \frac{2z}{1-z^2} = \frac{2x}{1-x^2} \cdot \frac{2y}{1-y^2} \cdot \frac{2z}{1-z^2}.$$

2801. Proposed by A. S. HATHAWAY, Rose Polytechnic Institute.

A dog at the center of a circular pond makes straight for a duck which is swimming along the edge of the pond. If the rate of swimming of the dog is to the rate of swimming of the duck as $n:1$, determine the equation of the curve of pursuit and the distance the dog swims to catch the duck.

2802. Proposed by WARREN WEAVER, Throop College of Technology.

Consider two circles, each of radius k, with centers at $(0, 0)$ and $(k', 0)$ respectively, where k' is less than k. Through the point $(k', 0)$ draw a ray making an angle θ with the positive x-axis. Call the intersection of this line with the first circle A, and with the second circle B. Extend the line through the point $(k', 0)$ in the opposite direction, and call the intersection of this extension with the first circle A', and with the second circle B'. Prove that the sum of the two segments AB and $A'B'$ is independent of k, and depends only upon k', i. e. the shift of the circles, and θ.

2803. Proposed by S. A. COREY, Des Moines, Iowa.

In the November, 1918, number of the *Proceedings of the Edinburgh Mathematical Society* (Vol. 36, part 2, page 103), Professor Whittaker gives the following formula for the solution of algebraic and transcendental equations:

The root of the equation

$$0 = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots,$$

which is smallest in absolute value, is given by the series

$$-\frac{a_0}{a_1}-\frac{a_0^2a_2}{a_1\begin{vmatrix}a_1 & a_2\\ a_0 & a_1\end{vmatrix}}-\frac{a_0^3\begin{vmatrix}a_2 & a_3\\ a_1 & a_2\end{vmatrix}}{\begin{vmatrix}a_1 & a_2\\ a_0 & a_1\end{vmatrix}\begin{vmatrix}a_1 & a_2 & a_3\\ a_0 & a_1 & a_2\\ 0 & a_0 & a_1\end{vmatrix}}-\frac{a_0^4\begin{vmatrix}a_2 & a_3 & a_4\\ a_1 & a_2 & a_3\\ a_0 & a_1 & a_2\end{vmatrix}}{\begin{vmatrix}a_1 & a_2 & a_3\\ a_0 & a_1 & a_2\\ 0 & a_0 & a_1\end{vmatrix}\cdot\begin{vmatrix}a_1 & a_2 & a_3 & a_4\\ a_0 & a_1 & a_2 & a_3\\ 0 & a_0 & a_1 & a_2\\ 0 & 0 & a_0 & a_1\end{vmatrix}}-\cdots.$$

In case of any algebraic equation with imaginary or complex roots the above formula clearly fails. State the conditions under which the formula may be relied upon to give correct results.

2804. Proposed by T. H. GRONWALL, Washington, D. C.

Show that for $|x| < 1$

$$\frac{1}{\sqrt{1-x^4}}\int_0^x\frac{dx}{\sqrt{1-x^4}}=x+\sum_1^\infty\frac{3\cdot 7\cdots(4n-5)(4n-1)}{5\cdot 9\cdots(4n-3)(4n+1)}x^{4n+1},$$

$$\left(\int_0^x\frac{dx}{\sqrt{1-x^4}}\right)^2=x^2+\sum_1^\infty\frac{3\cdot 7\cdots(4n-5)(4n-1)}{5\cdot 9\cdots(4n-3)(4n+1)}\cdot\frac{x^{4n+2}}{2n+1}.$$

2805. Proposed by C. N. MILLS, Brookings, S. Dakota.

Derive the expression for volume

$$v=\int\int\int\rho^2\sin\phi d\rho d\phi d\theta.$$

In Byerly's *Integral Calculus*, page 183, revised edition, is a method by revolution, and in Czubers's *Integralrechnung*, page 200, is a method using the Jacobian determinant.

Required, a simple method one might use in developing the volume integral in polar coordinates.

2806. Proposed by R. E. MORITZ, University of Washington.

An anthropologist told me recently that large numbers of Russian peasants, whose knowledge of numbers is limited to multiplication and division by 2, employ the following method of multiplication which they were taught by a priest.

(1) Write the two numbers to be multiplied in the same horizontal line.

(2) Multiply the first number by 2 and write the product under the number so multiplied.

(3) Divide the second number by 2, discarding the remainder 1 when it occurs, and write the quotient under the number so divided.

(4) Treat the product and quotient thus obtained in the same manner as the original numbers. Continue this process until the quotient 1 is obtained.

(5) Strike out all the numbers on the left for which the corresponding numbers on the right are even.

(6) Add the remaining numbers on the left. Their sum is the required product.

Problem: Prove that this rule is correct.

SOLUTIONS OF PROBLEMS.

442. (Geometry) [1914, 156; 1919, 268]. Proposed by J. B. SMITH, Hampden-Sidney College.

If any three straight lines AD, BE, CF, be drawn from the corners of the triangle ABC to the opposite sides a, b, c, they will enclose an area. If Δ, Δ'' be the areas of the triangles ABC, DEF, show that

$$\frac{\Delta''}{\Delta}=\frac{(AF\cdot BD\cdot CE-AE\cdot CD\cdot BF)^2}{(ab-CE\cdot CD)(bc-AE\cdot AF)(ca-BF\cdot BD)},$$

where the signs of the factors are to be determined by the following rule: Each segment being measured from one of the corners of the triangle ABC, along one of the sides, is to be regarded as positive or negative according as it is drawn towards or from the other corner in that side.

By means of the equality (3), and analogues to (2) and (3)

$$\frac{CB'}{CF} = \frac{ca_2}{ca - c_2a_1}, \qquad \frac{A'B'}{CF} = \frac{c(a_1b_1c_1 - a_2b_2c_2)}{(ca - c_2a_1)(bc - b_2c_1)},$$

we deduce from (4)

$$\frac{\Delta''}{\Delta} = \frac{(a_1b_1c_1 - a_2b_2c_2)^2}{(bc - b_2c_1)(ca - c_2a_1)(ab - a_2b_1)}.$$

340 (Calculus) [1913, 196; 1919, 213]. Proposed by C. N. SCHMALL, New York, N. Y.

A pencil of parallel rays of light is incident upon a lens whose faces have the radii r_1, r_2, respectively. Show that the distance of the principal focus from the center of the first face of the lens will be a maximum or a minimum when

$$\frac{r_1}{r_2} = \frac{(\mu - 1)^{1/2}}{1 + (\mu - 1)^{1/2}},$$

where μ has its usual meaning.

SOLUTION BY H. S. UHLER, Yale University.

It is necessary to remark at the very beginning that the given result appears incorrect as it does not involve the thickness of the lens, and the statement of the problem is ambiguous since it does not specify whether the first or the second principal focus is meant.

By straightforward analysis, involving Snell's law, but too long to deserve publication in this place, I found

$$x = t + \frac{r_2}{\mu - 1} - \frac{\mu r_2^2}{(\mu - 1)[\mu(r_1 + r_2) - (\mu - 1)t]}, \tag{1}$$

where, to fix the ideas, a double convex lens was contemplated, $x \equiv$ distance from the center of the first (or incidence) face to the second principal focus (the focus beyond the emergence face), $t \equiv$ axial thickness of lens, $\mu \equiv$ index of refraction of the material (glass) of the lens with respect to that of the surrounding medium (air), and r_1 and r_2 denote the arithmetical values of the radii of curvature of the first and second faces of the lens, respectively.

Formula (1) may be checked by referring to James P. C. Southall's *The Principles and Methods of Geometrical Optics*, either edition, page 275, equation (170); and by changing his d, n, and r_2 to t, μ, and $-r_2$, respectively.

Now what quantity is to be the independent variable? Let it be assumed that t is to vary while μ, r_1, and r_2 remain constant. Then the necessary condition for a maximum or a minimum is

$$\frac{dx}{dt} = 1 - \frac{\mu r_2^2}{[\mu(r_1 + r_2) - (\mu - 1)t]^2} = 0,$$

which leads directly to

$$t = \frac{\sqrt{\mu}[r_1\sqrt{\mu} + r_2(\sqrt{\mu} \pm 1)]}{\mu - 1}. \tag{2}$$

To determine which sign has physical meaning we may proceed as follows: $x - t$ must not be negative, otherwise the principal focus will fall inside the material of the lens. Substituting in (1) the expression for $(\mu - 1)\,t$ given by (2) we find

$$x - t = \frac{r_2}{\sqrt{\mu} - 1} \quad \text{or} \quad -\frac{r_2}{\sqrt{\mu} + 1},$$

according as the upper or lower sign in (2) be taken. Since, under all ordinary conditions, $\mu > 1$ the upper sign alone is acceptable.

Continuing,

$$\frac{d^2x}{dt^2} = -\frac{2\mu(\mu - 1)r_2^2}{[\mu(r_1 + r_2) - (\mu - 1)t]^3}$$

which, for the critical value given by (2), reduces to

$$\left(\frac{d^2x}{dt^2}\right)_2 = +\frac{2(\mu - 1)}{r_2\sqrt{\mu}}$$

Since this result is positive the thickness given by (2) corresponds to a minimum focal length, as might be anticipated from purely physical considerations.

It is thus clear that a reasonable interpretation of the statement of the problem leads to an analytical condition which is quite different from the given answer. The thickness of the lens cannot be considered negligible (as is often done in very elementary discussions of problems in geometrical optics) because the numerator of (2) cannot vanish when the upper sign is taken. In other words, t cannot be avoided in the final condition.

In conclusion, attention may be called to the fact that each of $\partial x/\partial r_1 = 0$, $\partial x/\partial r_2 = 0$, $\partial x/\partial \mu = 0$ leads to a result which bears no relation to the given answer and hence t is the only sensible independent variable.

349 (Calculus) [1913, 312]. Proposed by C. N. SCHMALL, New York City.

If $y = a \cos (\log x) + b \sin (\log x)$, eliminate the constants a and b and obtain the equation

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0.$$

SOLUTION BY T. E. MERGENDAHL, Tufts College.

Taking the derivative of y, as given in the problem, and multiplying the result by x, we have

$$x\frac{dy}{dx} = - a \sin \log x + b \cos \log x.$$

Applying the same process to both sides of this equation, we find

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} = -(a \cos \log x + b \sin \log x) = -y,$$

or

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0.$$

Also solved by W. W. BEMAN, T. M. BLAKSLEE, P. J. DA CUNHA, H. T. DAVIS, H. H. DOWNING, G. H. GRAVES, P. HANSEN, A. M. HARDING, W. L. MISER, A. PELLETIER, ELIJAH SWIFT, and H. S. UHLER.

385 (Calculus) [1915, 161; 1919, 73]. Proposed by H. B. PHILLIPS, Massachusetts Institute of Technology.

If $f(x)$ is continuous between a and x, show that

$$\lim_{n \doteq \infty} \frac{\underline{|n}}{(x-a)^n} \int_a^x \cdots \int_a^x f(x)dx^n = f(a).$$

T. H. GRONWALL, New York City, offers the following criticism and completion of the solution already published.

First, the passage to the limit for $n \doteq \infty$, which forms the last step (p. 74), requires justification (this is however easily done by using Taylor's theorem with remainder term). Second, the problem as proposed assumes only that $f(x)$ is continuous, so that the use of the *derivatives* of $f(x)$ is not permissible, since they may not exist. The following proof requires $f(x)$ to be bounded in the interval from a to x and to be continuous only at the point a, but not in the whole interval. It is readily shown (integration by parts and complete induction) that

$$\int_a^x \cdots \int_a^x f(x)dx^n = \frac{1}{(n-1)!}\int_a^x (x-\xi)^{n-1} f(\xi)d\xi;$$ [1]

writing $\xi = a + t(x - a)$ and $f(\xi) = \varphi(t)$, the formula to be proved becomes

$$\lim_{n \doteq \infty} n \int_0^1 (1-t)^{n-1}\varphi(t)dt = \varphi(0)$$

[1] A proof of this relation may be found in Goursat-Hedrick's *Mathematical Analysis*, Vol. 2, pt. II, p. 36.—EDITORS.

(under the assumption that $\varphi(t)$ is continuous at $t = 0$), or what is the same

$$\lim_{n \doteq \infty} n \int_0^1 (1 - t)^{n-1}[\varphi(t) - \varphi(0)]dt = 0.$$

Let $\epsilon > 0$ be as small as we please; $\varphi(t)$ being continuous at $t = 0$, it is then possible to choose a δ, $0 < \delta < 1$, and so small that $|\varphi(t) - \varphi(0)| < \epsilon/2$ for $0 \leqq t \leqq \delta$. Let M be greater than $|\varphi(t) - \varphi(0)|$ for $0 \leqq t \leqq 1$. Then, decomposing the integral into two parts and using the first mean value theorem,

$$\left| n \int_0^1 (1 - t)^{n-1}[\varphi(t) - \varphi(0)]dt \right| \leqq \left| n \int_0^\delta \right| + \left| n \int_\delta^1 \right|$$

$$\leqq n \frac{\epsilon}{2} \int_0^\delta (1 - t)^{n-1}dt + nM \int_\delta^1 (1 - t)^{n-1}dt$$

$$< \frac{\epsilon}{2} + M(1 - \delta)^n.$$

Now N may be chosen so large that $M(1 - \delta)^n < \epsilon/2$ for $n > N$, so that finally

$$\left| n \int_0^1 (1 - t)^{n-1}[\varphi(t) - \varphi(0)]dt \right| < \epsilon \text{ for } n > N,$$

which completes our proof.

192 (Number Theory) [1913, 196; 1919, 214]. Proposed by the late ARTEMAS MARTIN.

Find rational values of v, w, and x that will satisfy simultaneously the conditions

$$(m^2 + n^2)(v^2 + w^2 + x^2)^2 - 4m^2n^2 v^2 + m^2n^2(m^2 + n^2) = 0,$$

$$(m^2 + n^2)(v^2 + w^2 + x^2)^2 - 4m^2n^2w^2 + m^2n^2(m^2 + n^2) = 0,$$

$$(m^2 + n^2)(v^2 + w^2 + x^2)^2 - 4m^2n^2 x^2 + m^2n^2(m^2 + n^2) = 0,$$

m and n being known rational quantities.

Discussion by H. S. Uhler, Yale University.

The following analysis shows that the problem is impossible.

Any set of values of v, w, and x that fulfil the given conditions must also satisfy the sum of the three equations, that is, must satisfy

$$3(m^2 + n^2)(v^2 + w^2 + x^2)^2 - 4m^2n^2(v^2 + w^2 + x^2) + 3m^2n^2(m^2 + n^2) = 0.$$

From this equation, we have

$$v^2 + w^2 + x^2 = \frac{mn}{3(m^2 + n^2)} [2mn \pm i\sqrt{9m^4 + 14m^2n^2 + 9n^4}].$$

In general, the right hand member is irrational, and the sum of the squares of any number of rational numbers cannot have an irrational value; therefore, the problem is impossible.

We may proceed farther and find explicit expressions for v, w, and x. By inspection, or by forming the differences between the given conditions, taken in pairs, we see that $v^2 = w^2 = x^2$. Consequently,

$$x^2 = \frac{mn}{9(m^2 + n^2)} [2mn \pm i\sqrt{9m^4 + 14m^2n^2 + 9n^4}],$$

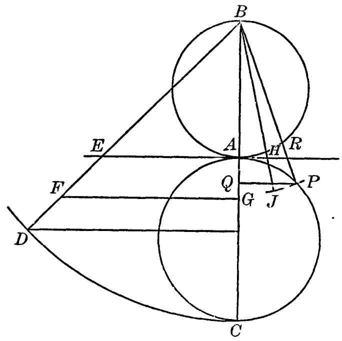
or

$$x = \pm \frac{1}{3}\sqrt{\frac{mn}{2(m^2 + n^2)}} \{\sqrt{3(m^2 + n^2) + 2mn} \pm i\sqrt{3(m - n)^2 + 4mn}\}.$$

Obviously, these four roots are irrational, in general.

2734 [1918, 444]. Proposed by E. L. REES, University of Kentucky.

Given two circles tangent to each other externally. From the extremity of a diameter through the point of tangency draw a secant such that the segment between the circles shall be equal to a given segment.



SOLUTION BY THE PROPOSER.

Let PB be the required position of the secant, assuming the problem having been solved. Draw PQ perpendicular to AC and let $BP = y$, $AQ = x$, $AB = d$, $AC = d'$, and $RP = a$, the given segment. Then we have, $y^2 = (d + x)^2 + x(d' - x)$. Also $(y - a)/d = (d + x)/y$. Eliminating x, we get $y^2 - 2a'y - d^2 = 0$, where $a' = a(2d + d')/2(d + d')$. Hence,

$$y = a' \pm \sqrt{a'^2 + d^2}.$$

The analysis suggests the following construction: With B as a center, BC as a radius, describe the arc cutting the horizontal diameter produced of the lower circle in D. Draw BD cutting the common tangent in E. Take $EF = a$. Draw FG perpendicular to AC. Then AG will equal a'. Make $AH = AG$. Draw BH and produce it to J, making $HJ = AH$. With B as a center and BJ as radius, describe an arc cutting the lower circle in P. Then BP is the required secant.

Also solved by P. J. DA CUNHA.

2741 [1919, 35]. Proposed by H. L. OLSON, New Hampshire College.

Prove or disprove the following statement: If the three sides and the area of a triangle are integers, at least one of the three altitudes is an integer.

I. SOLUTION BY FRANK IRWIN, University of California.

The statement is not true since the triangle whose sides are 5, 29, 30 has area 72, and altitudes 144/5, 144/29, and 24/5.

II. REMARKS BY J. L. RILEY, Stephenville, Texas.

On page 12 of Carmichael's *Diophantine Analysis* we find the following theorem:

A necessary and sufficient condition that rational numbers x, y, z shall represent the sides of a rational triangle is that they shall be proportional to numbers of the form $n(m^2 + h^2)$, $m(n^2 + h^2)$, $(m + n)(mn - h^2)$, where m, n, h are positive rational numbers and $mn > h^2$.

In deriving this result it is pointed out that if $x = n(m^2 + h^2)$, $y = m(n^2 + h^2)$, and $z = (m + n)(mn - h^2)$, the area is $hmn(m + n)(mn - h^2)$ and the altitude upon z is $2hmn$. If m, n, and h are integers ($mn > h^2$) we have a series of triangles for which the statement of Mr. Olson is true.

2742 [1919, 36]. Proposed by C. N. SCHMALL, New York City.

In Gregory's *Examples in the Differential and Integral Calculus*, 1841, Chap. VII, p. 124, ex. 22, I find the following celebrated problem: "To find a point within a triangle from which if lines be drawn to the angular points their sum may be the least possible." The author remarks that "the direct solution of this problem is long and complicated, etc." Required a simple, brief solution.

I. SOLUTION BY F. V. MORLEY, Johns Hopkins University.

Consider the three points, α, β, γ (complex variable) as on a unit or base circle. Then th sum of the distances from the point x is

$$\sum^{3} \sqrt{(x - \alpha)\left(\bar{x} - \frac{1}{\alpha}\right)} \tag{1}$$

where $x - \alpha = \rho t$, the distance times the turn, or direction factor, and $\bar{x} - 1/\alpha = \rho/t$, the conjugate expression. Since (1) is to be a minimum, D_x and $D_{\bar{x}}$ must vanish. As they are interdependent, either will suffice.

$$(2) \qquad D_{\bar{x}} = \sum^{3} \sqrt{\frac{x - \alpha}{\bar{x} - 1/\alpha}} = \sum^{3} t = 0.$$

This means that sum of the three roots (the symmetric function s_1) vanishes in the equation

$$(3) \qquad t^3 - s_1 t^2 + s_2 t - s_3 = 0.$$

The condition $D_x = 0$ involves $s_2 = 0$, so that equation (3) reduces to

$$t^3 - s_3 = 0$$

the roots of which are

$$t, \ \omega t, \ \omega^2 t,$$

when ω is a cube root of 1. That is, the three angles at x are all equal to 120°.

The argument may also be presented in its physical, instead of geometrical, aspect. The method in each case is the same.

Consider α, β, γ as centers of attraction for constant forces of equal magnitude. The total potential of x will then be the sum of the distances from x to the three points. For the total potential to be a minimum is to have equilibrium, and the resultant force in any direction must vanish. Hence, the three angles at x are all equal to 120°. It is to be noted that the partial derivative ($D_{\bar{x}}$) of potential, expresses the resultant force with direction factor included.

The well known construction for (the Fermat point) x is to draw on each side $\alpha\beta$, $\beta\gamma$, $\gamma\alpha$, an equilateral triangle, outward, calling the free vertices γ', α', β'. The points γ', α, x, β, are concyclic, and it follows that the line $\gamma' x$ bisects the angle $\alpha x \beta$ and coincides with the line $x\gamma$. x will therefore be the intersection of the three lines $\alpha\alpha'$, $\beta\beta'$, $\gamma\gamma'$.

II. Solution by R. E. Moritz, University of Washington.

Let $P_i = (x_i, y_i)$, $i = 1, 2, 3$, represent the angular points of the triangle expressed in Cartesian coördinates, $P = (x, y)$ any point within the triangle, and θ_i the angle which the line P_iP makes with the positive direction of the x-axis. Then, if D denotes the sum of the distances of P from the angular points, we have

$$D = \Sigma_i \sqrt{(x - x_i)^2 + (y - y_i)^2}.$$

In order that D may be a minimum the partial derivatives of D with respect to x and y must be separately equal to zero; hence,

$$\Sigma_i \frac{x - x_i}{\sqrt{(x - x_i)^2 + (y - y_i)^2}} = 0, \qquad \Sigma_i \frac{y - y_i}{\sqrt{(x - x_i)^2 + (y - y_i)^2}} = 0,$$

that is,

$$\cos\theta_1 + \cos\theta_2 + \cos\theta_3 = 0, \qquad \sin\theta_1 + \sin\theta_2 + \sin\theta_3 = 0,$$

or

$$\cos\theta_1 + \cos\theta_2 = -\cos\theta_3, \qquad \sin\theta_1 + \sin\theta_2 = -\sin\theta_3.$$

By squaring and then adding the resulting equations we get $2\cos(\theta_1 - \theta_2) = -1$, from which $\theta_2 - \theta_1 = 120°$. Similarly, $\theta_3 - \theta_2 = 120°$, $\theta_1 - \theta_3 = 120°$.

The problem, therefore, resolves itself into that of finding a point within the triangle at which the three sides subtend equal angles. This point is readily found as follows:

Describe on two sides of the triangle segments of circles containing each an angle of 120°. The intersection point of these circles is the required point. If one of the angles of the triangle is equal to or greater than 120° the intersection point falls on or without the triangle. In that case no point within the triangle satisfies the required conditions.

The following kinematic solution, while perhaps not simpler than the foregoing geometrical solution, has the advantage of being applicable to the more general problem, which I believe has never been solved, namely,

Given n points in a plane, to find a point from which if lines be drawn to the n points their sum may be the least possible.

Draw the triangle to a given scale on a drawing board and insert thumb tacks at the points

marking the vertices. Take a flexible string and fasten one end of it to one of the thumb tacks, say to A. Pass the other end of the string through a ring R, then around the thumb tack B, then back through the ring, then around the thumb tack at C, then once more through the ring R and then back to A. Now pull the free end of the string taut until the ring assumes a fixed position. The center of the ring R will be the required point. The proof is obvious.

III. Note by William Hoover, Columbus, Ohio.

The following extract from a paper by P. G. Tait[1] meets the requirement fully and is of interest historically:

The following problem, originally proposed by Fermat to Torricelli, *To find the point the sum of whose distances from three given points is the least possible*, seems to have given considerable trouble to the older mathematicians, and even in modern times (see *Gregory's Examples*, p. 126) to have been solved in a very tedious manner. Simpler solutions have since been given (*e.g.*, *Cambridge and Dublin Mathematical Journal*, VIII, p. 92), but none, to my knowledge, so direct, as that indicated by Quaternions. The object of this note is to show the simplicity of the quaternion method.

If α, β be the vectors of two of the given points, the origin being the third, and if ρ be the vector of the required point, we must have (by the conditions of the problem)

$$T\rho + T(\alpha - \rho) + T(\beta - \rho) \text{ a minimum.}$$

Hence,

$$S[U\rho - U(\alpha - \rho) - U(\beta - \rho)]d\rho = 0,$$

for all values of $Ud\rho$. Hence the versor sum in square brackets must vanish identically. The immediate interpretation is, that *lines parallel* to ρ, $\rho - \alpha$, $\rho - \beta$ form an equilateral triangle. The required point is therefore in the same plane as the three given points; and their distances, two and two, subtend equal angles at it, which is the well-known solution.

Equally simple is the quaternion solution of the same problem if more than three points be given. Let their vectors, to any origin, be α, β, γ, etc., and let ρ be the vector of the sought point. We have

$$\Sigma \cdot T(\alpha - \rho) = \text{minimum},$$

from which, as above, $\Sigma U(\alpha - \rho) = 0$.

Hence, *if unit forces act at the required point, in the lines joining it with the given points, these forces are in equilibrium*. Or, in another form, *a closed equilateral gauche polygon may be drawn whose sides are parallel to the lines joining the sought point with the given ones*."

IV. Solution by Otto Dunkel, Washington University.

Suppose that at P, a point inside the triangle ABC, we have a minimum with $AP = r_1$, $BP = r_2$, $CP = r_3$. Then r_1 must be the minimum distance from A to the ellipse with foci B and C and passing through P, since for any point Q on the ellipse $BQ + CQ = r_2 + r_3$. But the shortest distance from an external point to a closed convex curve is orthogonal to the curve and we know from the properties of the ellipse that r_1 in this case must make equal angles with the focal rays r_2 and r_3. Applying the same argument to r_2, we see that r_1, r_2, r_3 must make equal angles with one another.

V. Remarks and Historical Notes by R. C. Archibald, Brown University.

This problem has been already discussed in the Monthly by Professors Jackson[2] and Johnson.[3] Professor Jackson's discussion is about the same as Gregory's. The analytic treatment by Professor Moritz is practically identical with that given in 1853 by A. Cohen[4] and in 1902 by E.

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[1] *Proceedings of the Royal Society of Edinburgh*, Vol. 6 (1866–69), 1869, pp. 165–166; also in P. G. Tait, *Scientific Papers*, Vol. 1, 1898, pp. 76, 77.

[2] Volume 24 (1917), pp. 42–44.

[3] Volume 24 (1917), pp. 243–244.

[4] *Cambridge and Dublin Mathematical Journal*, Vol. 9, p. 92.

Goursat,[1] and his kinematic solution is given in the latter part of Tait's paper of 1867 from which Professor Hoover quoted. Tait concludes: "This kinematical process, equally with the quaternion one whose form directly suggests it, gives easily the solution of the more general problem,—To find a point such that m times its distance from A, together with n times its distance from B, etc., may be a minimum." (For three points this is solved in Stegmann–Kiepert, *Lehrbuch der Differentialrechnung*, 7. Aufl. 1895, p. 260.) The problem of finding a point in a plane the sum of whose distances from any number of given points is a minimum was solved by Tédenat in *Annales de mathématiques pures et appliquées* (1810–11), pp. 285–291.

The celebrated problem of our question was formulated by Fermat[2] in the seventeenth century. According to Viviani[3] he suggested it to Torricelli (before 1648, for Torricelli died in 1647). Torricelli discovered three solutions, one by "plane loci" (that is, with ruler and compasses), two others by "solid loci" (that is, by means of conic sections). He afterwards proposed it to Viviani in the following terms: "A triangle, each of whose angles is less than one-third of four right angles[4] being given; to find a point from which, if straight lines be drawn to its three angles, their sum shall be a minimum."

Viviani solved the problem after repeated efforts (he says, non nisi iteratis oppugnationibus tunc nobis vincere datum fuit) and his solution is given in the appendix to his *Geometria divinatio*.[5]

The problem was treated by Thomas Simpson[6] in 1750. He gave the following construction for determining the Fermat point: Describe on BC a segment of a circle to contain an angle of 120°, and let the whole circle BCQ be completed. From A to Q_1 the middle point of the arc $BA'C$ draw AQ intersecting the circumference of the circle in P, which will be the point required. Because of this construction Simson has been credited with the theorem: If on the sides of a triangle ABC, equilateral triangles $A'BC$, $B'CA$ and $C'AB$ be described externally AA', BB' and CC' are concurrent[7]—a construction which Mr. Morley cites above.

Simpson treats also the more general problem: Three points A, B, and C being given, to find the position of a fourth point V, so that if lines be drawn from thence to the three former, the sum $a \cdot AP + b \cdot BP + c \cdot CP$, where a, b, c denote given numbers, shall be a minimum.

The further history of generalizations is very extensive.

Also solved by P. J. Daniel, W. W. Gorsline, R. A. Johnson, H. M. Roeser, J. Rosenbaum, L. Weisner and the Proposer.

[1] *Cours d'analyse mathématique* tome 1; English ed. by Hedrick, 1904, pp. 130–131. See also G. Humbert, *Cours d'Analyse*, tome 1, 1903, pp. 193–196.

[2] *Oeuvres*, Tome 1, Paris, 1891, p. 153: "Datis tribus punctis, quartum reperire, a quo si ducantur tres rectæ addata puncta, summa trium harum rectarum sit minima quantitas." See also Tome 3, 1896, p. 136.

[3] V. Viviani, *De maximis et minimis geometria divinatio*. Florentiæ, MDCLIX, p. 144.

[4] This careful statement is necessary for constructions indicated in solutions above. If one of the angles, A say, is equal to, or greater than, 120°, in a certain sense its vertex is to be considered as the minimum point; compare the discussion of this by Goursat, *l.c.*, by Bertrand in *Journal de mathématiques pure et appliquées*, tome 7 (1843), pp. 155–160, and by Sturm in *Journal für die reine und angewandte Mathematik*, Vol. 97 (1884), p. 51.

[5] *L.c.*, pp. 145–150. Viviani's method of solution is reproduced in D. Cresswell's *Maxima and Minima*, Cambridge, 1812, pp. 120–121; second edition, 1817, pp. 121–122.

[6] *Doctrine and Applications of Fluxions*, London, 1750, § 36.

[7] The first formulation of the result as here stated seems to have been by T. S. Davies in the *Gentleman's Diary* for 1830, p. 36. It is here shown also that $AA' = BB' = CC'$. Problems closely related to this are frequently published, *e.g.*, in *School Science and Mathematics*, Feb., 1918, pp. 170–171; Jan., 1919, pp. 86–87; April, 1919, pp. 374–375.

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CONTENTS

EDITORIAL CORRESPONDENCE AND BOOKS FOR REVIEW should be addressed to the EDITOR-IN-CHIEF, R. C. ARCHIBALD, Brown University, Providence, R. I.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 27 King Street, Oberlin, Ohio.

(*a*) Let C be the segment AB. Then any point E on it is the limit of the positions of P as P approaches it. In particular, A and B are limiting points.

(*b*) Let C be AB with the middle point D not included. That is, as P moves in C it takes every position on AB except D, and AP takes every value from 0 to 4 except the value 2. Then D is the limit of the positions of P as P approaches it. This is an example of a point which is not on C and yet is the limit of points on C.

(*c*) Let C be AB with the points between E and D not included. That is, as P moves in C it takes every position on AB except those between E and D; and AP takes every value from 0 to AE and from 2 to 4 inclusive. Then E is the limit of the positions of P as it is approached from the left but not as it is approached from the right; and D is the limit as approached from the right but not from the left. Hence at E or D a unique limit, according to the definition, does not exist. If, however, C were AE only, E would be a unique limit.

(iv) If the limit L of the positions of P is itself a position of P, that is if L is on C, then the curve C is *continuous* at L. But at any point where L is not on C, or where no unique limit exists, the curve C is *discontinuous*.

Thus in (iii) (*a*) above, C is continuous at every point. In (iii) (*b*) C is discontinuous at D; and in (iii) (*c*) AE and DB are continuous at every point, but C is discontinuous at E and D.

(v) A curve is continuous over any segment of it when it is continuous at every point of that segment; and the motion of P as it generates the segment is a continuous motion.

3. Motion of a line in the plane.

Let P and P' be two distinct variable points which generate two curves C and C'. Then the line l, which is determined by P and P', is a variable line, and *moves* in the plane. A limit of l is defined as the line LL', when P and P' simultaneously approach distinct limits L and L' on C and C'.

If C'' is any other curve in the plane which intersects l in P'' and LL' in L'', then P'' approaches the limit L'' when P and P' approach the limits L and L'.

The motion of a line in a plane is continuous wherever it actually occupies its limiting position.

The following particular cases may occur:

(i) P and P' move continuously, and l occupies its limiting position LL' at every stage. The movement of l is then continuous.

(ii) P' is a fixed point and P moves continuously on a curve C which does not contain P'; then, in general, l rotates continuously about P'.

(iii) Either P or P' has a point of discontinuity in its motion; l will have, in general, a corresponding position of discontinuity.

(iv) If L and L' become coincident the line LL' is indeterminate in direction, and does not define the limit of l. In this case the limit is defined by LL'' where L'' is the limit of P'' on C''. But as l at this point is indeterminate in direction,

it need not occupy its limiting position LL'', and its motion is therefore discontinuous. The following examples will illustrate this:

(*a*) Let P and P' move on two straight lines c and c' (Fig. 2), which intersect at L, in such a way that the ratio LP: LP' is constant; and let P'' move on a line c'' which does not pass through L. The line l moves parallel to itself; and as P and P' approach coincidence at L, l approaches the limit LL''. But as l is not determined at L, its motion is not continuous there.

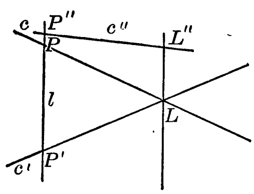


FIG. 2.

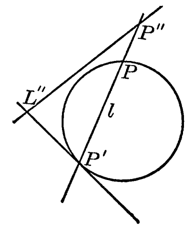


FIG. 3.

(*b*) P moves on a circle c, and P' is a fixed point on c (Fig. 3). As P moves into coincidence with P', l approaches the limiting position $P'L''$ which is defined to be the tangent to the circle at P'. But l is not determined when P is at P', and the rotation of l about P' at this point is therefore discontinuous.

If we fix, by definition, the position of l when P coincides with P' to be that of the tangent at P', we make the rotation of l about P' to be continuous at every point as P moves over the whole circumference.

4. Variable quantities.

The movement of points and lines in the plane implies the variation of quantities such as length, area, and angle. Thus if A of the triangle ABC moves in the plane while B and C remain fixed, the lines AB and AC rotate about B and C; and the lengths AB and AC, the area ABC, and the angles A, B, and C are variable quantities.

The definitions of § 2 practically involve the following definitions concerning variable quantities:

(i) If the value of a variable quantity, as it approaches a fixed value v, may be made to differ from v by less than any assignable quantity however small, whether it approaches through values greater or less than v, then v is the *limit* of the values of the quantity as it approaches v. (Cf. Goursat-Hedrick, *Mathematical Analysis*, vol. I, § 1.)

(ii) If the limit of a variable quantity and its value at a point are the same, the variation is continuous at that point. (Cf. Pierpont, *l. c.*, vol. I, § 339.)

(iii) The variation of a quantity is continuous over any interval when it is continuous at every point of that interval. (Cf. Goursat-Hedrick, *l. c.*, vol. I, § 3.)

Thus for example in Fig. 1 the length AP is continuous over the interval from 0 to AB in 2 (iii) (*a*); but it is discontinuous at D in (*b*), and at E and D in (*c*), as P moves from A to B.

5. In general, continuous movements of points and lines in the plane imply continuity in the variation of the quantities involved. Thus, for example, the distance AP from a fixed point A to a point P which moves continuously along a

curve is a continuous variable. Similarly the angle AOP between a line OP, which rotates continuously about a fixed point O, and a fixed initial position OA is a continuous variable. In each case the variable attains its limiting value when the point or the line reaches its limiting position.

But examples are given in 3 (iv) (a) and 3 (iv) (b) of continuous movements of points which imply discontinuous movements of lines and hence discontinuous variation in angle. In these cases the points reach their limiting positions where at the same time the angle need not attain its limiting value.

Similarly a continuous rotation of a line may determine a discontinuous movement of a point. Thus let O be a fixed point and l a fixed line (Fig. 4); and let a line m rotate continuously about O and intersect l in P. As m rotates about O, P moves continuously along l except when l and m are parallel. The movement of P and the distance AP are discontinuous at this point.

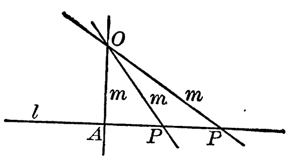


FIG. 4.

On the other hand, if the point at infinity be defined as usual as a unique point on l, the movement of P along l, which is discontinuous when P is the point at infinity, implies a continuous rotation of m about O.

6. In a plane geometric figure some points may be considered fixed while others vary, as for example in the triangle ABC of § 4. Similarly in the quadrangle $ABCD$ we may assume the points A, B, C to remain fixed and D to move freely in the plane. The form of the quadrangle then changes. The angles at C, D, A and the lengths CD and DA are variable quantities, and are functions of D in the sense that their values depend on the position of D. If D moves continuously these quantities vary continuously, except possibly where D coincides with one of the other points. When D approaches a limit, all the variable elements which depend on it approach at the same time their respective limits; and if no discontinuities exist they all have the values of their limits when D occupies its limit.

As D moves along the curve DD'' in Fig. 5, the angle BCD changes to BCD''. In its variation it passes through a zero value when D is at D'; and on the principle that a continuously increasing or decreasing variable changes sign on passing through a zero value, we say that BCD and BCD'' have opposite signs.

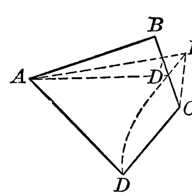


FIG. 5.

7. Suppose a finite set of variables to approach their respective limits at the same time; and suppose a constant relation to exist among these variables for all simultaneous sets of values preceding the limit. Then this relation will exist also at the limit, provided no discontinuities appear. This is evident, for it is merely stated that a certain relation exists among a set of values when it exists for values which differ from them by as little as we please. The principle thus stated is important in that it leads to generalizations among theorems of geometry. The following examples are given in illustration:

(i) ABC is a triangle inscribed in a circle, with A and B fixed, and C variable, on the circumference (Fig. 6). Then, from elementary geometry, the angle ACB is constant for all positions of C provided C remains on either the one side or on the other of AB.

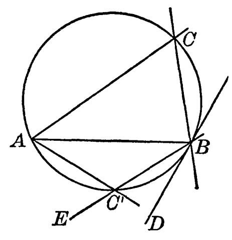


Fig. 6.

But assuming, as in 3 (iv) (*b*), that when C is at B, CB is the tangent BD at B, and that therefore all the elements involved are continuous as C moved into coincidence with B, we have from the above principle that the angle ABD is equal to the constant angle C. Similarly the angle $AC'E$ is seen to be equal to ABD. It follows therefore that the angle ACB is constant as C moves about the whole circumference.

This general statement includes the several theorems usually given in this connection in elementary geometry. It depends however on the assumption that CB occupies its limiting position BD when C comes to B.

The theorems thus connected are:

(*a*) Two angles at the circumference of a circle, standing on the same arc, are equal.

(*b*) The angles between a tangent and a chord from the point of contact are equal to the angles in the alternate segments.

(*c*) An exterior angle of a quadrilateral inscribed in a circle is equal to the opposite interior angle.

(ii) The rectangles on the segments of two intersecting chords of a circle are equal; that is, $PA \cdot PB = PC \cdot PD$.

By moving P continuously, we find that the theorem takes the following additional forms:

(*a*) The square on the ordinate from a point on the diameter of a circle is equal to the rectangle on the segments of the diameter.

(*b*) The rectangles on the segments of two intersecting secants are equal.

(*c*) If a secant and a tangent be drawn from the same point, the rectangle on the segments of the secant is equal to the square on the tangent.

(*d*) The squares on the two tangents from a point are equal.

(iii) The square on a side of a triangle, opposite an acute angle, is equal to the sum of the squares on the other two sides diminished by twice the rectangle on one of these and the projection of the other on it; that is, $AB^2 = BC^2 + CA^2 - 2BC \cdot DC$.

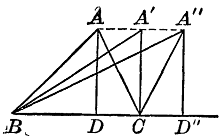


Fig. 7.

The principle of this theorem includes also each of the following:

(*a*) When A moves to D (Fig. 7), the line segment BA is divided externally at C; and for this divided line segment, $BA^2 = BC^2 + CA^2 - 2BC \cdot CA$.

(*b*) When A moves to A', and DC vanishes, the theorem is: The square on the hypotenuse of a right angled triangle is equal to the sum of the squares on the other two sides.

(*c*) When A moves to A'', $D''C$ is negative and the angle BCA'' is obtuse. Then $A''B^2 = BC^2 + CA''^2 + 2BC \cdot CD''$.

(*d*) When A moves to D'', the line segment BA is divided internally at C; and for this divided segment $BA^2 = BC^2 + CA^2 + 2BC \cdot CA$.

(iv) $ABCD$ is a complete quadrangle with opposite pairs of sides AB and CD, AC and BD, and AD and BC; and E and F are the middle points of one of these pairs, say BD and AC respectively. Then from elementary geometry,

$$AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2 + 4EF^2.$$

This theorem, in principle, includes the following:

(*a*) When D moves to B, $AB^2 + BC^2 = 2BF^2 + 2AF^2$. That is, the sum of the squares on two sides of a triangle is equal to twice the square on half the third side and twice the square on the median to the third side.

(*b*) When B and D move to H, any point on the llne-segment AC, $AH^2 + HC^2 = 2AF^2 + 2FH^2$. That is, if a line segment be divided equally at one point and unequally at another, the sum of the squares on the unequal segments is equal to twice the sum of the squares on half the line and on the part between the points of section.

(*c*) When D moves to C, and E therefore to the middle point G of BC, $AB^2 = 4FG^2$. That is, the square on the base of a triangle is equal to four times the square on the line joining the middle points of the sides.

Various other properties of the triangle and of the divided line segment may be obtained as particular cases of this general theorem.

In the same way also properties of any polygon may be found in special forms as properties of any polygon of a smaller number of sides, or of the divided line segment. Examples (iii) and (vii) further illustrate the same idea.

(v) The three radical axes of three circles taken two and two are concurrent. This theorem takes particular forms as one, two, or three of the circles become points.

If the three circles become points, the theorem states that the right bisectors of the sides of a triangle are concurrent.

(vi) S is a circle, tangent to four mutually external circles S_1, S_2, S_3, and S_4; and t_{12} is a common tangent to S_1 and S_2, etc. Then by Casey's theorem[1]

$$t_{12}t_{34} \pm t_{13}t_{42} \pm t_{14}t_{23} = 0.^2$$

This theorem takes particular forms as one, two, three, or four of the circles become points. Thus, for example,

(*a*) If the four circles which touch S become points, the result is Ptolemy's theorem on the concyclic quadrangle.

[1] *Proceedings of the Royal Irish Academy*, 1866. See also J. L. Coolidge, *A Treatise on the Circle and Sphere*, Oxford, 1916, p. 38.—EDITOR.

[2] "Here all the t_{ij}'s denote common direct tangential segments, or those connecting two pairs [of circles] with no common member denote direct tangents and the other four transverse, or those which lack one subscript denote direct, and those which include it transverse tangential segments" (Coolidge).

(*b*) If the four circles which touch S become points and S becomes a straight line, we have the common theorem on four collinear points A, B, C, D, viz: $AB \cdot CD + BC \cdot AD + CA \cdot BD = 0$.

(vii) A, B, C, D, E, F, are six points in any order on a conic section; and $ABCDEFA$ is an inscribed hexagram. If AB and DE, BC and EF, CD and FA intersect in X, Y, Z, respectively, then XYZ is a straight line, the Pascal line of the hexagram.

By moving vertices into coincidence, and assuming that their joins become tangents to the conic, we obtain this theorem in forms adapted to the pentagram, the tetragram, and the triangle.

Thus, for example, the hexagram becomes a triangle if B moves to A, D to C, and F to E. The theorem then becomes: If a triangle be inscribed in a conic, the sides intersect the tangents at the opposite vertices collinearly. The line of collinearity is the Pascal line of the triangle. Or the theorem may be stated thus: If a triangle be inscribed in a conic, and another circumscribed at the vertices of the first, the two triangles are in perspective. The axis of perspective is the Pascal line of the inscribed triangle.

This list of examples may be increased indefinitely from both elementary and advanced geometry. It will be found everywhere that general principles may be discovered to underlie groups of theorems which in our ordinary teaching of the subject have had no relationship to one another.

A NEW PROOF OF THE LAW OF TANGENTS.[1]

By WM. F. CHENEY, Jr., Berkeley, Cal.

1. In the triangle ABC, with angles A, B, and C, and sides a, b, and c, assume b greater than c. Lay off AD along b, equal to c, and draw BD; then

$$\angle DBA = \tfrac{1}{2}(180^\circ - A) = \tfrac{1}{2}(B + C);$$

[1] Other geometrical discussions of the law of tangents may be found in the following sources: W. E. Johnson, *Treatise on Trigonometry*, London, 1889, p. 96; R. Levett and C. Davison, *Elements of Plane Trigonometry*, London, 1892, pp. 170–171 (also in J. W. Mercer, *Trigonometry for Beginners*, Cambridge, 1906, pp. 259–260); E. Brand, *Journal de mathématiques élémentaires* (de Longchamps), 1895, pp. 153–154; E. M. Langley, *Journal de mathématiques élémentaires* (de Longchamps), 1896, pp. 3–4 (construction introducing Wallace's Line); E. W. Hobson, *A Treatise on Plane Trigonometry*, second edition, Cambridge, 1897, pp. 155–156; E. J. Wilczynski, *Plane Trigonometry and Applications*, Boston, 1914, pp. 105–106; and J. W. Young and F. M. Morgan, *Plane Trigonometry and Numerical Computation*, New York, 1919, pp. 47–48.

The law of tangents in the usual form (except for notation) was first given by Vieta in the seventeenth century; *Francisci Vietæ opera mathematica*, Lugduni Batavorum, 1646, p. 402: "Vt adgregatum crurum ad differentiam eorundem, ita prosinus dimidiæ summæ angulorum ad basin ad prosinum dimidiæ differentiæ."

Cf. A. von Braunmühl, *Vorlesungen über Geschichte der Trigonometrie*, Teil 1, 1900, p. 188; Teil 2, pp. 44–45; and other places referred to under heading "Tangentensatz" in indexes.—Editor.

and

$$\angle\ CBD = \tfrac{1}{2}(B - C).$$

2. Draw AP, the perpendicular bisector of BD, and produce it to meet BC at Q; then

$$AP = BP \cdot \tan \tfrac{1}{2}(B + C);$$

and

$$PQ = BP \cdot \tan \tfrac{1}{2}(B - C).$$

3. Draw PR parallel to BC, bisecting CD at R; then

$$AR = \tfrac{1}{2}(b + c);$$

and

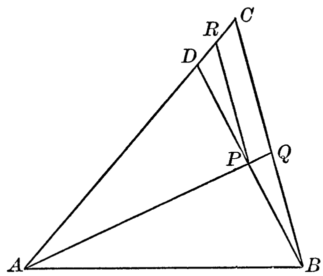
$$RC = \tfrac{1}{2}(b - c).$$

4. But

$$\frac{AR}{RC} = \frac{AP}{PQ};$$

therefore

$$\frac{b + c}{b - c} = \frac{\tan \frac{1}{2}(B + C)}{\tan \frac{1}{2}(B - C)}.$$



QUESTIONS AND DISCUSSIONS.

EDITED BY W. A. HURWITZ, Cornell University, Ithaca, N. Y.

DISCUSSIONS.

Professor J. E. Trevor gave an example in the December number of the MONTHLY of an instance in which a theorem of analysis is of importance in thermodynamics. Another case of this sort is contributed by him as the first discussion in the present number. Corresponding to a change of independent variable, the form of a function is altered; it is readily shown that the rate of change can be represented as an ordinary derivative of another function closely related to the first. The notion is of use in connection with the heats of dilution of a solution. Instances in which mathematical conceptions arise in connection with applied science are always of value; even those mathematicians whose personal interests are principally in so-called "abstract" fields cannot afford to neglect the phases of mathematics sometimes termed "practical."

In the study of analytic geometry beyond the more elementary portions determinants play an extremely important part; a glance at any treatise on projective or metric analytic geometry, on differential geometry, or on non-Euclidean geometry analytically treated, will reveal page after page filled with resultants, discriminants, Hessians, linear dependence, and other ideas based on determinants. In the elementary parts of the subject, however, determinants

play a negligible rôle. In most texts in common use, the determinantal form for the area of a triangle is a solitary instance. Professor Foraker in the second discussion below gives an introduction to the use of determinants in the elementary field. Most of his work consists of the statement of formulae; one theorem is proved, relating to the collinearity of the circumcenter, centroid, and orthocenter of a triangle. It is scarcely to be expected that the material would be suitable for ordinary class use, but it will be of interest to a few students in any class.

The third discussion is a plea for the more general use of the history of mathematics in connection with secondary and collegiate instruction. Attracted by an announcement of the plans of the National Committee on Mathematical Requirements, Mr. Laurin Zilliacus of the Bedales School in Petersfield, England, has written a letter on the subject to Professor H. W. Tyler, a member of the committee. With the permission of the writer, the recipient, and Professor J. W. Young, chairman of the committee, we reproduce nearly the entire letter, retaining the personal form in which it is written. The fact that Mr. Zilliacus has actual experience of the growth of interest and enthusiasm in a class through the use of historical material is of more value than any amount of mere theorizing on the question.

Another article basing pedagogic recommendations on the result of personal experience is found in the last discussion, by Professor Ettlinger, on the use of graphical methods in trigonometry. It is nearly certain that all teachers make some use, in their teaching, of the ideas suggested by Professor Ettlinger; not all, however, have carried these ideas so far, or treated them with such emphasis. One incidental point in the paper moves the editor to a comment which he has long hoped to see made. It is pointed out that a correctly drawn figure is of great aid in attacking a problem. This is undeniably true in trigonometry, and especially in elementary geometry. Is it not also true, on the other hand, that sometimes a figure drawn with carefully planned *inaccuracy* is of extreme importance and inspiration?

I. Heats of Dilution.

By J. E. Trevor, Cornell University.

The "heats of dilution" of a solution are mathematical curiosities, in that these quantities are defined with reference to changes of the form of a function. When $\psi(x)$ is a given function, and δx is an arbitrary positive increment of the independent variable x, let it be supposed that a certain operation changes the value of a quantity φ from the value

$$\varphi_1 = \psi(x) + a \cdot \delta x$$

to the value

$$\varphi_2 = \psi(x + \delta x),$$

where a is a constant. The change of value $\varphi_2 - \varphi_1$ is due to the change of

form of the function $\varphi(x, \delta x)$, and the rate of change of φ per unit increment of x in $\psi(x)$ is

$$\lim_{\delta x \to 0} \frac{\psi(x + \delta x) - \psi(x) - a \cdot \delta x}{\delta x} = \frac{d(\psi - ax)}{dx}$$

The heats of dilution of a solution are defined by such limits.

Consider constant masses M_1, M_2 of two component substances, such as salt and water or water and alcohol, capable of forming a homogeneous liquid mixture. Let the state of the body constituted of the masses M_1, M_2 be such that any mass m_j of the jth component exists separately from the mixture of the masses $M_j - m_j$ and M_k, where M_k is the mass of the other component and $M_j \geq m_j \geq 0$. In the case $m_j = 0$, *i.e.*, when the body is a homogeneous "solution" σ in a state of stable thermodynamic equilibrium under the pressure p at the temperature θ, let $E(p, \theta, M_1, M_2)$ be the energy and $V(p, \theta, M_1, M_2)$ be the volume of the solution, where p, θ, M_1, M_2 are variable parameters. When the body is in a state σ_1 with separated parts, each in stable equilibrium at p, θ, its energy E_1 and volume V_1 are the sums of the energies and volumes of the parts. The "enthalpy" G of the solution σ and the enthalpy G_1 of the body in the state σ_1 are defined by the equations

$$G = E + pV, \qquad G_1 = E_1 + pV_1,$$

from which it follows that

$$E - E_1 = (G - G_1) - p(V - V_1).$$

When the separate parts are brought together, under the constant pressure p, the state σ_1 is transformed into the state σ and the work $-p(V - V_1)$ is absorbed by the body. Hence, by the energy law, the heat absorbed by the body is the quantity $G - G_1$. The particular case of this process realized when $m_j = M_j$ is the formation of the solution from its components. If g_j is the enthalpy of unit mass of the jth component, the enthalpy G_1 of the separate components is $M_1 g_1 + M_2 g_2$, and the "heat of mixing" ΔG of the solution is

$$\Delta G = G - M_1 g_1 - M_2 g_2, \tag{1}$$

where g_1, g_2 are functions of p, θ.

Consider now the dilution of the solution whose mass is $M_1 + M_2$, by addition of the mass δM_j of the jth component at p, θ. The value of the enthalpy of the body composed of the masses $M_1 + M_2$ and δM_j is

$$G(p, \theta, M_1, M_2) + g_j(p, \theta)\delta M_j \tag{2}$$

before the operation, and it is

$$G(p, \theta, M_j + \delta M_j, M_k) \tag{3}$$

after the operation. Both before and after the operation, the enthalpy of the body is a function of the constant quantities p, θ, M_j, M_k, δM_j. In the opera-

tion the enthalpy of the body changes in value because the form of the function changes from the form (2) to the form (3). Suppressing the constants p, θ, M_k, this change of value is

$$G(M_j + \delta M_j) - G(M_j) - g_j\delta M_j.$$

Since this expression denotes the heat absorbed by the body (M_j, M_k, δM_j) during the dilution, we have that the heat absorbed per unit mass of diluent added is the limit

$$\lim_{\delta M_j \to 0} \frac{G(M_j + \delta M_j) - G(M_j) - g_j\delta M_j}{\delta M_j} = \frac{\partial G}{\partial M_j} - g_j.$$

This quantity is the "heat of dilution" Δ_j of the solution (M_1, M_2) for dilution by the jth component of the mixture.

Now by differentiating (1) we find

$$\frac{\partial \Delta G}{\partial M_j} = \frac{\partial G}{\partial M_j} - g_j.$$

Hence Δ_j is equal to the derivative $\partial\Delta G/\partial M_j$ of the heat of mixing. Further, since it is known that $G(p, \theta, M_1, M_2)$ is homogeneous of degree one in M_1, M_2, and hence by (1) that ΔG is homogeneous of degree one in these variables, we have

$$\Delta G = M_1\Delta_1 + M_2\Delta_2.$$

It thus appears that the heat of mixing of a solution is a linear function of the two heats of dilution.

II. Determinants in Elementary Analytic Geometry.

By F. A. Foraker, University of Pittsburgh.

The general purpose of this paper is to indicate how some results in elementary analytic geometry may be conveniently expressed in determinant form. We shall consistently use the notation

$$(1) \qquad \Delta = \begin{vmatrix} x_1, & y_1, & 1 \\ x_2, & y_2, & 1 \\ x_3, & y_3, & 1 \end{vmatrix}, \qquad D = \begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix};$$

and shall in the usual way indicate the cofactors of elements of determinants by corresponding capital letters. The points P_1, P_2, P_3 will be understood to have coördinates (x_1, y_1), (x_2, y_2), (x_3, y_3) respectively. It is well known that the area of the triangle $P_1P_2P_3$ is $\frac{1}{2}\Delta$, and that therefore the points P_1, P_2, P_3 are collinear if and only if $\Delta = 0$.

The equation of the line joining P_1 and P_2 is

$$(2) \qquad \begin{vmatrix} x, & y, & 1 \\ x_1, & y_1, & 1 \\ x_2, & y_2, & 1 \end{vmatrix} = 0;$$

in particular, the equation of a line having intercepts a, b is

$$(3)\qquad \begin{vmatrix} x, & y, & 1 \\ a, & 0, & 1 \\ 0, & b, & 1 \end{vmatrix} = 0.$$

The equations of lines respectively parallel and perpendicular to (2) through P_3 are

$$(4)\qquad \begin{vmatrix} x - x_3, & y - y_3, & 0 \\ x_1, & y_1, & 1 \\ x_2, & y_2, & 1 \end{vmatrix} = 0$$

and

$$(5)\qquad \begin{vmatrix} y_3 - y, & x - x_3, & 0 \\ x_1, & y_1, & 1 \\ x_2, & y_2, & 1 \end{vmatrix} = 0.$$

The latter equation may also be written:

$$(6)\qquad \begin{vmatrix} x, & y, & 1 \\ x_3, & y_3, & 1 \\ X_3, & Y_3, & 0 \end{vmatrix} = 0.$$

The equation of a line through P_1 with slope m is

$$(7)\qquad \begin{vmatrix} x, & y, & 1 \\ x_1, & y_1, & 1 \\ 1, & m, & 0 \end{vmatrix} = 0;$$

the normal form of the equation of a line may be written

$$(8)\qquad \begin{vmatrix} x, & y, & 1 \\ p, & 0, & \cos\alpha \\ 0, & p, & \sin\alpha \end{vmatrix} = 0,$$

provided $p \neq 0$.

All these equations are of the general type

$$(9)\qquad \begin{vmatrix} x, & y, & 1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix} = 0,$$

or may be readily reduced to that type. Conversely, (9) represents a straight line unless $A_1 = B_1 = 0$. From (9) any of the usual forms may be derived; it is seen at once that the intercepts and the slope are

$$a = -\,C_1/A_1, \qquad b = -\,C_1/B_1; \qquad m = -\,A_1/B_1.$$

The equation of a circle through the points P_1, P_2, P_3 is

$$(10)\qquad \begin{vmatrix} x^2 + y^2, & x, & y, & 1 \\ x_1^2 + y_1^2, & x_1, & y_1, & 1 \\ x_2^2 + y_2^2, & x_2, & y_2, & 1 \\ x_3^2 + y_3^2, & x_3, & y_3, & 1 \end{vmatrix} = 0;$$

this may be written

$$\Delta(x^2+y^2) - |\, x_i^2+y_i^2,\ y_i,\ 1\,|\, x - |\, x_i,\ x_i^2+y_i^2,\ 1\,|\, y = |\, x_i^2+y_i^2,\ x_i,\ y_i\,|,$$

where the meaning of the abbreviated determinants is obvious. Thus the circumcenter of the triangle $P_1P_2P_3$ has the coördinates

$$C_x = \frac{1}{2\Delta}|\, x_i^2+y_i^2,\quad y_i,\quad 1\,|, \qquad C_y = \frac{1}{2\Delta}|\, x_i,\quad x_i^2+y_i^2,\quad 1\,|. \tag{11}$$

The median through the vertex P_1 of the triangle $P_1P_2P_3$ has the equation

$$\begin{vmatrix} x, & y, & 1 \\ x_1, & y_1, & 1 \\ x_2+x_3, & y_2+y_3, & 2 \end{vmatrix} = 0, \tag{12}$$

which may be written in the form

$$\begin{vmatrix} x, & y, & 1 \\ x_1, & y_1, & 1 \\ x_1+x_2+x_3, & y_1+y_2+y_3, & 3 \end{vmatrix} = 0.$$

From symmetry it is seen at once that the three medians intersect in a single point, the centroid, having the coördinates

$$G_x = \tfrac{1}{3}(x_1+x_2+x_3), \qquad G_y = \tfrac{1}{3}(y_1+y_2+y_3). \tag{13}$$

The altitudes of the triangle $P_1P_2P_3$ through P_1 and P_2 have respectively the equations

$$\begin{vmatrix} x, & y, & 1 \\ x_1, & y_1, & 1 \\ y_2-y_3, & x_3-x_2, & 0 \end{vmatrix} = 0, \qquad \begin{vmatrix} x, & y, & 1 \\ x_2, & y_2, & 1 \\ y_1-y_3, & x_3-x_1, & 0 \end{vmatrix} = 0.$$

Solving, we find

$$x = \frac{\begin{vmatrix} x_1x_3-x_1x_2+y_1y_3-y_1y_2, & y_3-y_2 \\ x_2x_3-x_1x_2+y_2y_3-y_1y_2, & y_3-y_1 \end{vmatrix}}{\begin{vmatrix} x_3-x_2, & y_3-y_2 \\ x_3-x_1, & y_3-y_1 \end{vmatrix}}$$

If we interchange rows, change the signs in certain columns, and border with the quantities indicated, we have

$$x = \frac{\begin{vmatrix} x_2x_3-x_1x_2+y_2y_3-y_1y_2, & y_1-y_3, & 0 \\ x_1x_3-x_1x_2+y_1y_3-y_1y_2, & y_2-y_3, & 0 \\ x_1x_2+y_1y_2, & y_3, & 1 \end{vmatrix}}{-\begin{vmatrix} x_1-x_3, & y_1-y_3, & 0 \\ x_2-x_3, & y_2-y_3, & 0 \\ x_3, & y_3, & 1 \end{vmatrix}};$$

so that finally

$$x = \frac{\begin{vmatrix} x_2x_3 + y_2y_3, & y_1, & 1 \\ x_3x_1 + y_3y_1, & y_2, & 1 \\ x_1x_2 + y_1y_2, & y_3, & 1 \end{vmatrix}}{-\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}}$$

In like manner we may obtain y. The results involve the three subscripts symmetrically; therefore the point must be the intersection of each pair of altitudes. As the common intersection of the three altitudes, the orthocenter, we thus have

$$(14) \qquad \begin{aligned} O_x &= -\frac{1}{\Delta}\begin{vmatrix} x_2x_3 + y_2y_3, & y_1, & 1 \\ x_3x_1 + y_3y_1, & y_2, & 1 \\ x_1x_2 + y_1y_2, & y_3, & 1 \end{vmatrix}, \\ O_y &= -\frac{1}{\Delta}\begin{vmatrix} x_1, & x_2x_3 + y_2y_3, & 1 \\ x_2, & x_3x_1 + y_3y_1, & 1 \\ x_3, & x_1x_2 + y_1y_2, & 1 \end{vmatrix}. \end{aligned}$$

From (11), (13), (14) we have

$$S = \Delta(3G_x - 2C_x - O_x) = \begin{vmatrix} r_1 & y_1 & 1 \\ r_2 & y_2 & 1 \\ r_3 & y_3 & 1 \end{vmatrix},$$

where

$$\begin{aligned} r_1 &= x_1(x_1 + x_2 + x_3) - (x_1^2 + y_1^2) + (x_2x_3 + y_2y_3) \\ &= (x_2x_3 + x_3x_1 + x_1x_2) + (y_2y_3 - y_1^2), \end{aligned}$$

with similar expressions for r_2, r_3. Hence

$$S = (x_2x_3 + x_3x_1 + x_1x_2)\begin{vmatrix} 1, & y_1, & 1 \\ 1, & y_2, & 1 \\ 1, & y_3, & 1 \end{vmatrix} + \begin{vmatrix} y_2y_3, & y_1, & 1 \\ y_3y_1, & y_2, & 1 \\ y_1y_2, & y_3, & 1 \end{vmatrix} - \begin{vmatrix} y_1^2, & y_1, & 1 \\ y_2^2, & y_2, & 1 \\ y_3^2, & y_3, & 1 \end{vmatrix}.$$

The first determinant vanishes; each of the others is equal to $(y_3 - y_2)(y_1 - y_3)(y_2 - y_1)$; therefore $S = 0$. We may similarly examine the y-coördinates; we find that

$$\begin{aligned} 3G_x - 2C_x - O_x &= 0, \\ 3G_y - 2C_y - O_y &= 0. \end{aligned}$$

Thus we have proved the theorem: *The centroid lies one-third of the distance from the circumcenter to the orthocenter, along the line joining them.*

If we have the equations of three lines given in the form

$$(15) \qquad a_ix + b_iy + c_i = 0 \qquad [i = 1, 2, 3],$$

the lines will form a triangle if D, C_1, C_2, C_3 are all different from zero. The

vertices of the triangle are then

$$(16)\qquad \begin{aligned} x_{23} &= A_1/C_1, & y_{23} &= B_1/C_1. \\ x_{31} &= A_2/C_2, & y_{31} &= B_2/C_2. \\ x_{12} &= A_3/C_3, & y_{12} &= B_3/C_3. \end{aligned}$$

The coördinates of the circumcenter are seen by (11) and (16) to be[1]

$$(17)\qquad \begin{aligned} C_x &= \frac{1}{2D^2}\begin{vmatrix} \dfrac{A_i^2 + B_i^2}{C_i}, & B_i, & C_i \end{vmatrix}, \\ C_y &= \frac{1}{2D^2}\begin{vmatrix} A_i, & \dfrac{A_i^2 + B_i^2}{C_i}, & C_i \end{vmatrix}. \end{aligned}$$

The coördinates of the centroid are

$$(18)\qquad C_x = \tfrac{1}{3}\left(\frac{A_1}{C_1} + \frac{A_2}{C_2} + \frac{A_3}{C_3}\right), \qquad C_y = \tfrac{1}{3}\left(\frac{B_1}{C_1} + \frac{B_2}{C_2} + \frac{B_3}{C_3}\right).$$

The coördinates of the orthocenter are

$$(19)\qquad \begin{aligned} O_x &= -\frac{1}{C_1C_2C_3D^2}\begin{vmatrix} (A_2A_3 + B_2B_3)C_1^2, & B_1, & C_1 \\ (A_3A_1 + B_3B_1)C_2^2, & B_2, & C_2 \\ (A_1A_2 + B_1B_2)C_3^2, & B_3, & C_3 \end{vmatrix}, \\ O_y &= -\frac{1}{C_1C_2C_3D^2}\begin{vmatrix} A_1, & (A_2A_3 + B_2B_3)C_1^2, & C_1 \\ A_2, & (A_3A_1 + B_3B_1)C_2^2, & C_2 \\ A_3, & (A_1A_2 + B_1B_2)C_3^2, & C_3 \end{vmatrix}. \end{aligned}$$

The area of the triangle is $D^2/(2C_1C_2C_3)$.

A few of the above results are well known; the others are easily proved. This paper is intended only as an introduction to work along this line and is not supposed to be exhaustive in any sense.

III. The History of Mathematics in Elementary Instruction.[2]

By Laurin Zilliacus, Bedales School, Petersfield, England.

The General Education Board has recently announced the appropriation of $16,000 to be used by the National Committee on Mathematical Requirements in financing a study looking to improvements in the mathematical curriculum of the secondary schools of America. If there is to be any recasting of curricula, I earnestly hope that a full use will be made of the history of mathematics. A short perusal of this history, as given in popular works such as Ball's *History of Mathematics,* would, I feel sure, cause a majority of present-day teachers of mathematics to modify considerably both the outlines of their courses and their

[1] Since $|A_1B_2C_3| = |a_1b_2c_3|^2$.

[2] Extract from a letter to Professor H. W. Tyler.

methods of presenting the subject. The particular lines of change which would result I am not qualified to predict, but I think it would be safe to say in general that (1) there would be a reduction in unnecessary drudgery as a true perspective of values was obtained; (2) the fundamental principles (*e.g.*, compact number notation, symbolic statement, variation, limits, deductive reasoning) would stand out more clearly and comprehensively when it was seen how they had been gradually pulled out from confusion, instead of their being, as at present, presented as part of a complete and arbitrary structure; (3) there would be a living connection between mathematics and "practical" life in the minds of children who were shown how each has waited on the other throughout the ages.

My own experience in a school of boys and girls of 11 to 19 years of age has convinced me that the coördination of mathematics with experimental science and the historical treatment of mathematics go hand in hand and together yield most fruitful results. A couple of lectures in the school on the history of mathematics roused great interest and led to many questions and some independent reading. A chart was displayed of the "Stream of Mathematical Knowledge" from the earliest known times, showing how the thin trickle of isolated facts from the Egyptians was swelled by the Greeks into a great river of knowledge; how the river branched and wandered, increasing here and shrinking there, through the Arabs (with a great tributary from the Hindoos), the Romans, the Byzantines and the Moors, until with the advent of the printing press and the flight of the Byzantine Greeks from Constantinople the streams reunited in Europe, and swelled on every hand—there the chart ended, but it is to be continued later. I found that the children were delighted as with a story, and that mathematics acquired a new meaning for them when linked up with historical familiars such as Athens, Alexandria, the Mohammedan conquests, Haroun al Raschid, the Moors of Spain, etc., etc.

In particular classes, too, the history was of great service. My difficulties with a Mechanics class vanished when a discovery in books (Cox's *Mechanics*) enabled me to outline the development of our views of the solar system from the theory of epicycles through the Copernican system, Galileo's trouble-bringing discoveries and Kepler's Laws to Newton's great work. The class was deeply interested, and grasped Newton's Laws, circular motion, energy and momentum in a way I have never before experienced with a class.

A beginner's class in geometry was much helped by a short account of how Thales and his successors took experimentally determined facts from the Egyptians, investigated their explanation and deduced other useful facts from them: the meaning of a proof was quickly grasped, and very rapid progress made in what they regarded as an intellectual game. They themselves discovered experimentally the property of angles in a semicircle, and were probably as eager as Thales to find the explanation and proof.

Summing up the benefits that struck me even during my short experience of teaching mathematics with a due regard for its history, I should say they are as follows:

For the teacher: (1) The relative importance of various aspects of the subject becomes easier to determine, and hence what should be omitted and what emphasized.

(2) Indications are obtained of the order in which to introduce new knowledge, since the history of the world's progress in mathematics is not without similarity to that of an individual (*e.g.*, the late appearance in history of an understanding of the minus sign in algebra indicates where it should be attempted in the school study of this subject).

(3) The difficulties of beginners are more readily seen, for they are much the same throughout time. (A striking instance of this is the confusion between determinate and indeterminate problems—once pointed out, it never recurs.)

For the taught: (4) An understanding of mathematics as a body of methods and knowledge that have grown and are still growing by the labours of men, and not as a curiously pointless and arbitrary set of rules and definitions.

(5) A better grasp of underlying principles as the course of history makes them stand out more and more clearly.

(6) A connection of mathematics with interesting and useful inventions.

(7) The interest that any subject gains that is packed with anecdotes and stories.

IV. An Introduction to Plane Trigonometry by Graphical Methods.

By H. J. Ettlinger, University of Texas.

It is not an uncommon experience of teachers of trigonometry to find students, who are plunged without warning into the definitions of the trigonometric functions, completely bewildered by the terms, sine, cosine, etc. It may be a good plan in taking a cold bath to immerse suddenly, but it is equally a good plan in teaching trigonometry to lead the student into the subject by a method which will directly connect it with his previous study of plane geometry.

This summer in a course in trigonometry the writer developed at some length a graphical introduction, based on the use of the ruler, compasses, and protractor. This graphical substratum continued throughout the course as an essential element.

As a student, the writer recalls a frequent remark of one of his instructors, Professor Roever of Washington University, that the element of value contained in an accurately drawn figure is usually completely overlooked. The habit of drawing by means of instruments careful and accurate figures to represent mathematical situations should be developed as early as possible. Often such a figure suggests a solution; always it will provide a check. It serves admirably as an example of what modern mathematical rigor means, viz: clearness.

The writer puts forward no claim to originality in this paper. A number of the ideas are to be found in the early chapters of Wilczynski and Slaught's *Plane Trigonometry* (Allyn and Bacon). In the course as given the past summer,

however, the text was supplemented by further development and additional problems.

The course opens with a review of plane geometry, covering the theorem of Pythagoras, similar triangles, and the construction of triangles. These theorems are treated in detail. The special case of similar right triangles is given careful attention. The construction of oblique triangles for the four cases is recalled. After pointing out the impossibility of measuring certain distances directly and the magnitude of error introduced when the measurement is possible, we proceed immediately to the graphical solution of triangles.

The use of a ruler and compasses does not require explanation. For this work a ruler graduated to tenths of an inch is a distinct advantage, since with a little practice and the use of fine lines the student can soon estimate accurately to hundredths of an inch. A centimeter scale graduated to millimeters has its merits also.

The rotation definition of an angle is now introduced and defined in terms of the arc subtended on a unit circle. For many purposes it is convenient to make the angle and this arc synonymous. For instance, it removes at once the mystery connected with the notation $\theta = \text{arc} \sin y$. It also simplifies the radian unit of measurement. The use of a protractor to measure any angle should be explained.

The student now accurately represents the actual situation in space on a sheet of paper. This he accomplishes by constructing a similar triangle to a suitable scale, and from his knowledge of similar triangles he readily sees why he obtains the correct values for the unknown sides and angles from the figure by direct measurement. For the solution of right triangles, cross-section paper is of great assistance.

The area of any triangle is found by dropping a perpendicular from a vertex to the opposite side as base and measuring the length of the altitude thus obtained. The area is then computed by the usual formula.

The construction of an accurate figure and the careful measurement of the unknown parts of the triangle continues throughout the course in combination with a solution by computation. Later in the course, after the theory of logarithms has been explained and logarithms used in computation, the slide rule is presented as a graphical table of logarithms. It is the belief of the writer that the graphical introduction facilitated the understanding of the principle of the slide rule.

The transition from the graphical method to the use of trigonometric functions is easily made by pointing out the limitations of accuracy to the above constructions. Triangles having small angles and situations demanding more accurate results can be cited. The need for a computation method is appreciated. A course given without this kind of an introduction must necessarily plunge the student into the definition of the trigonometric ratios similar to the arbitrary definitions of the theory of functions. This point of view a freshman cannot understand.

After the trigonometric functions have been defined, the class constructs a

two-place table of sines and cosines by drawing the angles and measuring the lengths. The graph of the sine curve, and cosine also, is drawn, using the data just obtained. This establishes at once the connection between sines and cosines and waves.

The construction of an angle when one of the ratios is given is a useful application of the graphical method. The angle is measured by means of the protractor, and the other ratios are scaled off. Squared paper is very helpful for this work. Finally, it should be pointed out that at this time the inverse trigonometric functions should be introduced as nomenclature for the angles just constructed, and not, as is usually done, at the very end of the course.

It occurs to the writer that some teachers might contend that in a short course in plane trigonometry there is no time to give to the considerations discussed in this paper. In reply the writer can only cite his experience of this summer. The course covered twenty-eight lecture periods, each fifty minutes in length. Out of these, four were devoted to hour quizzes, and two more, to review at the end of the course. In addition to the graphical introduction, all the topics of the average text on plane trigonometry were covered, including proofs for all the formulæ used, trigonometric identities, and trigonometric equations. The only curtailment necessary was in the time ordinarily devoted to the last topic.

From the point of view of interest it may be stated that this introduction makes a strong appeal to all classes: the future engineer always on the alert for practical methods, the bright student studying mathematics for its own sake, as well as the "prerequisite for the A.B." individual. The method, moreover, is valuable *per se* inasmuch as it is the one used during the late war to solve problems in aerial navigation, artillery and orientation, and plane sailing where first approximations were desired. These fields alone provide a wealth of simple problems.

RECENT PUBLICATIONS.

REVIEWS.

Mathematical Essays and Addresses.

The Human Worth of Rigorous Thinking: Essays and Addresses. By Cassius J. Keyser. New York, Columbia University Press, 1916. 314 pp. Price $1.75.

Whitehead and Russell's imposing *Principia Mathematica* furnishes a systematic treatise of the most recent and horoughgoing philosophy of mathematics and as such is addressed to those well grounded in mathematics and logic; Shaw's smaller book, *The Philosophy of Mathematics,* presents the various aspects of mathematics to graduate students. A pleasant task has been assigned to the reviewer, that of describing and evaluating Professor Keyser's delightful essays

and addresses in which he interprets to what are largely non-mathematical audiences the inner spirit of mathematics. Only a mathematician can appreciate the difficulty of his undertaking, and only one who reads or listens to these may know the pleasantness and the effectiveness of the interpretation. These essays are in part known to many through their appearance in such widespread journals as *Science, The Hibbert Journal,* and *The Journal of Philosophy, Psychology and Scientific Methods.*

The style in which these essays are rendered must be at once the satisfaction and the despair of all mathematicians—the satisfaction because the subject has so able an exponent, the despair because so very few indeed—let us say one-half of one per cent.!—of the profession have had their tongues touched by so live a coal from the altar.

Strictly speaking, the title applies only to the first three addresses: "The human worth of rigorous thinking," "The human significance of mathematics," "The humanization of the teaching of mathematics," and to the closing address "Mathematics." Pointing out in his first address that mathematics may be characterized by its aim of thinking rigorously whatever is, or may become, rigorously thinkable, or by the collected results of this rigorous thinking, he examines "the just claims to human regard of perfect thought and the spirit of perfect thinking." To comprehend this fully, one must have come to know what rigorous thinking is through a close study of some one or more distinctively mathematical treatises, of the rôle of rigorous thinking in the history of civilization, and of the numberless applications of mathematics to engineering and the natural sciences. This last study makes it evident that "all thinkers and especially students of natural science are engaged, both consciously and unconsciously, both intentionally and unintentionally, in the mathematization of concepts—that is to say, in so transforming and refining concepts as to fit them finally for the amenities of logic and the austerities of rigorous thinking." As examples of this are adduced the concepts of continuity, function, and infinity. At this phase of the development of the subject he takes issue with Bergson and William James in their attack on the method of concepts, the modern method of science, contending that instead of the aim and ideal of intellect being the understanding and subjugation of matter, the essential function of concepts is something more than this, viz., to quote Diotima, "I am persuaded that all men do all things, and the better they are, the better they do them, in the hope of the glorious fame of immortal virtue." He instances further the great difficulty or utter impossibility which these men have on psychological grounds of explaining such contradictions as those, for example, involved in bridging over the transition from sensations, finite in number, to concepts, infinite in multiplicity. What the intellect has done is rather to create for itself a world of conceptual magnitudes which is free from the contradictions of the world of the senses; these magnitudes form the subject-matter of science. The aim of this creative intellect is to preserve and to promote the life of the intellect itself, to think in accordance with the laws of its being, to bring all its laws and methods into complete harmony. Thus "science

and especially mathematics, the ideal form of science, are creations of the intellect in its quest of harmony."

The second address, delivered at the University of California in 1915 before the American Association for the Advancement of Science and affiliated organizations, is too well known to demand a full analysis. Various approaches to the study of the human significance of mathematics are suggested, the historical, the utilitarian, the logical, and the spiritual, the last furnishing the theme for the address, viz., the relation of mathematics to the supreme ideals of mankind. The sovereign passion of mankind is for release from life's limitations and the tyranny of change. Our human aspirations find their highest unity in the quest of a stable world. This quest is to be made, not in the world of sense, but in that of concepts. When on this road toward perfection thought has attained a high degree of refinement, precision, and rigor, we call it mathematical thought. Yet all thought, mathematical or non-mathematical, refined or crude, possesses the unity of being human. Along with the great contributions of theology, philosophy, jurisprudence, art and science to the wealth of the world's knowledge, mathematics has shown to the world that "there exists a stable world of pure thought, a divinely ordered world of ideas, accessible to man, free from the mad dance of time, infinite and eternal." At the risk of evoking a disclaimer from Professor Keyser, one may fairly say that in his protest against the limitation of the motivation of the intellect to a superiority over the material he shares much in common with the ideals of the pragmatists,[1] while his theory of science, and of mathematics in particular, is almost identical with that of Poincaré.

The thesis of the third address, more briefly summarized, is that hope of improvement in the teaching of mathematics, in secondary schools and colleges alike, lies in the possibility of humanizing it, in recognizing that it is not sufficient to say that mathematics possesses its high position as a great human enterprise because it has given the world a metrical and computatory art essential to the conduct of daily life, furnishes countless applications to engineering and the natural sciences, and is an excellent means of giving mental discipline. It is not sufficient even to say with the mathematician that mathematics is the science of pure thought. Rather is it to be said and to be shown in our teaching that its human significance penetrates all fields, it lives in all the activities of men and of nature.

The next four articles, the thirteenth, and the last may be classed together, treating as they do such general mathematical notions as the dimensions of the universe, hyperspace and infinity. Advisedly avoiding the philosophical questions as to the nature of space and accepting it merely as "a vast region or room around us, an immense eternity, locus of all suspended or floating objects of outer sense," the speaker in "The walls of the world" attempts a bold task in seeking to interpret even to an audience of scientists Pascal's characterization of space: "Space is an infinite sphere whose center is everywhere and whose surface is nowhere." A grasp of the notions of infinite sequences and of the equality of

[1] R. B. Perry, *Present Philosophical Tendencies*, third impression, p. 268.

two such sequences, such as would be sufficient for the comprehension of Pascal's statement, is doubtless not to be gained from a single hearing; fortunate is it that the address is available in a form which will allow those interested to reflect more carefully on the notions involved. While Lucretius's argument for the infinite extent of the universe is in a clear fashion shown to depend on a confusion of the terms "boundless" and "infinite," the suggestion is made through a reference to infinities of higher order that our reason may enable us to go in thought beyond the confines even of our infinite universe.

The next article, "Mathematical emancipations" (1906), and the thirteenth, "Concerning multiple interpretations of postulate systems and the 'existence' of hyperspace" (1913), offer a more scientific treatment of the same subject as the foregoing. The former defines "dimension" with due accuracy and develops instances of magnitudes of successively increasing dimensionality, leading up to the fact that ordinary space is not inherently and uniquely three-dimensional. For example, the plane is a three-dimensional space of circles since each circle is determined by three parameters; and on the other hand ordinary space is by the same test a four-dimensional space of lines or of spheres. This interpretation of hyperspace, which is of course essentially that familiar to mathematicians, has the merit of being simple, of making it unnecessary to pass beyond the bounds of the universe or to transcend the limits of intuition. Hyperspace of points however exists not for imagination or intuition, but only for thought and reason. The question of imagining configurations in a hyperspace of points is a question of psychology, not of mathematics, but it can be readily shown that the structure of a fourfold figure can be traced out by its significant analogy with a three-dimensional figure.[1] On the score of accuracy it should be remarked that "Jove nods" when color is named as an example of three-dimensionality in its dependence on the three primary colors; this is true only in the sense of homogeneous coördinates, for a given color depends manifestly only on the two independent ratios of these three components.

The thirteenth article considers the comparative advantages of the languages of geometry and analysis, and argues that n-dimensional geometry has every kind of existence that may be attributed to ordinary geometry of space.

To the assumptions of modern science that the universe is an organic system of order and that it is the sole system of law and order, "The universe and beyond" opposes the view that above every nature there is a supernatural, a hypercosmic. In support of this view it is noted that while mathematics may be defined as

[1] In the presentation of this subject the reviewer has commonly advanced from the straight line to the square, and from this to the cube, pointing out that in our picture of the cube right angles and lines of equal length do not always appear as such; it is then noted that if a hypercube of four dimensions is to be constructed, one would erect a cube on each of the six faces of this cube extending into the fourth dimension, these cubes being "adjacent" in sets of three, and the "outer vertices," eight in number, would determine the last of the eight cubes which bound the hypercube; the number of edges, faces (squares), and vertices are enumerated by analogy with two and three dimensions, and finally *the picture of the hypercube* (the projection into three-dimensional space) *is given*, this comprising a cube entirely outside a second cube with the pairs of corresponding vertices joined by eight lines.

the science of measurement, direct or indirect, and of position, as the science of operations, etc., as characterized by its content, we may also define it by its method as in the well-known definition of Benjamin Peirce; and since pure mathematics is intrinsically concerned with accuracy, correctness and completeness in its logic rather than with the truth of its applications to our universe, which is the domain of natural science, it may conceivably deal with that which transcends the sensible universe, as when for example one deals in euclidean and non-euclidean geometries with inconsistent systems of consistent relationships.

The last article, "Mathematics," supplements the characterization of mathematics as given in the preceding paper by a presentation of the growth of mathematics as a science through the critical movement grounded in the study of the theory of functions and of the foundations and in a study of symbolic logic which discloses the basis of logic as identical with the basis of mathematics. The background of Keyser's writing seems clearly to be the philosophy of mathematics embodied in Whitehead and Russell's *Principia Mathematica,* the first two volumes of which he has reviewed in the twelfth chapter of the present collection. (The reader who desires to follow the discussion intensively needs to refer for Russell's latest interpretations to his recently published *Introduction to Mathematical Philosophy.*) Hamilton's and Schopenhauer's indictments of mathematics are rejected by Keyser as consisting of malicious misrepresentation; those of Huxley as based on actual ignorance of the nature of mathematical thinking in its relation to observation and causation. It is not surprising that a writer so zealous for the purity and for the spiritual values of mathematics should in the closing part of this paper disparage the term "applied mathematics" and distinguish sharply between mathematics and natural science. This indicates a divergence, partly of emphasis, partly of definition, from the usage of the many who view mathematics as inclusive of its applications.

Incidentally taking issue with the views of Professor Royce as to the possibility and validity of existence proofs of the infinite, the seventh article considers "The axiom of infinity." Mankind by the need of considering the indefinitely small and the indefinitely large has been forced through mathematics, as through ethics and philosophy, to treat the infinite. This search has brought high rewards, even if it were found to be an insoluble problem. The studies of Riemann, Bolzano, Dedekind and Cantor are suggested as having for the first time in history obtained a logical hold on infinity. The contention is rightly made that when some modern critics characterize Bolzano's definition of infinity based on the inexhaustibility of a class as a negative definition, or of the infinite as the really positive and the finite as the negative, the distinction is not essential but represents merely an arbitrarily chosen point of view. Yet, to the reviewer's mind, Professor Keyser is guilty of this very "verbal legerdemain" when he insists that Royce, Russell and others are wrong in saying that the concept of the infinite denies the axiom of the whole and part because the common-sense axiom is applicable only when there is a number telling now many; this contention stands or falls solely with the particular definition given to the terms involved.

The most interesting portion of this essay is the argument[1] that it is impossible to prove the existence of the infinite, that the demonstrations of such existence invôlve reasoning in a circle, and that one must instead adopt the axiom of infinity which states that "conception and logical inference alike presuppose absolute certainty that an act which the mind finds itself capable of performing is intrinsically performable endlessly, or, what is the same thing, that the assemblage of possible repetitions of a once performable act is equivalent to some proper part of the assemblage."

The remaining articles are contributions to educational discussions. "The permanent basis of a liberal education" points out that amid the unceasingly changing state of knowledge there are certain great invariant facts which should be enshrined in any plan providing a liberal education; such facts are the immense past of mankind, the material universe in which we find ourselves, our dependence on the world of ideas, the social character of our world, the fact of the human future, and the discipline of beauty.

The address on "Graduate mathematical instruction . . ." calls upon university departments to offer such courses as will enable students to gain a general knowledge of the problems and methods of these fields even though specializing in other departments. It is maintained on the ground of experience and observation that there is an opportunity for such courses and a distinct duty of the universities in the matter, in spite of the fact that the intrinsic technicality of each subject may make it difficult and in part impossible for the instructor to make it known to the non-specialist. The reviewer may in this connection instance the philosopher Josiah Royce, who in the later part of his life with the wealth of his intellectual equipment studied the fundamentals of modern mathematics with the double effect of showing in his writings an unusually intelligent and an unusually sympathetic grasp of mathematics in its philosophical bearings. How happy a contribution might be made to a clean-cut organizing and harmonizing of the various systems of philosophy if only more frequently mathematicians would bring to that study the instrumentality of their critico-logical training! Keyser would demand as a prerequisite only a year of collegiate mathematics, since in his view the material most valuable for such avocational instruction does not depend on the calculus. The course would comprise a presentation of the nature of mathematics, of postulate systems and the rôle they have played in ancient and modern times, the function concept, the limit concept, the calculus, the theory of point sets, invariants, and groups.

Three other chapters are of more general import, one of which, "Mathematical productivity in the United States" (1902) might wisely have been supplemented by the encouraging contrast which a summary of present conditions would furnish.

W. D. CAIRNS.

[1] C. J. Keyser, "The axiom of infinity and mathematical induction," *Bulletin of the American Mathematical Society*, Vol. 9, May, 1903.

NOTE ON PRIME NUMBERS.

In my review of the first volume of Dickson's *History of the Theory of Numbers* in this MONTHLY (*1919*, 402), I quote from page 425 of the *History* the conjecture of M. Cantor (correctly reproduced from his paper) that three successive primes are not in arithmetical progression unless one of them is 3. Through the kindness of Professor F. H. Loud my attention has been called to the fact that the conjectured theorem is false. In fact from a table of primes it is easy to find sets of three consecutive primes with common difference 6, for example. The four primes 251, 257, 263, 269 are successive and are in arithmetical progression.

R. D. CARMICHAEL.

Plane Trigonometry and Numerical Computation. By J. W. YOUNG and F. M. MORGAN. New York, Macmillan, 1919. 12mo. 7 + 122 pp. Price $1.25.

The Preface: "Ever since the publication of our *Elementary Mathematical Analysis* (The Macmillan Co., 1917) we have been asked by numerous teachers to publish separately, as a textbook in plane trigonometry, the material on trigonometry and logarithms of the text mentioned.

"The present textbook is the direct outcome of these requests. Of course, such separate publication of material taken out of the body of another book necessitated some changes and an introductory chapter. As a matter of fact, however, we have found it desirable to make a number of changes and additions not required by the necessities of separate publication. As a result fully half of the material has been entirely rewritten, with the purpose of bringing the text abreast of the most recent tendencies in the teaching of trigonometry.

"There is an increasing demand for a brief text emphasizing the numerical aspect of trigonometry and giving only so much of the theory as is necessary for a thorough understanding of the numerical applications. The material has therefore been arranged in such a way that the first six chapters give the essentials of a course in numerical trigonometry and logarithmic computation. The remainder of the theory usually given in the longer courses is contained in the last two chapters.

"More emphasis than hitherto has been placed on the use of tables. For this reason a table of squares and square roots has been added. Recent experience has emphasized the applications of trigonometry in navigation. We have accordingly added some material in the text on navigation, have introduced the haversine, and have added a four-place table of haversines for the benefit of those teachers who feel that the use of the haversine in the solution of triangles is desirable. The material can, however, be readily omitted by any teacher who prefers to do so."

NOTES.

The fifth quinquennial *Table des matières* of *Revue semestrielle des publications mathématiques* was published in 1918. It covers volumes 21–25 (1913–1917).

Noordhoff of Groningen published in 1918 the second volume of the *Oeuvres complètes de Thomas Jan Stieltjes* which are being issued under the auspices of the Amsterdam mathematical society. It is a quarto volume of over six hundred pages. The first volume, which was about one-third smaller, appeared in 1914.

The fourth edition of Appell's *Traité de mécanique rationnelle*, tome 1, was published in 1919. It contains about five pages more than the third edition.—Recent publications by R. de Montessus de Ballore are: *Introduction à la théorie des courbes gauches algébriques* (Paris, Croville-Morant, 1918); the sixth enlarged and thoroughly revised edition of the elementary algebra part of C. de Comberousse's *Cours de mathématiques* (Paris, Gauthier-Villars, 1919); and the *Yearbook of the Universities* to which we have already made reference (*1919*, 358).

Recent catalogues of second hand mathematical books: No. 97 (495 titles in mathematics and physics), Galloway and Porter, Cambridge, England; No. 773 (3336 titles in exact and applied science), Henry Sotheran and Co., 140 Strand, W. C. 2, London. The latter catalogue includes the library of Olaus Henrici.

Among the articles in *Revista matemática Hispano-Americana*, September, 1919 are: "El tratado de la logarítmica de Juan Caramuel" by D. F. Diéguez pp. 203–212 (including facsimiles of two pages of tables); and 'La teoría de los grupos en la enciclopedia matemática" by G. A. Miller, pp. 229–232. Cf. *1919*, 302.

The frontispiece of *Popular Astronomy* for November, 1919 is a group picture of members and visitors at the Ann Arbor meeting of the American Astronomical Society, September 2–5. Among those in the group are the following members of the Association: E. W. Brown, W. D. MacMillan, F. R. Moulton, and E. D. Roe, Jr. Elsewhere in the issue is a fine portrait of Benjamin Baillaud (director of the Paris observatory, and first president of the International Astronomical Union), and a picture of the Palais des Académies where the meetings of the International Research Council were held in Brussels last July.

Science, for December 19, records that at a recent meeting of the corporation of Yale University it was voted "to extend the sincere congratulations of the corporation to Professor Ernest Brown on the completion of his monumental work on the *Tables of the Motion of the Moon*, and to assure him that the university considers the work that he has done on these volumes as among the most important scientific contributions ever made by an officer of Yale University."

G. Bell and Sons, London, have announced the publication of *Nomography* by S. Brodetsky, and of *Differential Equations and their Applications* by H. Pioggio.— Dunod, Paris, has published a French translation, by A. E Gérard, of Silvanus P. Thompson's *Calculus Made Easy* (first published anonymously in 1910) with the title: *Le calcul intégral et différentiel à la portée de tout le monde.*

Engineering Education. Essays for English. Selected and edited by R. P. Baker (New York, Wiley, 1919) includes the following: Under the heading The Origins of Engineering Education, "Evolution of the scientific investigator" by S. Newcomb, pages 3–28 (reprinted from the Smithsonian Institution *Report*, 1904); under the heading The Bases of Engineering Education—Mathematics, "The place of mathematics in engineering practice" by W. H. White, pages 103–112 (reprinted from *Nature*, September 19, 1912), and "On the relation of mathematics to engineering" by A. Ranum, pages 113–121 (reprinted from the *Sibley Journal of Engineering*, January, 1914). Each essay is preceded by a brief biographical sketch of the author.

The following paragraph appeared in *Nature* for November 27:

"The unfailing energy of Prof. Pearson's department at University College, London, has now resulted in the production of a series of tracts published by the Cambridge University Press.

The objects of this new series are not only to publish new tables (as well as to republish old and inaccessible tables), but also in due course to issue works on interpolation, mechanical quadratures, calculating machines, and other matters of importance to the practical computer. The first of the series is before us, and is entitled *Tables of the Digamma and Trigamma Functions*, by ELEANOR PAIRMAN. The work contains tables of the logarithmic derivate of the Gaussian Π-function and of its derivate, in addition to some useful miscellaneous information concerning these two functions. The functions are tabulated to eight places of decimals at intervals of 0.02 from 0 to 16, with tables of second differences. There seems no doubt that this series will be of extreme value to computers, and we must feel deep gratitude to Prof. Pearson for using the resources at his disposal in producing it. Finally, it should be said that the appearance of the first of the series is up to the standard which we have grown accustomed to expect from the Cambridge University Press."

ARTICLES IN CURRENT PERIODICALS.

ANNALS OF MATHEMATICS, 2d series, volume 21, no. 1, September, 1919: "Investigation of a class of fundamental inequalities in the theory of analytic functions" by J. L. W. V. Jensen [translation from the Danish by T. H. Gronwall], 1–29; "Functions of limited variation in an infinite number of dimensions" by P. J. Daniell, 30–38; "A new sequence of integral tests for the convergence and divergence of infinite series" by R. W. Brink, 39–60; "Calculation of the complex zeros of the function $P(z)$ complementary to the incomplete gamma function" by P. Franklin, 61–63; "Total differentiability" by E. J. Townsend, 64–72.

ATHENAEUM, 1919, November 14: "Einstein's theory of gravitation" by X., 1189 [Last paragraphs: "Einstein supposes that space is euclidean where it is sufficiently remote from matter, but that the presence of matter causes it to become slightly non-euclidean—the more matter there is in the neighbourhood, the more space will depart from Euclid. By the help of this hypothesis, together with his previous theory of relativity, he deduces gravitation—very approximately, but not exactly, according to the Newtonian law of the inverse square.

"The minute differences between the effects deduced from his theory and those deduced from Newton are measurable in certain cases. There are, so far, three crucial tests of the relative accuracy of the new theory and the old.

"(1) The perihelion of Mercury shows a discrepancy which has long puzzled astronomers. This discrepancy is fully accounted for by Einstein. At the time when he published his theory, this was its only experimental verification.

"(2) Modern physicists were willing to suppose that light might be subject to gravitation, *i.e.*, that a ray of light passing near a great mass like the sun might be deflected to the extent to which a particle moving with the same velocity would be deflected according to the orthodox theory of gravitation. But Einstein's theory required that the light should be deflected just twice as much as this. The matter could only be tested during an eclipse among a number of bright stars. Fortunately a peculiarly favourable eclipse occurred this year. The results of the observations have now been published, and are found to verify Einstein's prediction. The verification is not, of course, quite exact; with such delicate observations that was not to be expected. In some cases the departure is considerable. But taking the average of the best series of observations, the deflection at the sun's limb is found to be 1.98″, with a probable error of about 6 per cent., whereas the deflection calculated by Einstein's theory should be 1.75″. It will be noticed that Einstein's theory gave a deflection twice as large as that predicted by the orthodox theory, and that the observed deflection is slightly *larger* than Einstein predicted. The discrepancy is well within what might be expected in view of the minuteness of the measurements. It is therefore generally acknowledged by astronomers that the outcome is a triumph for Einstein.

"(3) In the excitement of this sensational verification, there has been a tendency to overlook the third experimental test to which Einstein's theory was to be subjected. If his theory is correct as it stands, there ought, in a gravitational field, to be a displacement of the lines of the spectrum towards the red. No such effect has been discovered. Spectroscopists maintain that, so far as can be seen at present, there is no way of accounting for this failure if Einstein's theory in its present form is assumed. They admit that some compensating cause *may* be discovered to explain the discrepancy, but they think it far more probable that Einstein's theory requires some essential modification. Meanwhile, a certain suspense of judgment is called for. The new law has been so amazingly successful in two of the three tests that there must be something valid about it, even if it is not exactly right as yet.

"Einstein's theory has the very highest degree of aesthetic merit: every lover of the beautiful must wish it to be true. It gives a vast unified survey of the operations of nature, with a technical simplicity in the critical assumptions which makes the wealth of deductions astonishing. It is a case of an advance arrived at by pure theory: the whole effect of Einstein's work is to make physics more philosophical (in a good sense), and to restore some of that intellectual unity which belonged to the great scientific systems of the seventeenth and eighteenth centuries, but which was lost through increasing specialization and the overwhelming mass of detailed knowledge. In some ways our age is not a good one to live in, but for those who are interested in physics there are great compensations."]

Journal of the Indian Mathematical Society, volume 11, no. 4, August, 1919: "Progress report" by D. D. Kapadia, 121; "Mr. S. Ramanujan, B.A., F.R.S.," 122 and portrait frontispiece ["Was born of poor parents in December, 1888, at Erode in the Madras Presidency . . . In 1910 he went to Madras with two good sized note books filled with his researches in mathematics and with the help of some friends obtained a clerk's post in the Harbour Trust Office. This enabled him to remain in Madras and work at mathematics with the help of books and periodicals which were made accessible to him there. In 1912, having one day seen the remark in Mr. G. H. Hardy's tract on *Orders of Infinity* that the precise order $\rho(x)$ had not yet been ascertained and having himself arrived at a result relating to it, he opened correspondence with Mr. Hardy sending him some of his results on continued fractions and theory of numbers. Mr. Hardy was struck with the grandeur of the results and wrote to him an appreciative letter asking for more of his results in other branches of mathematics. On receiving these which were mainly in definite integrals and elliptic functions, Mr. Hardy discovered in Ramanujan a great mathematician and asked him if he would go to Cambridge . . . The University of Madras granted Mr. Ramanujan a special scholarship of the annual value of £250 for three years and he sailed for England in March, 1914 . . . An account of his work while in England has already appeared in the February, 1917 number of this *Journal*. . . . In recognition of his abilities in mathematics he was elected a fellow of the Royal Society in 1917, and in 1918 he was granted a Fellowship of Trinity. The University of Madras has also been pleased to give him an annual grant of £250 for another period of five years without imposing any conditions regarding residence or work. Since May 1917 he has not been keeping good health. He returned to India in March last for the sake of his health and is now residing . . . very near the place of his birth"]; "On the cartesian oval" by V. R. Aiyar, 123–144; "Some difficulties met with in reading mathematics without a teacher: a complaint against text-books" by W. A. Garstin, 145–154; "Astronomical notes" by T. P. B. Sastri, 155; Problems and Solutions, 156–160; "Mathematics (in brief) from current periodicals," i–iii.

Mathematics Teacher, volume 12, no. 1, September, 1919: "An experiment in motivation" by W. S. Schlauch, 1–9; "Scales for the study of children's characteristics" by E. S. Smith, 10–16; "Applied mathematics in high schools: some lessons from the war" by W. E. Breckenridge, 17–22; "A junior high school course in mathematics" by Emily Renshaw, 23–27; "A simple method of reconstructing a hyperbolic paraboloid" by E. J. Cuy (Couyundjopoulos), 28–29; "The national committee on mathematical requirements," 30–32; Editorials, Book Reviews, Notes and News, 33–40.

Messenger of Mathematics, volume 48, no. 9, January, 1919: "Note on the deflection of beams" by W. H. Macaulay, 129–130; "Laws of facility of error" by A. R. Forsyth, 131–144—No. 10, February: "On Napier's circular parts" by W. W. Johnson, 145–153 [Historical]; "Theorems in the expansion of polynomials, obtained by an application of the calculus of residues" by E. A. Milne, 153–159; "The dissection of rectilineal figures" by W. H. Macaulay, 159–160.

Nouvelles Annales de Mathématiques, volume 78, June, 1919: "Réduction à une forme normale d'un système d'équations différentielles simultanées linéaires à coefficients constants" by H. Vogt, 201–209; "Le théorème de Feuerbach dans les cubiques" by Malgouzon, 210–213; "Démonstration du théorème de Chasles sur les arcs égaux de lemniscate" by F. Balitrand, 213–215; "Sur les conditions pour qu'une fonction $P(x, y) + iQ(x, y)$ soit monogène" by M. Fréchet, 215–219; "Groupes de points sur l'hyperbole équilatère" by J. Ser, 220–228; Questions and solutions, 230–240.

Pennsylvania School Journal, volume 67, April, 1919: "The Courtis tests in arithmetic" by W. A. Boyer, 480–481.

Proceedings of the American Philosophical Society, volume 58, 1919: "Graphical representation of functions of the Nth degree" by F. E. Nipher, 236–240.

Revue de Métaphysique et de Morale, volume 26, no. 5, September–October, 1919: Etudes critiques—"Les Principes de l'analyse mathématique par Pierre Boutroux" by M. Winter, 649–667 [Last sentences: "Le livre que nous venons d'analyser donnera au jeune géomètre une vue d'ensemble de la science à laquelle il travaille. L'ouvrage de M. Boutroux intéressera également le philosophe; les grands courants de la pensée qui se dissimulent sous les théories mathématiques y sont discrètement décrits: l'auteur nous signale les heurts, les chocs qui en resultent. Il n'a pas cru devoir rester 'au-dessus de la mêlée'. Il a pris parti, il a défendu les idées qui nous semblent les plus conformes au développement de la science."]

Revue de Philosophie, année 19, no. 4: July–August, 1919: "Mathématique et métaphysique" by P. M. Périer, 384–395 (to be continued); "Pierre Duhem (1861–1916)", 457–462 [Quotations: "M. Duhem Pierre-Maurice-Marie naquit à Paris le 10 juin 1861. Elève du collège Stanislas, il se voue à l'étude des sciences et se prépare à l'Ecole Normale Supérieure. Il y est admis à l'âge de vingt et un ans. Son intelligence lucide, son travail obstiné, le font bientôt classer parmi les meilleurs sujets. Après ses trois années d'études, le nouvel agrégé, sorti premier du concours, aurait pu avoir hâte de jouir de la situation qu'il s'etait créée. Les deux années suivantes, il demanda et obtint la faveur de rester à l'Ecole en vue de perfectionner sa formation scientifique. . . . Peu d'hommes cependant ont été de tous points plus meritants que M. Duhem. Magnifiquement doué du point de vue intellectuel, à la fois *historien, philosophe* et surtout *savant,* il sut allier à la science abstraite des dispositions artistiques, qui ne laissaient pas de surprendre dans l'homme du chiffre et de l'abstraction. Partout où il s'offrait à lui, dans l'art, en littérature, dans l'architecture ou dans la nature, le beau ne le trouva jamais indifférent. Sa faculté esthétique n'explique-t-elle pas le charme singulier et la coloration de sa parole et surtout de sa claire composition?"].

Revue Philosophique de la France et de l'Etranger, année 44, September–October, 1919: "L'imagination pure et la pensée scientifique" by J. Segond, 297–321.

Revue scientifique, année 57, no. 17, 6–13 September, 1919: "Le temps et sa mesure" (suite) by M. Moulin, 517–527.

School Review, volume 27, no. 8, October, 1919: "Geometry by analysis" by H. O. Barnes, 612–618.

School and Society, volume 10, August 23, 1919: "What and how far have military courses and training contributed to the college curricula?" by P. P. Boyd, 219–224—October 11: "The National Committee on Mathematical Requirements," 435–437.

School Science and Mathematics, volume 19, no. 8, November, 1919: "A practical method for demonstrating the error of mean square" by H. F. Roberts, 677–692 [bibliography on page 692]; "How x came to stand for unknown quantity" by F. Cajori, 698–699 [Quotation: "As a matter of fact, there is no evidence, worthy of serious consideration, to show that x was used as the symbol for unknown quantity before the publication of Descartes' *Géométrie,* in 1637"]; "A 'flu' dream in mathematics" by W. A. Austin, 701–713; "Horner's method" by Elizabeth Sanford, 726 [First of five verses:

"Have you ever asked why asylums were filled
With people whose sweet dispositions were killed?
It's because of one man, with a Mother Goose name,
Who invented a system to drive folks insane.
That's Horner."];

"Force, work and power—their relation. Note on relation of work to heat added" by S. A. Garlick, 727–731; "Nineteenth meeting of the Central Association of science and mathematics teachers," 751; Problems and solutions, 755–758; "New outline map of the United States on the Lambert projection," 760; "The National Committee on Mathematical Requirements," 763–764; Review by G. A. Miller of Cajori's *History of Mathematics* (new edition, New York, 1919), 768, 770.

Science, new series, volume 50, November 7, 1919: Review by H. S. White of T. Smith and R. W. Cheshire's *Constructional Data for Small Telescope Objectives* and *Additional Data, etc.* (London, 1915–1916), 437–439—November 21: "The history of science and the American Historical Association" by E. J. Benton, 478; "The deflection of light by gravitation and the theory of relativity," 478–479—November 28: "The historical point of view in the teaching of science" by G. A. Miller, 489–493 [Address before the Missouri State Teachers Association. First two paragraphs: "The teachers of Missouri should take special interest in the history of science

at the present time in view of the fact that the American Association for the Advancement of Science is expected to meet soon in this state and the question of forming a special section of this association for the purpose of considering topics in the history of science is to be raised during this meeting. Teachers of mathematics have an additional reason for taking an unusually keen interest in this subject just now in view of the appearance during the past summer of two very important books on the history of their subject.

"One of these is entitled 'History of the Theory of Numbers' and was prepared by Professor L. E. Dickson of the University of Chicago, while the other bears the more general title 'A History of Mathematics' and was prepared by Professor Florian Cajori, of the University of California, who holds the unique position of a regular professorship of the history of mathematics in a university. The former book is the first volume of the most complete history of number theory ever written and marks an epoch in American mathematical literature, while the latter is technically only a 'revised and enlarged edition' of a book which appeared a quarter of a century ago under the same title, but the changes are so extensive that it too may be regarded as practically a new work."]

SCIENTIFIC MONTHLY, volume 9, no. 5, November, 1919: "The controversy on the origin of our numerals" by F. Cajori, 458–464. [The article closes with the following summary: "The following are the outstanding facts:

1. The earliest reliable record of the use of our numerals with the zero is an inscription of 867 A.D. *in India.*
2. The validity of the testimony of early Arabic writers ascribing to India the numerals with the zero is shaken, but not destroyed.
3. There is not a scintilla of evidence in the form of old manuscripts or numeral inscriptions to support the Greek origin of our numerals.
4. At present the hypothesis of the Hindu origin of our numerals stands without any serious rival. But this hypothesis is by no means firmly established.

As a by-product of the discussion of recent years we must admit that, on the evidence presented, the claim that our numerals and the zero were used in India as early as the fifth century must be abandoned; our earliest apparently reliable evidence belongs to the ninth century. We must also abandon the claim that the early Hindus used the abacus, the rule of 'double false position,' and the process of 'casting out the nines.' These corrections are due to G. R. Kaye."]

UNIVERSITY OF TEXAS BULLETIN, no. 1904, January, 1919: *A comparison of the premiums of the Teachers Insurance and Annuity Association with those of other legal reserve companies*, by E. L. Dodd, 19 pp.

UNDERGRADUATE MATHEMATICS CLUBS.

EDITED BY U. G. MITCHELL, University of Kansas, Lawrence.

THE MATHEMATICS CLUB OF THE UNIVERSITY OF KANSAS, Lawrence, Kansas. [*1918*, 35, 450, 459; *1919*, 208.]

The officers of the club for the year 1919–20 are as follows: President, Jessie Craig '20; vice-president, Beatrice Hagen '20; secretary-treasurer, Evelina Watt '20; reporter, Etna Morrison '20; faculty adviser, Professor Ellis B. Stouffer; program committee, the faculty adviser, the secretary, the reporter and Ruth Kelsey '20.

Below are given the programs for the year 1919–20, as announced in the printed folder issued by the club.

October 9, 1919: "Codes and Ciphers" by Professor Ulysses G. Mitchell.
October 23: "The game of nim" by Etna Morrison '20.

November 13: "De Moivre's theorem" by Lucile Noah '20.
December 11: "Hyperbolic functions" by Marie McKinney '21.
January 8, 1920: "Sailing quicker than the wind" by Marie Shaklee '21; "Famous problems of antiquity" by Nadene Weibel '21.
January 22: "Mathematics of the calendar" by Lillie Strand '21.
February 12: "Life and works of Newton" by Hilda Bushnell '21; "The Leibniz-Newton controversy" by Wayne Stevenson '20.
February 26: "Mathematical machines" by Victor Mellenbruch Gr., and Eran O. Burgert Gr.
March 11: "Mathematics in France" by Professor Solomon L. Lefschetz; "Mathematicians of America" by Professor Cyril A. Nelson.
March 25: "Flatland" by Nina McLatchey Gr. Play "Flatlanders" by members of the club.
April 8: "Origin of number symbols" by Marie Power '20.
April 22: "Geometric fallacies" by Marie Brown '21.
May 13: "Magic squares" by Pearl L. Mellenbruch Gr.
May 27: Annual picnic.

THE MATHEMATICAL CLUB OF ROCKFORD COLLEGE, Rockford, Illinois. [*1918*, 188, 409.]

The officers of the club for the year 1918–19 were: President, Ruth Gleasman '19; vice-president, Dorothy Jamison '20; secretary-treasurer, Lucy E. Brown '20.

The following programs were given during the year.
September, 1918: Initiation of new members.
November: "The Ueberfeld Horses," from Maeterlinck's *The Unknown Guest*, by Lea Gordon '20.
December: "Problems in paper cutting" by Aline Bartholomew '20.
January, 1919: "Comet's tails" by Professor Bessie I. Miller.
February: "Magic squares and mathematical tricks" by Dorothy Jamison '20.
March: Geometrical recreations—"Geometrical fallacies" by Anna Foster '21; "Geometrical paradoxes" by Bohnmilla Hrdlicka '21; "Geodesics" by Elizabeth Rearick '20.
April: "Short cuts in calculation" by Margaret Dodd '21.
May: "The theory of probability" by Ruth Gleasman '19.

Membership in the club is limited to students who have taken or are taking elective work in mathematics. At present there are seventeen members. The officers for the year 1919–20 are: President, Dorothy Jamison '20; vice-president, Virginia Schneider '20; secretary-treasurer, Frances Regan '21.

THE UNIVERSITY OF SASKATCHEWAN MATHEMATICAL SOCIETY, Saskatoon, Saskatchewan. [*1918*, 270, 460.]

On account of the interruption of the University work by the influenza epidemic no meeting of the society was held during the year 1918–19 before

Christmas. The first meeting of the new year was held in February, 1919. Gladys Shannon '20 presented a paper on "The concept of infinity in geometry." The paper was followed by considerable discussion and it was decided that the subject should be given further consideration at the second meeting.

The second meeting was held at the residence of Professor Lloyd L. Dines and about twenty members were present. The discussion of the subject of the first meeting was continued by John H. Simester '20, Lillian Williamson '21, Helen Fernald '21 and Gladys Shannon '20, as well as by nearly all of the other members present.

The third and last meeting for the year 1918–19 was held at the residence of Dean George H. Ling and about fifteen members were present. John H. Simester '20 presented a paper on "The logic of algebra" and after a discussion of it several interesting problems were presented by members of the club. A resolution was introduced and passed unanimously that hereafter the society should be called "The Shuttleworth Mathematical Society of the University of Saskatchewan" in honor of the late Roy Shuttleworth, who was the first president of the club and who fell in France in August 1918 after having won the Military Medal for distinguished services.

The following officers were elected for the year 1919–20: President, John H. Simester '20; vice-president, Gladys Shannon '20; secretary-treasurer, Andrew M. Ridout '21.

THE MATHEMATICS CLUB OF VASSAR COLLEGE, Poughkeepsie, N. Y. [*1918*, 136, 456; *1919*, 264.]

The officers for the first semester of the year 1918–19 were: President, Helen Thompson '19; vice-president, Susan Burr '20; secretary-treasurer, Helen Lewis '20; faculty member of executive committee, Professor Elizabeth B. Cowley; student member of executive committee, Mildred Booth '20.

Below are given the programs for the year 1918–19.

October 4, 1918: Business meeting for election of officers for first semester. After the election the members of the club "took a hike."

November 6: "Spherical aberration" by Professor Henry S. White; "Some mathematical fallacies" by Kathleen Millay '21. The members of the club had an animated discussion over the fallacies. Some problems were proposed for solution before the next meeting.

December 3: "Air resistance against rotating projectiles" by Eliza Sommerville '20; "Underwater trajectories and phenomena of flight" by Mildred Werntz '20. It was announced that Amy Davison '21 had solved the problems proposed at the last meeting. It was also announced that a list of examples and questions that were intended to test personal efficiency had been placed on the bulletin board. They had been clipped from a newspaper.

January 16, 1919: A play "Flatlanders"[1] written by Kathleen Millay '21 and Lucille Free '21 was presented by members of the club.

[1] Published in full in the June, 1919, number of the MONTHLY, pp. 264–267.

February 27: Election of officers for second semester: President, Helen Thompson '19; vice-president, Susan Burr '20; secretary-treasurer, Helen Lewis '20; faculty member of executive committee, Professor Cowley; student member of executive committee, Mary Coye '20.

Subject of program "The fourth dimension." Mary Coye '20 explained it by the use of an analogy; Katherine Jaeger '19 gave historical data; Amy Davison '21 discussed some of the theoretical aspects.

It was recommended that in the future some of the meetings be held in other places than in Rockefeller Hall (recitation hall).

April 10: "A brief account of the life of Lewis Carroll" by Pauline Chrisler '19; "Lewis Carroll as a mathematician" by Christie White '19; "An adaptation of Stephen Leacock's essay on 'A, B, and C'" (reading) by Jean Proutt '19.

The meeting was held in senior parlor. Mrs. H. S. White and some freshmen were invited as guests of the club. After the formal program and business session were over tea was served.

May 20: The club had a picnic near the "old lake." A short business meeting was held and then mathematical games were played until supper time.

The officers elected for the first semester of the year 1919–20 are: President, Mildred Booth '20; vice-president, Amy Davison '21; secretary-treasurer, Susan Burr '20; faculty member of the executive committee, Professor Cowley; student member of the executive committee, Mary Coye '20.

SUMMARY NOTES—1919.

The establishment of the Students Army Training Camps at various universities and the outbreak of the epidemic of influenza in the fall of 1918 interfered greatly with the work of undergraduate mathematics clubs. Of thirty-three clubs which reported during 1919 on the work of the school year 1918–19, nineteen—University of Alabama, University of Chicago, University of Colorado, Greenville College, Grinnell College, University of North Carolina, University of Oregon, Swarthmore College, Albion College, Columbia University, Denison University, Harvard University, University of Illinois, University of Kansas, University of Kentucky, University of Maine and University of Nebraska—do not mention any meetings during the period from September first 1918 to January first 1919 and the first eight of these clubs are reported as remaining inactive during the entire school year. Nearly all of these inactive clubs have reported that they expect their work during the year 1919–20 to be back to normal activity again.

One new club, that of the University of Iowa, was organized in April 1919.

During 1919 the MONTHLY has published reports of the activities of twenty different clubs. These reports included accounts of more than 100 meetings and a list of about 160 topics discussed. Discussions of two club topics, "The Number π" and "Codes and Ciphers," have been published and some suggestions given as to forms of entertainment for social meetings.

2810. Proposed by H. S. UHLER, Yale University.

In the expansion of the following determinant or eliminant, find the total number of terms, and the number of terms having the coefficients $+1$, -1, $+2$, -2, $+3$, -3, $+4$, -4, $+5$, -5, $+6$, -8, $+10$, respectively.

$$\begin{vmatrix} a & b & c & d & e & 0 & 0 & 0 \\ 0 & a & b & c & d & e & 0 & 0 \\ 0 & 0 & a & b & c & d & e & 0 \\ 0 & 0 & 0 & a & b & c & d & e \\ A & B & C & D & E & 0 & 0 & 0 \\ 0 & A & B & C & D & E & 0 & 0 \\ 0 & 0 & A & B & C & D & E & 0 \\ 0 & 0 & 0 & A & B & C & D & E \end{vmatrix}$$

2811. Proposed by J. L. RILEY, Stephenville, Texas.

Given the cube roots of 60, 61, 63, and 64, to find the cube root of 62 by the method of differences.

2812. Proposed by C. N. SCHMALL, New York City.

If $F(x, y, z)$ be a homogeneous function of x, y, z, which becomes $\phi(u, v, w)$ by the elimination of x, y, z, by means of the equations $\partial F/\partial x = u$, $\partial F/\partial y = v$, $\partial F/\partial z = w$; show that

$$\frac{\partial F}{\partial u}\bigg/x = \frac{\partial F}{\partial v}\bigg/y = \frac{\partial F}{\partial w}\bigg/z.$$

2813. Proposed by PAUL CAPRON, U. S. Naval Academy.

An ellipse having the major-axis $2a$ and the eccentricity ϵ, is revolved first about its major axis, forming a prolate spheroid, then about its minor axis forming an oblate spheroid. Show that the surfaces of these spheroids are, respectively,

$$2\pi a^2(1/\epsilon\sqrt{1-\epsilon^2}\sin^{-1}\epsilon + 1)$$

and

$$2\pi a^2\left[2 + 1/\epsilon(1-\epsilon^2)\log\left(\frac{1+\epsilon}{1-\epsilon}\right)\right].$$

SOLUTIONS OF PROBLEMS.

339 (Calculus) [June, 1913; May, 1919]. Proposed by T. H. GRONWALL, Washington, D. C.

To show that for any real value of x

$$\left|\frac{d^n}{dx^n}\left(\frac{\sin x}{x}\right)\right| \leqq \frac{1}{n+1}, \quad \text{and} \quad \left|\frac{d^n}{dx^n}\left(\frac{1-\cos x}{x}\right)\right| \leqq \frac{1}{n+1}.$$

I. Solution by Otto Dunkel, Washington University.

The function $y = \sin x/x$, $x \neq 0$; $y = 1$, $x = 0$, is single-valued and continuous for all values of x and it can be expressed as a power series

$$y = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots.$$

The power series shows that the function possesses all of its derivatives at $x = 0$, and it is easily seen from the power series or from the original form of the function that the derivatives exist for all other values of x. Hence, we may write the sequence of equations for finding the derivatives

$$xy = \sin x, \qquad y + xy' = \sin\left(\frac{\pi}{2} + x\right),$$

$$2y' + xy'' = \sin(\pi + x), \qquad \cdots, \qquad (n+1)y^{(n)} + xy^{(n+1)} = \sin\left(\frac{n+1}{2}\pi + x\right), \qquad \cdots$$

These equations, taken in turn, show that y, y', y'', $\cdots$, approach zero as x becomes infinite Thus $y^{(n)}$, for example, attains its maximum or minimum value for some finite value of x. Fo such a value of x we have $y^{(n+1)} = 0$ and

$$(n + 1)y^{(n)} = \sin\left(\frac{n+1}{2}\pi + x\right)$$

and hence

$$| y^{(n)} | \leqq \frac{1}{n+1}.$$

The proof of the second inequality may be carried through in the same manner.

II. Solution by H. S. Uhler, Yale University.

Let $r \equiv (\sin x)/x$ and $s \equiv \sin x$, then

$$r = \frac{1}{x}\cdot s, \tag{0}$$

$$\frac{dr}{dx} + \frac{1}{x}\cdot r = \frac{1}{x}\cdot\frac{ds}{dx}, \tag{1}$$

$$\frac{d^2r}{dx^2} + \frac{2}{x}\cdot\frac{dr}{dx} = \frac{1}{x}\cdot\frac{d^2s}{dx^2}, \tag{2}$$

$$\cdots\cdots\cdots$$

$$\frac{d^{n-1}r}{dx^{n-1}} + \frac{n-1}{x}\cdot\frac{d^{n-2}r}{dx^{n-2}} = \frac{1}{x}\cdot\frac{d^{n-1}s}{dx^{n-1}}, \tag{$n-1$}$$

$$\frac{d^nr}{dx^n} + \frac{n}{x}\cdot\frac{d^{n-1}r}{dx^{n-1}} = \frac{1}{x}\cdot\frac{d^ns}{dx^n}. \tag{n}$$

In order to find an explicit formula for the general derivative of r with respect to x in terms of n, x, $\sin x$, and $\cos x$ we may proceed as follows:

Case 1. *n even.* Multiply equations (n), $(n-1)$, $(n-2)$, $(n-3)$, $\cdots$, (2), (1), (0) by $+1$ $-\frac{n}{x}$, $+\frac{n}{x}\cdot\frac{n-1}{x}$, $-\frac{n}{x}\cdot\frac{n-1}{x}\cdot\frac{n-2}{x}$, $\cdots$, $+\frac{n}{x}\cdot\frac{n-1}{x}\cdots\frac{4}{x}\cdot\frac{3}{x}$, $-\frac{n}{x}\cdot\frac{n-1}{x}\cdots\frac{3}{x}\cdot\frac{2}{x}$, $+\frac{n}{x}\cdot\frac{n-1}{x}\cdots\frac{2}{x}\cdot\frac{1}{x}$, respectively, and add the results. We then find

$$\frac{d^nr}{dx^n} = \pm\frac{1}{x}\cdot F_1(x)\cdot\sin x \pm \frac{n}{x^2}\cdot F_2(x)\cdot\cos x, \tag{A}$$

where

$$F_1(x) \equiv 1 - \frac{n(n-1)}{x^2} + \frac{n(n-1)(n-2)(n-3)}{x^4} - \cdots \mp \frac{n(n-1)\cdots 4\cdot 3}{x^{n-2}} \pm \frac{n(n-1)\cdots 2\cdot 1}{x^n},$$

$$F_2(x) \equiv 1 - \frac{(n-1)(n-2)}{x^2} + \frac{(n-1)(n-2)(n-3)(n-4)}{x^4} - \cdots \pm \frac{(n-1)(n-2)\cdots 5\cdot 4}{x^{n-4}} \mp \frac{(n-1)(n-2)\cdots 3\cdot 2}{x^{n-2}}.$$

The upper or lower signs are to be used throughout according as n is of the form $4m$ or $4m - 2$, respectively; $m = 1, 2, 3, \cdots$. When $n = 4m$, the number of terms in $F_1(x)$ and $F_2(x)$ is $2m + 1$ and $2m$, whereas when $n = 4m - 2$ the number of terms in $F_1(x)$ and $F_2(x)$ is $2m$ and $2m - 1$, respectively.

Case 2. *n odd.* Multiply equations (n), $(n-1)$, $\cdots$, (1), (0) by the same factors as were used above, observing that the signs of the last three factors will now be $-$, $+$, $-$; and add.

$$\frac{d^nr}{dx^n} = \mp\frac{1}{x}\cdot F_3(x)\cdot\cos x \pm \frac{n}{x^2}\cdot F_4(x)\cdot\sin x \tag{B}$$

where

$$F_3(x) \equiv 1 - \frac{n(n-1)}{x^2} + \frac{n(n-1)(n-2)(n-3)}{x^4} - \cdots \pm \frac{n(n-1)\cdots 5\cdot 4}{x^{n-3}} \mp \frac{n(n-1)\cdots 3\cdot 2}{x^{n-1}},$$

$$F_4(x) \equiv 1 - \frac{(n-1)(n-2)}{x^2} + \frac{(n-1)(n-2)(n-3)(n-4)}{x^4} - \cdots \pm \frac{(n-1)(n-2)\cdots 4\cdot 3}{x^{n-3}} \mp \frac{(n-1)(n-2)\cdots 2\cdot 1}{x^{n-1}}.$$

The upper or lower signs are to be used throughout according as n is of the form $4m - 1$ or $4m - 3$, respectively. When $n = 4m - 1$, or $n = 4m - 3$ the number of terms in both $F_3(x)$ and $F_4(x)$ is $2m$ or $2m - 1$, respectively. In all cases, the total number of terms in the right hand members of formulæ (A) and (B) is $n + 1$.

The following properties of these formulæ will help to throw light on the general problem. Since $F_1(x)$, $F_2(x)$, $F_3(x)$, and $F_4(x)$ involve only even powers of x it is seen, at once, from the outstanding factors $\pm \dfrac{\sin x}{x}$, $\pm \dfrac{n \cos x}{x^2}$, and $\mp \dfrac{\cos x}{x}$, $\pm \dfrac{n \sin x}{x^2}$ that $d^n r/dx^n$, when numerically equal positive and negative values are substituted for x (n being always a positive integer), does not change sign for n even, and it does change sign for n odd. Hence the graph of $y = d^n r/dx^n$ is symmetrical with respect to the y-axis when n is even. On the other hand, the ordinates are equal in magnitude but opposite in sign for $x = +c$ and $x = -c$, when n is odd. When x is neither zero nor infinite it is evident that $d^n r/dx^n$ is finite, continuous, and single-valued. By reducing all terms of the right-hand members of (A) and (B) to the common denominator x^{n+1} we see that $d^n r/dx^n$ assumes the so-called indeterminate form 0/0 when $x = 0$. Differentiating the re-written numerators of (A) and (B), and dividing each result by $d/dx(x^{n+1})$, *i.e.*, by $(n + 1)x^n$, we find $\pm \dfrac{x^n \cos x}{(n+1)x^n}$ and $\pm \dfrac{x^n \sin x}{(n+1)x^n}$, respectively. Cancelling x^n and then substituting zero for x we obtain $\pm \dfrac{1}{n+1}$ and 0 for the limits of $d^n r/dx^n$ corresponding, respectively, to even and odd values of n. Finally, since $|\sin x| \not> 1$, and $|\cos x| \not> 1$, (A) and (B) show that $d^n r/dx^n$ vanishes as x becomes infinite, n remaining finite. It has thus been demonstrated that, for all values of x, $d^n r/dx^n$ is a finite, continuous, and single-valued function of x.

The properties of $d^n r/dx^n$ just proved show that the greatest numerical values of this derivative may be obtained from its algebraic maxima and minima, if it possesses any. Accordingly the next step in the argument is to establish the existence of maxima and minima.

A necessary condition that $d^n r/dx^n$ shall have maxima and minima is $d^{n+1}r/dx^{n+1} = 0$. Hence it must be shown that the $(n + 1)$th derivative can always vanish when neither n nor x is infinite. When n is even, $n + 1$ is odd so that formula (B) applies. When n is odd, $n + 1$ is even and formula (A) applies. It is not necessary to write out separately and in detail the equations for $d^{n+1}r/dx^{n+1} = 0$ since both assume the form

$$f(x)\cdot \sin x = xF(x)\cdot \cos x \qquad (C)$$

where $f(x)$ and $F(x)$ are polynomials in x^2, each having integral coefficients and a term free from x.

We are to show that the necessary condition (C) is an equation which always has real roots in addition to $x = 0$. Now $f(x) = 0$ and $F(x) = 0$ either have one or more real roots in common, or they have no common root. If such a root be admitted then a proof of the existence of a real root of condition (C) would be superfluous. Supposing that $f(x) = 0$ and $F(x) = 0$ have no common root we may demonstrate the existence of real roots of equation (C) in the following manner.

In the first place, assume that both $f(x) = 0$ and $F(x) = 0$ have real roots. Then take a positive value of x, R say, which is greater than the numerically largest root of $f(x) = 0$ and $F(x) = 0$. Also imagine the curves $y = f(x)\cdot \sin x$ and $y = xF(x)\cdot \cos x$ plotted to the same scale on the same diagram, and consider the possibility of the two loci having real intersections. For values of x between R and $+\infty$ the curves can cross the axis of x only when $\sin x$ and $\cos x$ vanish, for we have chosen R too large to permit either $f(x)$ or $F(x)$ to become zero. The curves $(x > R)$ intersect the x-axis at *alternate* points which have the constant interval $\pi/2$. In short, for $x > R$, the loci have the general nature of $y = \sin x$ and $y = \cos x$ except in so far as each trigonometric ratio is multiplied by a function of x that is finite, continuous, and single-valued ("damped" harmonic curves). It is clear, therefore, that for $x > R$ the loci intersect each other in an infinite number of discrete real points and hence that equation (C) has an infinite number of real roots.

The same argument applies if $f(x) = 0$ and $F(x) = 0$ have no real roots. In this case, however, we do not have to bother about R, as the possibility of one locus avoiding the other by crossing and recrossing the x-axis at the vanishing points of $f(x)$ and $F(x)$ no longer obtains.

For the sake of completeness the following comments seem appropriate. The question as to whether $f(x) = 0$ and $F(x) = 0$ have one or more common roots is not germane to the proof just presented. A similar remark applies to a discussion of the nature of the roots of these equations taken separately. Again, condition (C) cannot be fulfilled by the simultaneous vanishing of the members of the pairs $f(x)$, $\cos x$ and $F(x)$, $\sin x$. For, $\cos x$ and $\sin x$ vanish when, and only $(x \neq 0)$ when x is an integral multiple of $\pm (\pi/2)$ or $\pm \pi$, and it is a well-established fact,—upon which the proofs of the transcendentality of π depend (Lindemann, Hilbert),—that π cannot be a root of any "algebraic" equation, *a fortiori* for one having integral coefficients, *e.g.*, $f(x) = 0$ and $F(x) = 0$.

Having now established the existence of real roots of (C) we must next demonstrate the sufficiency of this condition, that is, we must show that $d^{n+2}r/dx^{n+2}$ cannot vanish when $d^{n+1}r/dx^{n+1} = 0$. The $(n+2)$th equation of the set given at the very beginning is

$$\frac{d^{n+2}r}{dx^{n+2}} + \frac{n+2}{x} \cdot \frac{d^{n+1}r}{dx^{n+1}} = \frac{1}{x} \cdot \frac{d^{n+2}s}{dx^{n+2}}.$$

Therefore, when $d^{n+1}r/dx^{n+1} = 0$, we have also

$$\frac{d^{n+2}r}{dx^{n+2}} = \frac{1}{x} \cdot \frac{d^{n+2}s}{dx^{n+2}},$$

where $d^{n+2}s/dx^{n+2}$ has the values $-\sin x$, $+\cos x$, $+\sin x$, $-\cos x$ corresponding, respectively, to $n = 4m$, $4m-1$, $4m-2$, $4m-3$. Now $(\cos x)/x$ cannot here vanish for a finite value of x because the necessary condition (C) is not fulfilled (as explained above) when $\cos x = 0$. Again $(\sin x)/x$ cannot vanish for a non-infinite value of x since, as before, condition (C) prevents $\sin x = 0$ for $x \neq 0$, and the ratio $(\sin x)/x$ approaches 1 for $x \doteq 0$. Consequently it has been shown that the necessary condition for the occurrence of maxima and minima of $d^n r/dx^n$ is also sufficient.

We are now prepared to discuss the values of $d^n r/dx^n$ at its maxima and minima. The $(n+1)$th equation of the original list would be

$$\frac{d^{n+1}r}{dx^{n+1}} + \frac{n+1}{x} \cdot \frac{d^n r}{dx^n} = \frac{1}{x} \cdot \frac{d^{n+1}s}{dx^{n+1}}.$$

Hence, when $d^{n+1}r/dx^{n+1} = 0$ we have also

$$\frac{d^n r}{dx^n} = \frac{1}{n+1} \cdot \frac{d^{n+1}s}{dx^{n+1}},$$

where $d^{n+1}s/dx^{n+1}$ has the values $+\cos x$, $+\sin x$, $-\cos x$, $-\sin x$ associated, respectively, with $n = 4m$, $4m-1$, $4m-2$, $4m-3$. Therefore, $|d^n r/dx^n|$ equals $1/(n+1)\cdot|\cos x|$ or $1/(n+1)\cdot|\sin x|$ when n is even or odd, respectively. Since $|\cos x| \leqq 1$ and $|\sin x| \leqq 1$ it follows that $|d^n r/dx^n|$ cannot exceed $1/(n+1)$. As we have seen that (C) is satisfied by $x = 0$ but not by any integral multiple of either $\pm(\pi/2)$ or $\pm \pi$, the final results of the analysis may be written:

$$\left|\frac{d^n r}{dx^n}\right|_{x=0} = \frac{1}{n+1}, \text{ for } n \text{ even,}$$

$$\left|\frac{d^n r}{dx^n}\right| < \frac{1}{n+1}, \text{ for } n \text{ odd, or for } n \text{ even and } x \neq 0.$$

REMARK. In the special case $n = 1$, I have followed the numerical values of

$$\frac{dr}{dx} = \frac{x \cos x - \sin x}{x^2}$$

from $x = 0$ through its greatest arithmetic maximum, which occurs when $x = \pm 2.081576\cdots$ (or $119° \; 15' \; 55.87'' \cdots$). The corresponding value of $|dr/dx|$ is $0.436182\cdots$, which is about 12.8 per cent. less than $1/(n+1) = 1/2$.

Attention must next be turned to

$$\frac{1-\cos x}{x} \equiv u.$$

Writing c in place of $1 - \cos x$ the general equation is found at once to be

$$\frac{d^n u}{dx^n} + \frac{n}{x} \cdot \frac{d^{n-1}u}{dx^{n-1}} = \frac{1}{x} \cdot \frac{d^n c}{dx^n}.$$

Using precisely the same multipliers and method as before, we obtain

$$\frac{d^n u}{dx^n} = \mp \frac{1}{x} \cdot F_1(x) \cdot \cos x \pm \frac{n}{x^2} \cdot F_2(x) \cdot \sin x + \frac{\lfloor n}{x^{n+1}}, \tag{A'}$$

$$\frac{d^n u}{dx^n} = \mp \frac{1}{x} \cdot F_3(x) \cdot \sin x \mp \frac{n}{x^2} \cdot F_4(x) \cdot \cos x - \frac{\lfloor n}{x^{n+1}}. \tag{B'}$$

When n is even, (A'), and x is given pairs of values $+ a$ and $- a$, $d^n u/dx^n$ assumes values that are equal in magnitude but opposite in sign. On the other hand, when n is odd this derivative alters neither in sign nor in magnitude when x is changed from $+ a$ to $- a$. Hence, when n is odd $y = d^n u/dx^n$ is symmetrical with respect to the axis of ordinates.

Since all the separate terms of formulæ (A') and (B') involve either $\sin x$, or $\cos x$, or $\lfloor n$ in the numerator and some positive integral power of x in the denominator we see that $d^n u/dx^n$ is finite for all values of x between 0 and ∞. When x becomes infinite $d^n u/dx^n = 0$. When $x = 0$ the indeterminate form 0/0, when treated in the usual way, gives $\pm \dfrac{x^n \sin x}{(n+1)x^n}$ for n even, and $\mp \dfrac{x^n \cos x}{(n+1)x^n}$ for n odd. Therefore, for $x = 0$, the nth derivative of u is equal to 0 or $\mp \dfrac{1}{n+1}$ according as n is even or odd, respectively.

For sake of brevity and variety, the question of the existence of maxima and minima may be settled by making use of the elementary properties of plane curves. The fraction c/x is finite, continuous, single-valued, with an infinite number of maxima and minima. Whenever u has a maximum or a minimum du/dx vanishes. In other words, du/dx changes sign for every value of x that corresponds to a maximum or a minimum of u. Also the general formulæ (A') and (B') show that any derivative is finite, continuous, and single-valued. Therefore, du/dx must likewise have an infinite number of maxima and minima. We then apply the foregoing argument to du/dx and d^2u/dx^2, and so on indefinitely. Obviously the same line of reasoning could have been applied to r and all of its x-derivatives. The earlier proof was presented *in extenso* because of its rigor and probable instructive value.

Proceeding as in the case of r, we find that, when $d^{n+1}u/dx^{n+1} = 0$,

$$\frac{d^n u}{dx^n} = \frac{1}{n+1} \cdot \frac{d^{n+1}c}{dx^{n+1}}.$$

Accordingly, at a maximum or minimum,

$$\left|\frac{d^n u}{dx^n}\right| \text{ equals } \frac{1}{n+1} \cdot |\sin x| \qquad \text{or} \qquad \frac{1}{n+1} \cdot |\cos x|$$

according as n is even or odd. Finally we conclude that

$$\left|\frac{d^n u}{dx^n}\right|_{x=0} = \frac{1}{n+1}, \text{ for } n \text{ odd},$$

$$\left|\frac{d^n u}{dx^n}\right| < \frac{1}{n+1}, \text{ for } n \text{ even, or for } n \text{ odd and } x \neq 0.$$

2732 [1918, 444]. Proposed by PAUL CAPRON, U. S. Naval Academy.

A conical cup, filled with fluid, stands with the vertex upward on a smooth horizontal surface. The inner and outer surfaces of the cup are similar cones of revolution, having altitudes h and $h(1 + x)$; the ratio of the specific weights of the material of the cone and the fluid is σ; the height of a barometer column of the fluid is h_0. Show that for equilibrium

$$\frac{h_0}{h}(1+x)^2 + \sigma x(1 + x + x^2/3) \nless 2/3.$$

Solution by the Proposer.

Let w be the specific weight of the fluid; then the atmospheric pressure is wh_0. Let a be the inner radius of the cone and P = the vertical component of the pressure between the fluid and the cone.

The weight of the fluid is $\frac{1}{3}\pi a^2 hw$, and

The weight of the cone is $\frac{1}{3}\pi a^2 h\sigma w[(1+x)^3 - 1]$.

The pressure between the horizontal surface and the cone is $\pi a^2 hw$.

The atmospheric pressure upon the cone has the vertical component $\pi a^2(1+x)^2 wh_0$.

From the equilibrium of the fluid,

$$P + \tfrac{1}{3}\pi a^2 hw = \pi a^2 hw.$$

From the equilibrium of the cone,

$$\tfrac{1}{3}\pi a^2 h\sigma w[(1+x)^3 - 1] + \pi a^2(1+x)^2 wh_0 \not< P = \tfrac{2}{3}\pi a^2 hw.$$

Hence, for equilibrium,

$$\frac{h_0}{h}(1+x)^2 + \sigma x(1 + x + x^2/3) \not< 2/3.$$

2733 [1918, 444]. Proposed by J. L. RILEY, Stephenville, Texas.

An ellipse of constant eccentricity passes through the focus of a parabola and has its foci on the curve. Find the envelopes of its axes.

Solution by William Hoover, Columbus, Ohio.

Let F be the focus, $(a, 0)$ of the parabola,

$$y^2 = 4ax; \tag{1}$$

F_1, F_2, the foci, (x_1, y_1), (x_2, y_2) of the ellipse in any one of its positions; $2A$, e, the major axis and constant eccentricity.

We have

$$FF_1 + FF_2 = 2A, \tag{2}$$

$FF_1 = x_1 + a$, and $FF_2 = x_2 + a$; and hence

$$2a + x_1 + x_2 = 2A. \tag{3}$$

Let the equation of the major axis of the ellipse be

$$y = mx + b. \tag{4}$$

Eliminate y from (4) and (1), and we have the quadratic giving the abscissas x_1, x_2 of F_1, F_2,

$$m^2x^2 - (4a - 2bm)x + b^2 = 0 \tag{5}$$

and so

$$x_1 + x_2 = (4a - 2bm)/m^2. \tag{6}$$

Similarly eliminating x,

$$my^2 - 4ay + 4ab = 0, \tag{7}$$

and then

$$y_1 + y_2 = 4a/m, \tag{8}$$

and (3) becomes

$$(2am^2 - 2bm + 4a)/m^2 = 2A. \tag{9}$$

Again,

$$(F_1F_2)^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 = 4A^2e^2. \tag{10}$$

From (5),

$$x_1x_2 = b^2/m^2, \tag{11}$$

and from (7),

$$y_1y_2 = 4ab/m. \tag{12}$$

From (6) and (11),

$$(x_1 - x_2)^2 = 16a(a - bm)/m^4, \tag{13}$$

and from (8) and (12).

$$(y_1 - y_2)^2 = 16a(a - bm)/m^2. \tag{14}$$

Substituting (13) and (14) in (10),

$$16a(1 + m^2)(a - bm)/m^4 = 4A^2e^2. \tag{15}$$

From (15) and (9) we obtain:

$$\frac{4a(1 + m^2)(a - bm)}{\{a(1 + m^2) + a - bm\}^2} = e^2. \tag{16}$$

An accommodating feature of this problem is the analytical form of the left member of (16). Let

$$a - bm = a(1 + m^2) \tan^2 \epsilon; \tag{17}$$

then (16) becomes

$$\frac{2 \tan \epsilon}{1 + \tan^2 \epsilon} = 2 \sin \epsilon \cos \epsilon = \sin 2\epsilon = e,$$

consistent with e not > 1 for the ellipse. Substituting b from (17) in (4), the equation of the major axis is the quadratic in m,

$$(x - a \tan^2 \epsilon)m^2 - ym + a(1 - \tan^2 \epsilon) = 0. \tag{18}$$

The condition for equal values of m in (18) is

$$y^2 = 4a(1 - \tan^2 \epsilon)(x - a \tan^2 \epsilon),$$

the envelope of the major axis.

The equation of the straight line through the center of the ellipse and perpendicular to (4) is

$$y - \frac{2a}{m} = -\frac{1}{m}\left(x - \frac{2a - bm}{m^2}\right);$$

or, substituting b from (17) and arranging,

$$ym^3 + (x - 2a - a \tan^2 \epsilon)m^2 - a \sec^2 \epsilon = 0. \tag{19}$$

Differentiating with respect to the variable parameter m, we find $m = -2(x - 2a - a \tan^2 \epsilon)/3y$ and (19) becomes

$$y^2 = \frac{4}{27a \sec^2 \epsilon}(x - a - a \sec^2 \epsilon)^3,$$

the envelope of the minor axis of the ellipse.

Also solved by the PROPOSER.

NOTES AND NEWS.

EDITED BY E. J. MOULTON, Northwestern University, Evanston, Ill.

Assistant Professor A. F. CARPENTER, of the University of Seattle, has been promoted to an associate professorship of mathematics.

At the University of California, Associate Professor C. A. NOBLE has been promoted to a full professorship of mathematics, and Assistant Professor B. M. WOODS to a full professorship of aërodynamics.

Mr. E. D. MEACHAM, of the University of Oklahoma, has been promoted to an assistant professorship of mathematics.

Assistant Professor H. H. CONWELL, of the University of Idaho, has been promoted to an associate professorship of mathematics.

Dr. R. A. ARMS, of Juniata College, has been appointed instructor in mathematics at the University of Pennsylvania.

Assistant Professor K. P. WILLIAMS, of Indiana University, after serving as captain with the Rainbow Division in France, has returned and been promoted to an associate professorship.

Associate Professor R. B. MCCLENON, of Grinnell College, has been promoted to a full professorship of mathematics.

Dr. J. V. DEPORTE has been appointed assistant professor of mathematics at the New York State College for Teachers, Albany.

Assistant Professor FLORENCE P. LEWIS, of Goucher College, has been promoted to an associate professorship of mathematics.

At Vassar College, Assistant Professor LOUISE D. CUMMINGS has been promoted to an associate professorship and Dr. MARY E. WELLS to an assistant professorship of mathematics.

Dr. J. W. CAMPBELL, of Wesley College, Winnipeg, has been appointed associate in mathematics and astronomy at the State University of Iowa.

New appointments at the University of Wisconsin for the present year, not previously announced, include: Messrs. C. D. EHRMAN, I. W. TRAYLER, J. E. DAVIS, H. L. OLSON, M. INGRAHAM, F. E. RECHART, and Misses E. M. F. FELTGES and B. ALBRIGHT to instructorships; and Messrs. T. M. DAHM, D. R. LAMONT, R. R. KNOEN, and D. J. STEWART to assistantships.

At the University of Minnesota, Mr. R. M. MATHEWS and Miss OLIVE ATWOOD have been appointed instructors in mathematics.

Assistant Professor T. E. MASON, of Purdue University, has been promoted to an associate professorship.

Dr. C. M. HEBBERT has been appointed instructor in mathematics at the University of Illinois.

Assistant Professor ARTHUR RAMSEY has been promoted to a professorship of mathematics in Grove City College.

Miss OTTILIA W. DUEKER has been appointed professor of mathematics and Dean of Women at Friends University, Wichita, Kansas.

Adjunct Professor J. J. LUCK, of the University of Virginia, has been promoted to an associate professorship of mathematics.

Professor F. J. HOLDER, of the University of Pittsburgh, has been appointed professor of mathematics and dean of the school of commerce at Mercer University, Macon, Ga.

Associate Professor R. P. STEPHENS, of the University of Georgia, has been promoted to a full professorship of mathematics.

Mr. J. J. TANZOLA has been appointed instructor in mathematics at Cooper Union.

Dr. C. C. BRAMBLE, of the U. S. Naval Academy, has been promoted to an assistant professorship of mathematics.

At the Armour Institute of Technology, assistant professor W. C. KRATHWOHL has been promoted to an associate professorship and Dr. W. L. MISER, of the University of Arkansas, has been appointed assistant professor of mathematics.

Dr. C. H. FORSYTH, of Dartmouth College, has been promoted to an assistant professorship of mathematics.

At the College of the City of New York, Mr. HARRY LANGMAN has been appointed instructor in statistics and Dr. H. F. MACNEISH instructor in mathematics.

Capt. L. L. BURCHNALL, scholar of Christ Church, Oxford, has been appointed lecturer in mathematics in the University of Durham.

Sir J. J. THOMSON, master of Trinity College, Cambridge, who recently resigned the Cavendish professorship of experimental physics, has been elected to the newly established professorship of physics. This professorship is without stipend, and will terminate with the tenure of office of the first professor unless the university determines otherwise.

Dr. J. PROUDMAN has been appointed professor of applied mathematics at the University of Liverpool.

Professor PAUL APPELL, dean of the faculty of sciences, Paris, has resigned from the office that he has held for sixteen years.

Dr. H. LEBESGUE, of the University of Paris, has been promoted to a professorship of the application of analysis to geometry. Professor LEBESGUE has been elected president of the Société mathématique de France for the year 1919.

At the University of Nancy T. GOT, professor in the Lycée Marseille, and M. LEAU, professor in the Lycée St. Louis, are each giving a course in mathematics.

The following have been appointed maîtres de conférences: J. CHAPELON and J. KAMPÉ DE FÉRIET at the University of Lille, M. JANET at the University of Grenoble, and P. HUMBERT and E. TURRIÈRE at the University of Montpellier. Dr. Humbert is to serve during the absence of Dr. A. Denjoy, who is at the University of Utrecht, and Dr. Turrière replaces Professor H. VILLAT, called to the University of Strassburg.

C. F. A. CURRIER, professor of history and political science at Mass. Institute of Technology since 1891 died Sept. 6, 1919, aged 62 years. "His natural bent was mathematics," and in early years he collaborated with Wentworth in preparing the "Teachers Solutions" for the series of mathematics text-books.

Dr. JAMES MACLAY, professor of mathematics at Columbia University since 1905, died on November 28, 1919, aged fifty-five years. He studied at the University of Berlin two years, received his doctorate from Columbia in 1899, was instructor in mathematics there 1895–1901, and adjoint professor 1901–1905.

Professor W. GROSS, of the University of Vienna, died October 29, 1918, at the age of thirty-two years.

The death is reported of Dr. J. KELLER of the technical school at Zurich.

Professor EMILE DUMONT, of the Institut Michot-Montgenost at Brussels, was killed in action in the late war.

Popular Astronomy reports that Dr. VIKTOR KNORRE, who was for many years an observer at the Berlin Observatory died on August 25, 1919, at the age of seventy-eight. He came of a family long connected with astronomy. His grandfather was an observer at Dorpat and his father was director of the observatory at Nikolajew in South Russia.

The deaths of the following professors are reported in *l'Enseignement mathématique* for October: ANTOINE GOB, of Athéné royal at Liège, at the age of fifty-one; E. BÖTTCHER, of the University of Leipzig, Aug. 5, 1919, at the age of seventy-two; O. DZIOBEK, 1919, and F. GRÄFE, December 2, 1918, both of the Charlottenburg Technical School and both at the age of sixty-three; E. NETTO of the University of Griessen, on May 2, 1919, at the age of seventy-two; R. STURM of the University of Breslau, on April 12, 1919, at the age of seventy-seven; K. T. REYE and J. WELLSTEIN, both formerly at the University of Strassburg, REYE, in June, 1919, at the age of eighty-one and WELLSTEIN on June 24, 1919, in his fiftieth year.

This MONTHLY announced the death of Professor H. G. ZEUTHEN (*1919*, 323 and 404) on the authority of *L'Enseignement mathématique* for June. He did not die till January 6, 1920.

Professor G. N. WATSON, of the University of Birmingham, has been elected fellow of the Royal Society of London.

Professor E. PASCAL, of the University of Naples, and Professor E. ALMANSI, of the University of Rome, have been elected national members of the Reale Accademia dei Lincei, of Rome. Professor G. FARRO, of the University of Paris, has been elected a corresponding member.

Professor E. GOURSAT, of the University of Paris, has been elected member of the Paris Academy of Sciences, in the section of mathematics, and Professor H. ANDOYER, of the Sorbonne, in the section of astronomy. Professor Goursat succeeds Professor E. PICARD, who has been chosen perpetual secretary of the academy, Professor E. COSSERAT, of the University of Toulouse, has been elected non-resident member of the academy.

Professor H. FEHR, of the University of Geneva, has been elected foreign correspondent of the Royal Society of Sciences of Liège.

Professor H. VON MANGOLDT was elected president of the Deutsche Mathematiker-Vereinigung for 1918–19. Professor FELIX KLEIN was honorary president for the same period, in honor of the fiftieth anniversary of his doctorate, celebrated on December 12, 1918; Professor KLEIN celebrated his seventieth birthday on April 25, 1919.

Professor J. M. TAYLOR, of Colgate University, recently completed fifty years as a teacher of mathematics in that institution. It was most fitting, therefore, that he should be selected as the one to be honored by the university in connection with its centennial celebration held in October, 1919. Only two honorary degrees were conferred on this occasion, namely Doctor of Science on Professor Taylor and Doctor of Laws on Elihu Root. Professor Taylor is the author of a series of mathematical texts for schools and colleges which have been widely used. Professor H. E. SLAUGHT reports that the calculus text of the series was being tried out in manuscript form when he was a sophomore at Colgate in 1881, and he wishes to testify that much of his inspiration and enthusiasm in mathematics may be traced to that course and others under Professor Taylor.

The following note is taken from *Nature* for December 18:

The appointment of Mr. G. H. HARDY, fellow and mathematical lecturer of Trinity College, Cambridge, to the Savilian professorship of geometry at Oxford reminds us that the present year marks the tercentenary of the foundation by Sir Henry Savile of the first university chairs of geometry and astronomy in Great Britain. Gresham in 1596 had inaugurated similar professorships in London, but the Gresham College never attained the importance it might have done, and London had to wait two centuries for her university. Both the famous Elizabethans, Gresham and Savile, performed valuable services for their Queen and country, and both were favourites at Court. Savile, who was born near Halifax, Yorkshire, in 1549 was, from 1585 until his death in 1622, warden of Merton College, Oxford, of which he had been made a fellow in 1570. He founded

the Savilian professorships in 1619, and the first holders of them were Briggs and Bainbridge, the former of whom had been the first Gresham professor of geometry. Briggs, who, like Savile, was born near Halifax, is best known for his notable works on logarithms and his intimacy with Napier, and the details of his life are generally familiar. Bainbridge did not rise to the same celebrity as his colleague, which may be partly accounted for by the fact that he was trained as a physician, and while Savilian professor of astronomy he was also Linacre reader in medicine.

A supplementary list (see *1919*, 420) of Degrees of Doctor of Philosophy in mathematics conferred in the academic year 1918–1919 is as follows:

University of California: CLYDE LYNNE EARLE WOLFE, "On the indeterminate cubic equation $x^3 + Dy^3 + D^2z^3 - 3Dxyz = 1$."

University of Illinois: JESSIE MARIE JACOBS, "The trilinear binary form as a cubic surface."

University of Indiana: GERTRUDE IONE MCCAIN, "Series of iterated linear fractional functions—character of the functions"; HAROLD EICHHOLTZ WOLFE, "A study of plane circle to circle transformations by means of tetracyclic coördinates."

University of Pennsylvania: WAYNE SENSENIG, "The invariant theory of involutions on conics."

At the fourth annual meeting of the Mathematical Association of America, Professor D. E. SMITH, of Columbia University, was elected president, and Professors HELEN A. MERRILL, of Wellesley College, and E. J. WILCZYNSKI, of University of Chicago, vice-presidents. The members of the Council, elected to serve for three years were Professors R. D. CARMICHAEL, E. R. HEDRICK, H. E. SLAUGHT, and J. W. YOUNG.

At the annual meeting of the American Mathematical Society in New York, December 30–31, 96 members were present and 38 papers read. The officers elected included the following: as vice-presidents, Professors C. N. HASKINS and R. G. D. RICHARDSON; as members of the council to serve until December, 1922, Professors T. H. HILDEBRANDT, E. KASNER, W. A. MANNING, and H. H. MITCHELL.

At the thirty-fourth annual meeting of the American Historical Association at Cleveland, Ohio, a "Conference on the History of Science" was held on December 31, 1919. The following papers were read: "History of Egyptian medicine" (illustrated) by Professor T. W. TODD, Medical School, Western Reserve University; "Peter of Abano, a medieval scientist" by Professor LYNN THORNDYKE, Western Reserve University; "The history of algebra" by Professor L. C. KARPINSKI, University of Michigan; "The problem of the history of science in the college curriculum" by Professor HENRY CREW, Northwestern University.

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The slogan of the Association is to include in its membership every teacher of collegiate mathematics in America and to make such membership worth while. Application blanks for membership may be obtained from the Secretary, W. D. Cairns, 27 King Street, Oberlin, Ohio.

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EDITORIAL CORRESPONDENCE AND BOOKS FOR REVIEW should be addressed to the EDITOR-IN-CHIEF, R. C. ARCHIBALD, Brown University, Providence, R. I.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 27 King Street, Oberlin, Ohio.

FOURTH ANNUAL MEETING OF THE MATHEMATICAL ASSOCIATION OF AMERICA.

The fourth annual meeting of the Association was held at Columbia University on Thursday and Friday, January 1 and 2, 1920, in conjunction with the annual meeting of the American Mathematical Society, which was held on Tuesday and Wednesday, December 30 and 31, 1919. There were 143 in attendance at the sessions, including the following 108 members of the Association:

C. R. ADAMS, Brown University.
JOSEPH ALLEN, College of the City of New York.
R. C. ARCHIBALD, Brown University.
C. S. ATCHISON, Washington and Jefferson College.

CLARA L. BACON, Goucher College.
RALPH BEATLEY, Horace Mann School for Boys.
A. A. BENNETT, University of Texas.
E. G. BILL, Dartmouth College.
C. L. BOUTON, Harvard University.
JOSEPH BOWDEN, Adelphi College.
H. S. BROWN, Hamilton College.
DANIEL BUCHANAN, Queen's University.
R. W. BURGESS, Brown University.

W. D. CAIRNS, Oberlin College.
MARY E. CASTER, Paterson, N. J.
J. A. CLARKE, W. Philadelphia High School for Boys.
J. B. COLEMAN, University of South Carolina.
G. M. CONWELL, N. Y. State College for Teachers.
J. L. COOLIDGE, Harvard University.
ELIZABETH B. COWLEY, Vassar College.
C. H. CURRIER, Brown University.

ELEANOR C. DOAK, Mount Holyoke College.
E. L. DODD, University of Texas.
FLETCHER DURELL, Lawrenceville School.

H. S. EVERETT, Bucknell University.

T. S. FISKE, Columbia University.
W. B. FITE, Columbia University.
J. A. FOBERG, Crane Junior College.
T. M. FOCKE, Case School of Applied Science.

A. S. GALE, University of Rochester.
W. V. N. GARRETSON, Rutgers College.
R. E. GILMAN, Brown University.
O. E. GLENN, University of Pennsylvania.
W. C. GRAUSTEIN, Harvard University.

W. M. HAMILTON, U. S. Nautical Almanac Office.
H. E. HAWKES, Columbia University.
OLIVE C. HAZLETT, Mount Holyoke College.
E. R. HEDRICK, University of Missouri.
A. A. HIMWICH, Physician, New York City.
ANNA M. HOWE, Dana Hall, Wellesley, Mass.
L. S. HULBURT, Johns Hopkins University.
W. A. HURWITZ, Cornell University.

DUNHAM JACKSON, University of Minnesota.
S. A. JOFFE, Asst. Actuary, Mutual Life Ins. Co.

EDWARD KASNER, Columbia University.

W. D. LAMBERT, Coast and Geodetic Survey.
MARCIA L. LATHAM, Hunter College.
D. D. LEIB, Connecticut College for Women.
FLORENCE P. LEWIS, Goucher College.
H. M. LUFKIN, Cornell University.

H. B. MITCHELL, Columbia University.
C. N. MOORE, University of Cincinnati.
FRANK MORLEY, Johns Hopkins University.
H. M. MORSE, Harvard University.
F. D. MURNAGHAN, Johns Hopkins University.

E. J. OGLESBY, New York University.
F. W. OWENS, Cornell University.
HELEN B. OWENS, Cornell University.

GEORGE PAASWELL, Civil Engineer, New York City.
ALEXANDER PELL, Bryn Mawr, Pa.
ANNA J. PELL, Bryn Mawr College.
S. S. PENN, New York, N. Y.
L. R. PERKINS, Middlebury College.

PATRICK RAFFERTY, Holy Cross College.
W. W. RANKIN, Jr., University of South Carolina.
W. R. RANSOM, Tufts College.
H. W. REDDICK, Cooper Union.
L. J. REED, Johns Hopkins University.
C. N. REYNOLDS, Jr., Wesleyan University.
J. B. REYNOLDS, Lehigh University.
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L. H. RICE, Tufts College.

PAUL SAUREL, College of the City of New York.
RALEIGH SCHORLING, Lincoln School.
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C. G. Simpson, Pennsylvania State College.
Mary E. Sinclair, Oberlin College.
H. E. Slaught, University of Chicago.
Clara E. Smith, Wellesley College.
D. E. Smith, Columbia University.
J. P. Smith, Georgetown University.
P. F. Smith, Yale University.
Sarah E. Smith, Mount Holyoke College.
W. M. Smith, Lafayette College.
Marion E. Stark, Wellesley College.

J. S. Taylor, Mass. Institute of Technology.
H. D. Thompson, Princeton University.
J. I. Tracey, Yale University.
A. B. Turner, College of the City of New York.
Bird M. Turner, Phebe Anna Thorne Model School, Bryn Mawr College.

P. H. Underwood, Ball High School, Galveston, Tex.
C. B. Upton, Teachers College.

H. S. Vandiver, Cornell University.

C. A. Waldo, Washington University.
Evelyn Walker, Hunter College.
J. L. Walsh, Graduate School, Harvard University.
H. E. Webb, Central High School, Newark, N. J.
Eula A. Weeks, Cleveland High School, St. Louis, Mo.
Mary E. Wells, Vassar College.
E. E. Whitford, College of the City of New York.
F. B. Wiley, Denison University.
W. A. Wilson, Yale University.
F. S. Woods, Mass. Institute of Technology.

J. W. Young, Dartmouth College.

S. D. Zeldin, Mass. Institute of Technology.

The general topic for all the sessions of this meeting was "Mathematics in Relation to the Allied Sciences." Speakers were invited from five different scientific fields; namely, Physiology, Crystallography, Physical Chemistry, Biometry and Vital Statistics, and Physics. In each session the number of papers was limited to two in order that ample time might be given to general discussion. The wisdom of this arrangement was fully justified as shown by the marked and prolonged interest in the discussion after each paper. It is worthy of note that among these five branches of science the one which seemed to call for the widest range of mathematical applications, as set forth by the various speakers, was in connection with research in Physiology. By common consent it was acknowledged that the Association is deeply indebted to these speakers for the information and inspiration contributed by them on this occasion. Credit is due to Professor H. E. Hawkes, chairman of the program committee, for the arrangement and successful outcome of the program.

On Wednesday evening, between the sessions of the Society and the Association, a joint dinner and social gathering was held in the new Students Hall of Barnard College where about 115 members and their friends were present. Professor E. W. Brown presided and short addresses were given by Professor Frank Morley, president of the Society, Professor H. E. Slaught, president of the Association, Professor Dunham Jackson, Professor T. S. Fiske, Professor R. G. D. Richardson, Professor H. E. Hawkes, and Professor J. L. Coolidge. The arrangements for the dinner and for all matters of comfort and convenience connected with the meetings were made by Professor T. S. Fiske, chairman of the committee on arrangements.

As usual at the annual meetings, interest in the election of officers of the Association grew in intensity and the balloting continued up to the hour of the business meeting on Friday morning. The total number of ballots cast was considerably larger than last year. It is hoped that the number taking advantage of this franchise privilege may steadily increase until a large percentage of the members shall become thus active.

At the close of the final session a formal and unanimous resolution was passed expressing appreciation of the hospitality of Columbia University, of the services of Professor Fiske as chairman of the committee on arrangements, and of Professor Hawkes as chairman of the program committee.

President Slaught presided at each of the three sessions. The following seven papers were read:

(1) "Mathematics for the physiologist and physician" by Dr. H. B. WILLIAMS, Assistant Professor of Physiology, College of Physicians and Surgeons.

(2) "The regular solids and the types of crystal symmetry" by Dr. P. L. SAUREL, Professor of Mathematics, College of the City of New York.

(3) "The mathematics of physical chemistry" by Professor G. B. PEGRAM, Dean of the School of Mines, Engineering, and Chemistry, Columbia University.

(4) "The mathematics of biometry" by Dr. L. J. REED, Associate Professor of Biometry and Vital Statistics, Johns Hopkins University.

(5) "An experiment in conducting freshman mathematics courses" by Dr. F. B. WILEY, Professor of Mathematics, Denison University

(6) Preliminary report of the National Committee on Mathematical Requirements by Dr. J. W. YOUNG, Professor of Mathematics, Dartmouth College.

(7) "Mathematics for students of physics" by Dr. LEIGH PAGE, Assistant Professor of Physics, Yale University.

Abstracts of the papers and discussions follow below, the numbers corresponding to the numbers in the list of titles:

(1) It seems to have been generally assumed that students intending to enter upon the study of medicine or the biological sciences need only the most elementary training in mathematics. The biological sciences in the main, however, rest entirely upon the fundamental sciences of physics and chemistry. Without adequate preparation in these sciences the biological student will be greatly handicapped in his studies and if he attempts in later life to enter upon serious original research in his chosen field he will meet almost insurmountable obstacles and be apt to commit most absurd mistakes.

The ideal preparation would be a very thorough training in physics, chemistry, and physical chemistry, with all the mathematical preparation which such training presupposes. Since this would be in itself the work of a life-time, some sort of compromise must be effected. A fairly definite list of the desirable courses in mathematics was given by Dr. Williams.

It was suggested that for students of medicine especially the usual courses in general biology, zoölogy, histology, comparative anatomy and the like be omitted from the preparatory course and mathematics, chemistry and physics be substituted. Given a good ground work in these fundamental sciences, the student will be able at any later time to extend his field of knowledge by independent reading without the help of instructors. Much of the work ordinarily taken in the academic courses by these students is duplicated in the medical school.

Medicine is both science and art. For those who aspire to know anything of the science, training in fundamentals is essential, and if one intends to participate in the advance of the medical sciences a very thorough training should be secured. The best work in physiology has always been done by men whose preliminary training was exceptionally good.

In the discussion of this paper, Professor Slaught told of a student holding the doctor's degree in zoölogy who is now taking a year's course in calculus because it was found necessary for later biological research.

Professor D. E. Smith gave voice to the general feeling of surprise on the part of the hearers that so extensive a training in mathematics was called for. In answer to his questions Dr. Williams stated (1) that students can ordinarily assimilate the material in plane and solid geometry most readily in their high-school years, can then best spare the time for that work and can at that period most easily get into their memory the facts of elementary mathematics; (2) that analytic trigonometry is a useful possession if a student has the time for its study, the speaker himself having used it in science related to medicine; and (3) that the hyperbolic functions are useful for the physiologist in alternating current theory, which is needful in certain researches in physiology.

In answer to inquiries made by Professor Hawkes, Dr. Williams gave further detailed suggestions. The theory of probability and of the errors of observation are important in their physiological bearings. Again, in the study of the nature of nerve impulse the simple physical theory proved inadequate to cover the facts and it was necessary to attack the investigation as a problem in electrical conduction on the theory that the nerve behaves like a submarine cable; measurements were made sufficient to compute the characteristics of the nerve and an artificial cable was constructed such that the curve of electrical response of the artificial cable could hardly be distinguished from that of the nerve studied. It was further found that the variations due to temperature changes agreed with the electrical theory. The studies on the heart by electrical means made by Eintoven of Leyden were possible because of his being well trained in mathematical investigation; his invention of an electrometer which is a well-known modification of the D'Arsonval galvanometer was directed by his study of mathetheory. Many times the physiologist meets with formidable mathematical notation in his reading of physics papers, and he should be able to command this as well as the purely physical material.

Professor Archibald drew attention to O. Fischer's article "Physiologische Mechanik (Bewegungsphysiologie)," 1904, with its bibliography of over four hundred titles, in *Encyclopädie der mathematischen Wissenschaften*.

Professor Jackson reported that Dr. Dryer of Minnesota in a recent address on the use of mathematics in medicine instanced the application of statistics to the study of blood pressure; when the blood volume in animals was compared with bodily weight, it was found that the former was proportional, not to the latter as formerly supposed, but more nearly to the bodily surface, this surface having been found experimentally to be approximately proportional to the two-

thirds power of the weight. There was no suggestion of any a priori reason for the relation between surface and weight such as would have been familiar to one trained in mathematics.

Professor C. N. Moore spoke of the curve representing the time-rate of healing of wounds, which follows an exponential law. Two observations made only a short time apart determine the curve and any important deviations from this curve occurring within the next few days indicate a pathological condition which calls for investigation.

Professor Hurwitz remarked that if the value of mathematics for the various sciences is to be emphasized, college teachers must be able to show such advantages not merely for the exceptional investigator, but as well for a considerable proportion of the students of science.

Professor Bennett related a case where the author of a paper at an educational gathering spoke of the area under a curve as representing the number of observations, and the chairman of the meeting remarked that the paper was very interesting but of course one could not expect to understand the mathematical portion! Professor Bennett also mentioned a limited study of periodic functions as desirable early in the course in algebra.

Professor Fiske closed the discussion by stressing the need of teaching more than the bare matter which the majority will probably use; rather is it our duty to continue as much as ever to give all the mathematical instruction which we find possible.

(2) The object of Professor Saurel's paper was threefold. In the first place, it was pointed out that, as the result of the work of Hessel, Bravais, Gadolin, Curie, and Lorentz, it was now possible to present in very simple form the solution of the problem of finding all the finite groups of operations that can be formed by means of rotations about axes passing through a given point and inversions with respect to that point. In the second place, it was shown that the solution of this problem leads directly to the enumeration of the types of crystal symmetry. Finally, a set of symbols and of names for the thirty-two different types of crystal symmetry was suggested which is believed to be simpler than any set in use at present.

Mr. Wintringham told of Jäger's work on crystal symmetry, the symmetry of animal and vegetable forms, including such studies as that of the pineapple.

Professor Archibald brought out the statement from Professor Saurel that Gadolin's contributions in respect of the thirty-two types of crystal symmetry were not essentially new, his axes of inverse symmetry being equivalent to Hessel's earlier treatment, viz., rotation followed by reflection on a plane, but were more in harmony with the modern methods of treatment. The study by the Greeks of the regular solids (fundamental in crystal classification) involved propositions on "golden section," and Professor Archibald pointed out: (1) that Kepler was enthusiastic regarding this "section" (which he termed "divine") and the manner in which it arose in connection with growths and the Fibonacci

series; and (2) that these elements were fundamental in a theory, likely to revolutionize the whole modern theory of design, described by Mr. Jay Hambidge in *Dynamic Symmetry: The Greek Vase,* soon to be issued by the Yale University Press.

(3) Dean Pegram stated that the student who has college calculus such as provided by the customary three-hour course for a year should find little difficulty mathematically with any of the literature in the most recent text books and treatises in undergraduate courses in physical chemistry. It would be of advantage to the student of physical chemistry if his course in calculus included some elementary work in the solution of differential equations. Students looking forward to advanced work in chemistry should certainly not be allowed to make the mistake of failing to take such a course in the calculus, and preliminary to this it needs to be assumed that they have had good training in algebra and analytic geometry. The subject of the mathematics of statistics as applied both in the theory of measurements, and more especially in the kinetic theory of gases, is of interest in chemistry, but the demands in this direction appear not to be large at the present time.

Teachers of mathematics in college can perform good service to their students who are looking forward to work in chemistry by utilizing preferably for illustrative purposes equations and processes of the type that occur frequently in physical chemistry. At this point in his paper Dean Pegram cited about fifteen types of algebraic and differential equations which the undergraduate student of physical chemistry should be able to handle. These are for the most part familiar to all teachers of elementary calculus, yet it is of value to have them grouped together as they occurred in Dean Pegram's paper.

The Mariotte-Guy Lussac law, $pv = RT$, with the modified forms:

$$\left(p + \frac{a}{v^2}\right)(v - b) = RT, \qquad \text{(Van der Waals)}$$

$$\left(p + \frac{a}{T(v + c)^2}\right)(v - b) = RT, \qquad \text{(Clausius)}$$

$$p(v - b) = RTe^{-(A/RTv)}. \qquad \text{(Dieterici)}$$

Adiabatic equation of a perfect gas, $pv^k = C$.

Expressions for saturated vapor:

$$\log p = a + b\alpha^t + c\beta^t, \quad \text{(Biot)}$$

$$\log p = \alpha - \frac{\beta}{T} - \frac{\gamma}{T^2}, \quad \text{(Rankine)}$$

$$\log p = k_1 + k_2 \log T - \frac{k_3}{T}. \quad \text{(Hertz-Nernst)}$$

First law of thermodynamics:

$$du = dq - dw = dq - pdv;$$

second law of thermodynamics:

$$ds = \frac{dq}{T},$$

with the incidental training needed here in dealing with partial differentiation, exact differentials and related topics.

Ordinary and partial differential equations of the following types: Monomolecular reaction:

$$\frac{dx}{dt} = k(a - x);$$

bimolecular reaction:

$$\frac{dx}{dt} = k(a - x)(b - x);$$

and similar forms involving three or more factors.

Catalyzed reaction:

$$\frac{dx}{dt} = \frac{k(a - x)}{x^{1/2}}.$$

Diffusion:

$$\frac{\partial c}{\partial t} = D\frac{\partial^2 c}{\partial x^2}.$$

The normal probability curve ($y = ke^{-\gamma^2 x^2}$) in finite or differential form, with a similar formula arising in the kinetic theory of gases:

$$dn = \frac{4n}{a^3\sqrt{\pi}} e^{-v^2/a^2} v^2 dv.$$

The following were mentioned as the chief texts of interest in this study: Lewes's *System of physical chemistry* (three volumes); Nernst's *Theoretical chemistry*, Washburn's *Physical chemistry*, Partington's *Higher mathematics for students of chemistry*, Mellor's *Higher mathematics for students of physics and chemistry*.

While a good course in calculus prepares a student to do work in physical chemistry, advanced students of chemistry in our universities are finding it very desirable and profitable to put much time on advanced courses in physics and in pure mathematics, since the development of chemistry is sure to be in the direction of more elaborate mathematical formulation of its theory.

In the discussion that followed, Dean Pegram stated that the extent to which the teacher of mathematics should go into the interpretation of these equations is conditioned quite largely by a number of circumstances, such as the student's ability and previous training, the instructor's training and inclination, the place of the mathematics courses in the curriculum relative to the courses in physical chemistry. If the mathematician gives to the student a facility in mathematical operations such as are suggested by these equations, he performs a valuable service. The pupil's interest may well be aroused by giving him the significance of the quantities appearing in these equations if this seems feasible.

Professor Hurwitz mentioned a teacher of thermodynamics who demands considerably more than the amount called for by the speaker, *e.g.*, differential geometry (elliptic, hyperbolic and parabolic points), the non-vanishing of the Jacobian as the condition for reversibility of a transformation. Professor Pegram, however, pointed out that his program had been proposed for undergraduate preparation, not necessarily for graduate students, who might well be expected to continue with more advanced courses suited to their particular needs.

In answer to a question by Professor Burgess, the speaker said that while the ability to fit curves to experimental data may wisely be acquired in undergraduate years if time allows, adequate preparation in the fundamental courses must certainly be assured.

Professor Jackson emphasized the fact that we must be very explicit in our instruction in the fundamental courses, commending the usual and evidently needful practice in thermodynamics of indicating not merely the variable with respect to which the partial differentiation is made, but as well the quantity which is held fast.

(4) In this paper Professor Reed attempted to outline the ideal course of mathematical training for students expecting to enter the field of biometry. In brief this mathematical training consists of courses in algebra, trigonometry, analytic geometry and calculus given in much the usual way, together with a course in the modern statistical methods of treating observed data. This latter should include a discussion of probability leading up to the normal curve, a brief discussion of mechanics treating in particular the principles of moments, and a consideration of methods of treating frequency curves, correlation, and "curve fitting." This statistical theory should be developed from the principles of moments rather than by the familiar "least squares" method as the former is more flexible. The above course would give the student in any of the natural sciences the mathematical background necessary for the treatment of the majority of his problems.

When questioned by Professor Hawkes, Dr. Reed stated that the year course in calculus and a semester course in statistics are sufficient as undergraduate preparation, but that further courses should be added in preparation for research.

Professor Fite emphasized the desirability of some limited amount of probability with examples other than those so commonly drawn from games of chance.

In reply to Professor Richardson, the speaker said that he had for several years given with apparent success a course in calculus, three hours per week for a semester, to students of chemistry, devoting a fair amount of time to drill in using Peirce's *Table of Integrals;* on the other hand Professor Richardson maintained that he had not been able to accomplish this in so short a time. Professor Ransom held that this question depends largely on the number of functions which the student needs to be able to handle, and that for the most part the student in his reading along scientific lines needs not so much to set up the differential equations, nor even to integrate the equations, since the author gives the integra-

tion for the cases most commonly met with; rather does he need to be able to understand the language, to know what it is all about. Professor Jackson expressed his agreement with Professor Richardson, said that even after a solid year's course in calculus the student too often fails to get an adequate grasp of the notions of derivative, integral, etc., just as we may in our despair have noted that the last notion the pupil gets in analytic geometry is the connection between the curve and the equation! On the question of brief courses in calculus, integration chiefly by use of tables, etc., Professor Slaught suggested that what is most needed is a certain maturity of judgment and power which can come only through much practice both in translating problems into the language of calculus and in actually carrying out the steps of the work both in differentiation and integration.

(5) Professor Wiley's paper was inserted at the close of the session on Thursday afternoon by unanimous consent. He has been conducting an experiment in handling freshmen by an individual method which contains some unique features. The plan involves: (1) a laboratory arrangement whereby each student works by himself definite assignments which he gets from a head assistant as rapidly as he is able to complete the work satisfactorily; (2) a corps of assistants chosen chiefly from among the best students in the upper classes, who are rewarded partly by the honor conferred upon them, partly by the benefit derived from the practice teaching involved, and partly, in some cases, by tuition exemption; (3) a scheme of credits based upon the completion of certain assignments and the passing of certain general and special tests. Each student has opportunity to confer with the regular instructor as well as with the special assistants, and groups of students working on practically the same assignments are encouraged to confer together and may meet at intervals for joint conference with the instructor.

The paper inspired numerous questions as to the mechanical details and as to the practicability of the scheme, the answers being given for the most part in the above abstract.

(6) Professor Young made a report of progress on behalf of the National Committee on Mathematical Requirements. In addition to news items of interest already published in previous issues of the MONTHLY, announcement was made of the fact that the U. S. Bureau of Education has agreed to publish the reports of the committee. Professor Young also made a plea for coöperation on the part of teachers of mathematics and mentioned several new reports which are in preparation by the committee. A preliminary report on a suggested revision of the work in mathematics of the first two years of the high school is ready for publication and will soon be issued by the Bureau of Education. Copies of this report will be mailed to all members of the Association and of the Society. This report has been discussed by a large number of secondary associations and clubs, and the committee has had the benefit of suggestions from such sources as well as from individuals all over the country. [Compare *1920,* 145–146].

To Dr. Durell's request that the committee would propose as an alternative a plan incorporating a course in algebra as such and a course in geometry as such, since there are many who object to unified courses, Professor Young replied that the report is meant to cover what the committee believes should be the content of the first two years of the high-school course and that the committee does not intend to suggest any preference as between separate and so-called correlated courses.

Professor Dodd expressed his apprehension that where the committee recommends that "more emphasis should be placed on the acquisition of insight and power and less time devoted to acquiring mere facility in the solution of formal exercises," the latter part might unfortunately be observed by teachers the country over without a like endeavor to adopt the first part.

Professor Young also gave a statement of the fundamental considerations on the basis of which it is proposed to revise the present college entrance requirements. Suggestions from any source are desired in this connection. The statement in full follows.

December 30, 1919.

To the Council of the Mathematical Association of America:

The National Committee on Mathematical Requirements was requested by your body at your meeting in December, 1918, and by the Council of the American Mathematical Society also in December, 1918, to undertake a reconsideration of the definitions of college entrance requirements as formulated by a Committee of the American Mathematical Society in 1903.

The National Committee is concerned with the study of a desirable reorganization of courses in mathematics in secondary schools and colleges and with proposals for the improvement in the teaching of this subject generally. The subject of college entrance requirements is only one phase of the work for which this committee was organized.

The National Committee felt that it could not profitably attack the problems connected with the revision of college entrance requirements until it was fully acquainted with present tendencies and movements in secondary education relating to mathematics, nor until its own ideas as to desirable changes in secondary school courses in mathematics had assumed a certain degree of definiteness. Accordingly, the consideration of college entrance requirements was postponed. At a meeting of the National Committee on November 1, 1919, however, the problems connected with the desirable mathematical preparation of college students were discussed. We are able, therefore, to make the following preliminary report of progress:

We call attention to the following facts as bearing on the problem under consideration.

(*a*) There is general recognition of the principle that secondary-school courses in mathematics, especially the introductory courses, should be planned so as to be of the greatest possible value to the pupils taking them, whether these pupils ultimately go to college or not. College entrance requirements should not

influence the organization or the teaching of such courses to the impairment of their general value.

(*b*) The growth of the junior high school movement is developing a group of courses in mathematics for the seventh, eighth, and ninth school years which may differ in organization and classification of subject matter from the traditional division into elementary algebra, plane geometry, etc. A proper articulation between secondary school and college may make it desirable, even though it be not necessary, to take such new organization of subject matter into account in formulating college entrance requirements.

The results secured under present entrance requirements are unsatisfactory mainly for two reasons:

(*a*) Too many students on entering college lack a sufficient understanding of mathematical principles and sufficient power in applying them.

(*b*) Too many such students are found to be weak in carrying out the simpler, but fundamentally important, operations of arithmetic and algebra.

In our opinion the fault is not as much with the topics listed in the definitions of the "units" of college entrance requirements as it is with the type of instruction generally given. The latter, moreover, is greatly influenced by the character of the prevailing type of entrance examination.

We believe that the definitions of "units" should be revised by eliminating from the requirements in algebra some of the technically more complicated and relatively less useful topics (for example, the extraction of the square root of a polynomial) and by specifying more definitely the extent to which the study of the remaining topics should be carried; in particular, by definitely limiting the difficulty and complexity of the formal manipulative exercises. This would allow more time for securing understanding of principles, for practice in applying them, and for thorough drill in the simpler fundamental operations, both numerical and algebraic.

As previously indicated, however, improvement in the preparation of students entering college must be sought primarily in the instruction in secondary schools. In general this instruction does not place sufficient emphasis on understanding and insight, and is too exclusively concerned with the development of mere skill or facility in formal manipulation. The colleges bear a considerable share of the responsibility for this condition, in that the prevailing type of entrance examination, which exerts a powerful influence on instruction, has overemphasized the formal exercise. Steps should, therefore, be taken to bring about a change in such examinations. They should be more consciously formulated to test the candidate's understanding of principles and his power of applying them, as well as his ability to perform the necessary simple fundamental operations accurately and expeditiously. Formal exercises should be in the minority, instead of the majority as at present. More emphasis should be given to verbal problems and these should be graded with the utmost care as to difficulty and as to the powers which they are intended to test.

There should, moreover, be added to the definition of the "units" a definite

statement as to the specific abilities which should be developed (such as the ability to understand and use a formula, the ability to interpret an algebraic result in a concrete setting, the ability to analyze and solve concrete problems, etc.).

The following principles have accordingly been tentatively adopted to govern our proposed reconsideration of college entrance requirements in mathematics:

1. The scholarship implied in the present customary requirement of $2\frac{1}{2}$ units should not be lowered.

2. Drill in algebraic manipulation should be limited to those processes and to the degree of complexity necessary for a thorough understanding of principles and required by the probable applications either in common life or in subsequent mathematics.

3. More emphasis should be given to such immediately useful elementary topics as:

(*a*) The understanding and use of the formula.

(*b*) The interpretation of graphic representation and the use of graphic methods.

4. More emphasis should be placed on the acquisition of insight and power and less time devoted to acquiring mere facility in the solution of formal exercises.

5. While specific minimum requirements in separate subjects like algebra, geometry, etc., are still necessary, adequate provision should be made, perhaps through the so-called comprehensive examination, for the pupils in those schools who are developing "general courses" in mathematics where these subjects are not taught in separate courses.

6. College entrance requirements should be stated not only in terms of subjects and topics but also in terms of specific mathematical abilities to be developed.

7. Means should be found to introduce a proper and desirable amount of flexibility into the requirements in order to encourage progress in secondary education through the experimental introduction of new topics and methods.

On the basis of these principles we hope to formulate our final report for submission to the Council not later than its next summer meeting.

Respectfully submitted, for the Committee,

J. W. Young,

Chairman.

(7) The subject of mathematical needs of students in physics, as presented in Dr. Page's paper naturally divides itself into two parts: first, the mathematical needs of the undergraduate student in physics, and second, those of the graduate.

The elements of algebra and trigonometry are essential prerequisites for an undergraduate course in physics. The parts of algebra of importance are the solution of equations of the first and second degree, simultaneous equations of the first degree, logarithms and exponentials, and the binomial theorem; Dr. Page pleaded for more thorough preparation in these topics and greater practice in

their application, even at the expense of those parts of the subject which the undergraduate is likely to have less occasion to use. At the very beginning of his course in physics the student takes up composition and resolution of vectors, and in order to obtain a clear idea of the significance of these directed quantities he must have a good working knowledge of the elements of trigonometry. It is more important that he should be able to distinguish readily sine from cosine when neither side of the angle is horizontal, than that he should be acquainted with a large number of trigonometrical relations which he may never have occasion to use.

Very early in his course in physics, the student takes up derivations which can be explained satisfactorily only by the use of the calculus. It seems unfortunate to the teacher of physics that the college course in general physics is usually given before the student is able to use the calculus; the student's comprehension of the calculus suffers when he is deprived of the excellent opportunity to apply it which is afforded by the study of physics. The tendency to bring into freshman courses in mathematics at least the elements of differentiation and integration was looked upon with favor, and hope was expressed that this movement will permeate down to the last year of high school.

The mathematical preparation of the man who intends to pursue graduate study and engage in research in physics cannot be too thorough. Its value to him is of two kinds: first, it provides him with tools which are essential for his work, and second, it trains him in those habits of logical thought which are exemplified preëminently in mathematical reasoning.

The first requirement of the graduate student in physics is a good course in advanced calculus and ordinary differential equations. Soon after must come courses in partial differential equations, and the study of Legendre's, Bessel's and Laplace's functions, in order to enable him to solve problems in electrostatics and allied subjects. The historical importance of Fourier's series involved in the solution of problems on the flow of heat has been greatly augmented by applications of this form of analysis to the theory of radiation. Some knowledge of the theory of probabilities is essential before courses in statistical mechanics and the kinetic theory of gases are undertaken.

Above all, emphasis was laid on the importance of vector analysis to the student of physics. With few exceptions, all the important quantities with which physics deals are either vectors or the scalar products of vectors. As mathematics is employed by the physicist as a tool in carrying out reasoning in physics, it is important that the notation employed should emphasize the physical rather than the mathematical significance of the logical processes involved. Hence the superiority of vector over scalar analysis. Of the current notations, none is comparable in simplicity and utility to that of Gibbs. One who has been brought up on this form of vector analysis and has made constant use of it cannot help feeling astonished that it has not been universally adopted by physicists.

In discussing this paper Professor Ransom held that while the speakers on

the program maintained that students expecting to go into scientific work need a wide acquaintance with mathematics, it is still to be said that many students do not know early enough what courses they need or even that there are mathematical subjects which serve their special purposes. Here is the opportunity for a survey course of a half year, where the instructor advisedly does the greater part of the work and suggests what will be of later use, a course which is profitable for sophomores, certainly for juniors and seniors. Such a course would help in furnishing a better knowledge of the *language* of mathematics. Professor Page replied that the student would doubtless not know of the requirements for, say, electrodynamics and the electromagnetic theory of light, where some such course as Gibbs's vector analysis is essential; he will, if wise, ascertain from his instructor in physics or from some other well informed person what the necessary preparation is. Professor Coolidge described a survey course at Harvard characterized by the students as "seeing mathematics"; the course was elected for the most part by those who had no adequate preparation in earlier mathematics and proved a disillusionment for those who had hopefully inaugurated the plan. Professor Hurwitz instanced a similar course at Cornell in which the instructor refused to make it an easy course, the result being that its enrollment has been very small. He added that uneducated laymen can today pick up a popular science journal and can from this acquire the essential principles of new methods in non-technical language; nevertheless only the exterior results can thus be transmitted, and it is our distinct duty to contribute as far as may be through mathematics also to the development of a well-informed public.

Professor Hawkes made the very important remark that much of what is sought through survey courses can be done by mathematical clubs. We need merely to refer to the past few volumes of the MONTHLY to see how much of this desirable work has actually been accomplished in the live universities and colleges of America. Dr. Helen Owens said that the junior mathematical club at Cornell considers application of mathematics to other subjects, thus enabling students to get preliminary notions of the mathematical courses needed in other branches. The experimental plan, which thus far has included applications to physics, statistics, actuarial work and chemistry, has proved very interesting.

A good natured controversy between Professors Bowden and Page as to whether Professor Gibbs was a pure mathematician or a member of the department of physics, turned the discussion to the question of a coördination between the two departments. Professor Page said that the pure mathematician does not keep closely enough in touch with the experimentalist and his work is apt to suffer so far as its immediate usefulness to the physicist is concerned.

Professor Woods emphasized the portion of the paper that called for those parts of differential geometry and other advanced mathematics which in recent years have become of the utmost importance to the physicist and chemist, *e.g.*, the Einstein theory. He also thinks it unfortunate to teach physics without calculus. Under a new plan at the Massachusetts Institute of Technology,

trigonometry is an entrance requirement, the freshman begins with calculus, even before he studies analytic geometry; this course is to be followed by a course in the beginnings of analytic geometry for one term, then further work in calculus.

Meeting of the Council of the Association.

The following seventy-two persons and two institutions, on applications duly certified, were elected to membership:

To individual membership.

W. C. Bartol, A.M. (Bucknell); Ph.D. (Adrian). Prof. of math. and astr., Bucknell Univ., Lewisburg, Pa.

Annette Bennett, A.M. (Columbia). Head of dept. of math., Eureka Coll., Eureka, Ill.

W. N. Birchby, A.M. (Colo. College). Pasadena, Cal.

L. I. Bonney, A.B. (Bates College). Asst. prof., Middlebury Coll., Middlebury, Vt.

F. E. Brasch. Asst. reference librarian, John Crerar Library, Chicago, Ill.

W. M. Brodie, M.E. (Va. Polytech. Inst.); A.M. (Columbia). Prof., Va. Polytech. Inst., Blacksburg, Va.

J. S. Brown, A.M. (Texas). Prof., Southwest Texas St. Normal College, San Marcos, Tex.

Margaret Buchanan, A.B. (West Va. Univ.). Grad. stud., Bryn Mawr Coll., Bryn Mawr, Pa.

S. E. Crowe, A.B. (Ohio State). Asst. prof., Mich. Agric. Coll., East Lansing, Mich.

W. L. Crum, Ph.D. (Yale). Instr., Yale Coll., New Haven, Conn.

R. D. Douglass, A.M. (Maine). Instr., Mass. Inst. of Tech., Cambridge, Mass.

W. F. Downey, A.B. (Amherst). In charge Collins Bldg., English High School Annex, Boston, Mass.

G. C. Evans, Ph.D. (Harvard). Prof., Rice Inst., Houston, Tex.

A. J. Fleisig, A. B. (St. Procopius). Instr., St. Procopius Coll., Lisle, Ill.

C. D. Garlough, A.M. (Illinois). Prof., Wheaton Coll., Wheaton, Ill.

H. H. Gaver, A.M. (Virginia). Instr., U. S. Naval Acad., Annapolis, Md.

R. E. Gilman, Ph.D. (Princeton). Asst. prof., Brown Univ., Providence, R. I.

W. W. Gorsline, B.S. (Chicago). Teacher of collegiate math. and surv., Crane Jun. Coll., Chicago, Ill.

W. C. Graustein, Ph.D. (Bonn). Asst. prof., Harvard Univ., Cambridge, Mass.

T. H. Gronwall, Ph.D. (Upsala); C.E. (Berlin). Math. and dynamics expert, Technical Staff, U. S. Ord., Washington, D. C.

V. G. Grove, A.M. (Kentucky). Asst. prof., Mich. Ag. Coll., E. Lansing, Mich.

H. E. Gudheim, M.E. (Royal Univ. of Tech., Stockholm). Assoc. prof. of graphics, Va. Polytech. Inst., Blacksburg, Va.

Howard Harding, B.M.E. (Michigan). With Rochester Railway and Light Co., Rochester, N. Y.

W. L. Hart, Ph.D. (Chicago). Asst. prof., Univ. of Minnesota, Minneapolis, Minn.

J. R. Hitt, B.S. (Miss. Coll.). Asst. prof., Mississippi Coll., Clinton, Miss.

Yun Huang Ho, A.B. (Cornell). Grad. stud., Cornell University, Ithaca, N. Y.

J. B. Jackson, A.M. (Columbia). Asso. prof., Univ. of South Carolina, Columbia, S. C.

J. S. W. Jones, D.Sc. (Washington Coll.). Prof., Washington Coll., Chestertown, Md.

G. B. King, A.B. (Ark. Cumberland College). Prof., Cumberland Coll., Clarksville, Ark.

J. R. Kline, Ph.D., (Pennsylvania). Asso., Univ. of Illinois, Urbana, Ill.

L. C. Knight, Ph.B. (Wooster). Asst. prof., College of Wooster, Wooster, Ohio.

Elmer Latshaw, Grad. preparatory engineering course (Drexel Inst.) Mech. designing, W. Philadelphia, Pa.

Lena R. Lewis, A.M. (Texas). Head of dept. of math., Thorp Spr. Chr. Coll., Thorp Springs, Tex.

H. M. Lufkin, S.T.B. (Phila. Div. Sch.). Asst., Cornell Univ., Ithaca, N. Y.

R. M. McDill, A.M. (Indiana). Prof., Hastings Coll., Hastings, Neb.

Martha P. McGavock, A.M. (Chicago). Head of dept. of math., Sullins Coll., Bristol, Va.

J. B. Meyer, M.S. (Purdue). Head of dept. of math., St. Normal School, Valley City, N. D.

Norman Miller, Ph.D. (Harvard). Lecturer, Queens Univ., Kingston, Can.

E. B. Mode, B.S. (Boston Univ.). Instr., Boston Univ., Boston, Mass.

H. M. Morse, Ph.D. (Harvard). Benjamin Peirce Instr., Harvard Univ., Cambridge, Mass.

G. W. Mullins, Ph.D. (Columbia). Asst. prof., Barnard Coll., New York, N. Y.

M. A. Nordgaard, A.M. (Maine). Asst. prof., Grinnell Coll., Grinnell, Iowa.

Yeisuke Ono. First Higher School, Tokio, Japan.

Helen B. Owens, Ph.D. (Cornell). Instr., Cornell Univ., Ithaca, N. Y.

S. F. Parson. Head of dept. of math., No. Ill. St. Normal School, De Kalb, Ill.

W. H. Pearce, A.M. (Michigan). Head of dept. of math., Central St. Normal School, Mount Pleasant, Mich.

H. P. Pettit, A.M. (Kentucky). Asst., Univ. of Illinois, Urbana, Ill.

E. C. Phillips, Ph.D. (Johns Hopkins). Prof. of math. and astr., Woodstock Coll., Woodstock, Md.

Hillel Poritsky. Asst. in math. and phys., Cornell Univ., Ithaca, N. Y.

C. Lois Rea, A.B. (Allegheny). Prof. of math. and sc., Cedarville Coll., Cedarville, Ohio.

F. W. Reed, Ph.D. (Virginia). Instr., Cornell Univ., Ithaca, N. Y.

J. N. Rice, Ph.D. (Catholic Univ.). Instr., Catholic Univ., Washington, D. C.

J. M. Robb, A.M. (Michigan). Head of math. dept., high school, Everett, Wash.

G. M. Robison, Ph.D. (Cornell). Instr., Cornell Univ., Ithaca, N. Y.

L. V. Romig, A.M., M.S. (Michigan). Prof., Augustana Coll., Rock Island, Ill.

C. D. Rose, A.M. (DePauw). Prof. of math. and astr., Nebr. Wesl. Univ., University Place, Neb.

Raleigh Schorling, A.M. (Chicago). Prin., High School, Lincoln School, New York, N. Y.

W. H. Schuerman, C.E. (Cincinnati). Dean of school of eng. and prof. of pure and appl. math., Vanderbilt Univ., Nashville, Tenn.

W. H. Sherk, A.M. (Chicago). Prof. of math., Univ. of Buffalo, Buffalo, N. Y.

G. A. Shook, Ph.D. (Illinois). Prof. of math. and physics, Wheaton Coll., Norton, Mass.

C. E. Shull, A.M. (Bridgewater). Head of dept. of math., Bridgewater Coll., Bridgewater, Va.

J. P. Smith, A.M. (Woodstock). Prof. of freshman math. and asst. Georgetown Univ. Observ., Washington, D. C.

M. G. Smith, Ph.D. (Illinois). Head of dept. of math., Greenville Coll., Greenville, Ill.

F. J. Taylor, A.B. (St. Thomas). Prof., College of St. Thomas, St. Paul, Minn.

J. S. Taylor, Ph.D. (California). Insr., Mass. Inst. of Tech., Cambridge, Mass.

J. A. Tobin, A.M. (Woodstock). Prof. of trig., anal. geom. and theoretical mech., Boston Coll., Newton, Mass.

R. S. Tucker, A.B. (Harvard). Instr., Harvard Univ., Cambridge, Mass.

C. B. Upton, A.M. (Columbia). Asso. prof. of math., Teachers College, Columbia University, New York, N. Y.

H. S. Vandiver. Instr., Cornell Univ., Ithaca, N. Y.

J. L. Walsh, A.M. (Wisconsin). Grad. student, Harvard Univ., Cambridge, Mass.

E. W. White, A.M. (Columbia.) Dean and prof. of math., Carson and Newman Coll., Jefferson City, Tenn.

C. O. Williamson, M.S. (Ohio Univ.). Instr. in appl. math., Coll. of Wooster, Wooster, Ohio.

To institutional membership.

New Hampshire College, Durham, N. H.

The College of Wooster, Wooster, Ohio.

The Council appointed the following Associate Editors of the Monthly for 1920:

Albert A. Bennett,	Cuthbert F. Gummer,	Ulysses G. Mitchell,
Edward L. Dodd,	Derrick N. Lehmer,	Elton J. Moulton,
Otto Dunkel,	Henry P. Manning,	David E. Smith,
Benjamin F. Finkel,	Raymond B. McClenon,	Horace S. Uhler.

It was voted that the incoming president of the Association appoint a committee of five members to report at the summer meeting of the Association on the question of its incorporation and on changes in the constitution necessary to carry such action into effect.

A committee of three was appointed to coöperate with a corresponding committee of the American Mathematical Society, if appointed, to consider the best policy with respect to holding meetings in conjunction in the future.

The invitation of Wellesley College for the Association to hold its summer meeting there in 1921 was referred to this committee.

It was voted that the Association hold its 1920 summer meeting in Chicago if suitable arrangements can be made. A formal invitation has since been received from the University of Chicago.

ANNUAL BUSINESS MEETING OF THE ASSOCIATION.

The secretary-treasurer announced the names of those elected to membership since the summer meeting of the Association. He reported also the death during 1919 of the following six charter members:

C. I. ALEXANDER, professor of mathematics, Texas Christian University (September 7);

C. J. BORGMEYER, professor of mathematics, St. Louis University (December 6);

F. E. CHAPMAN, Gulf Coast Military and Naval Academy (October 1);

W. E. ETZEL, rev. father, College of St. Thomas (February 3);

JAMES MACLAY, professor of mathematics, Columbia University (November 28);

L. G. WELD, director, Free School of Manual Training, Pullman, Ill. (November 28).

The election of officers for the year 1920 was conducted by mail and in person at this meeting, as provided by the constitution. The tellers (ELIZABETH B. COWLEY and A. S. GALE) appointed by the Council reported the result of the balloting as follows, 333 ballots having been cast:

For President: R. G. D. Richardson, 162 votes; D. E. Smith, 168 votes.

For Vice-Presidents: Helen A. Merrill, 198 votes; R. E. Moritz, 105 votes; H. L. Rietz, 164 votes; E. J. Wilczynski, 185 votes.

For additional members of the Council (to serve until January, 1923): R. D. Carmichael, 181 votes; E. L. Dodd, 106 votes; E. R. Hedrick, 215 votes; L. C. Karpinski, 115 votes; D. N. Lehmer, 119 votes; H. E. Slaught, 212 votes; Oswald Veblen, 169 votes; J. W. Young, 205 votes.

The following were accordingly declared elected:

President, D. E. SMITH, Columbia University.

Vice-Presidents: HELEN A. MERRILL, Wellesley College, and E. J. WILCZYNSKI, University of Chicago.

Additional members of the Council: R. D. CARMICHAEL, University of Illinois; E. R. HEDRICK, University of Missouri; H. E. SLAUGHT, University of Chicago; J. W. YOUNG, Dartmouth College.

In accordance with a provision of the constitution of the Association the Council filled two vacancies on its board by the appointment of E. L. DODD, University of Texas, and OSWALD VEBLEN, Princeton University. These vacancies were caused by the election of two members of the Council to offices in the Association.

The secretary was authorized to address a letter to the Director General of Railroads protesting against the arbitrary decision by which scientific societies were refused classification as educational organizations and thereby denied the privilege of reduced rates to members attending the meetings of such societies.

The secretary-treasurer made his financial report for the year, giving an account of all business transacted for the Association up to December 15, 1919. The report of the auditing committee (MARY E. SINCLAIR, H. E. SLAUGHT, and J. B. COLEMAN, chairman) was then made. The financial statement is printed in full below.

REPORT OF THE SECRETARY-TREASURER AS TREASURER, DEC. 15, 1919.

RECEIPTS.			EXPENDITURES.		
Balance Dec. 2, 1918		$3,728.11	Publisher's bills		$2,578.50
			President's office		174.77
1918 subscriptions	$ 11.10		Manager's office		25.24
1918 indiv. dues	121.72		Editor-in-chief's office		182.42
1918 instit. dues	12.75		Other editors' postage		18.35
1919 subscriptions	445.95		Committee on Membership		2.20
1919 indiv. dues	2,857.03		Committee on Dictionary		2.00
1919 instit. dues	404.50		Com. on Math. Requirements		58.12
Initiation fees	114.00		Secretary-Treasurer's office:		
Sale copies of MONTHLY	40.82		Postage	$203.50	
Sale reprints	13.71		Bond	5.00	
Advertising	533.32		Office supplies	11.70	
Exchange	.89		Express, telegrams	7.44	
Interest State Savgs. Bk.	84.22		Clerical work	227.75	
Interest Peoples Bk.	54.71		Part expense Register	20.00	
Interest Liberty Bonds	33.57		Printing	296.96	
			Chicago meeting	54.19	
			Michigan meeting	27.83	
Total 1919 receipts		$4,728.29	Paid to sections from initiation fees	33.50	
			Safety deposit rental	2.00	
Total assets up to 1920 business		$8,456.40	Pd. copies of MONTHLY	10.59	
					900.46
			Annals subvention		375.00
Total expenditures		4,317.06	Total expenditures		$4,317.06
			Checking account		500.79
Balance to the end of 1919 business		$4,139.34	State Savgs. Bk. Co. account		1,998.99
			Peoples Bkg. Co. account		1,081.29
Received on 1920 business		441.73	Liberty Bond		500.00
			Victory Bond		500.00
Book balance Dec. 15, 1919		$4,581.07	Bank balance Dec. 15, 1919		$4,581.07

When the accounts were closed on December 15, 1919, in order to furnish the auditing committee a complete record, there remained on the total business for the year 1919 the following items:

BILLS RECEIVABLE.

Advertising	$75.00
1919 dues unpaid	75.00
	$150.00

BILLS PAYABLE.

(Either paid in December or estimated.)

Publisher's bills, Sept.–Dec.	$1,450.00
Register of members	270.00
December *Annals* subvention	75.00
Init. fees due to sections	90.00
President's office	15.00
Manager's office	15.00
Editor-in-chief's office	30.00
Secretary-treasurer's office	130.00
Printing annual ballot, programs, etc.	125.00
Additional postage	50.00
	$2,250.00

If to the balance on 1919 business shown in this report, $4,139.34, there be added the amounts of bills receivable, $150 and there be subtracted the estimated amount of bills payable, $2,250, there results an estimated final balance on 1919 business of approximately $2,040. Of this surplus, about $1,000 was passed over to the Association by the management of the MONTHLY when the Association was organized, and this fund is kept by the Council of the Association as a reserve fund. The result for the year's finances is favorable, except for the fact faced one year ago but now made real, viz., an increased rate in the cost of printing dating from August 1, 1919, the exact rates not as yet having been announced by the publishers; this with the steadily decreasing amount derived from advertising makes it necessary to exercise all due care with regard to the expenses of the organization.

W. D. CAIRNS, *Secretary-Treasurer.*

THE NOVEMBER MEETING OF THE ILLINOIS SECTION.

The first program meeting of the Illinois Section of the Mathematical Association of America was held in room 418 Natural History Building, University of Illinois, on November 22, 1919. There were fifty-nine persons present including the following twenty-four members of the Association: Blumberg, Camp, R. D. Carmichael, Coble, Comstock, Crathorne, Emch, Foberg, Garlough, Kempner, Kustermann, Lytle, McNeill, Bessie I. Miller, G. A. Miller, Risley, Rosenbach, G. H. Scott, Shaw, Slaught, Steimley, E. L. Thompson, Townsend, and Wahlin.

The following program was given: (1) Address: H. E. Slaught, President of the Mathematical Association of America. (2) Round Table Discussion of the Topic:—Freshman Mathematics as related to varying admission credits from the high schools. Discussion led by C. E. Comstock, Bradley Polytechnic Institute, M. W. Coultrap, Northwestern College, G. C. Heritage, Crane Junior College. (3) Address: The Training of Mathematics Teachers, E. B. Lytle, University of Illinois. Discussed by Malcolm McNeill, Lake Forest College, Bessie I. Miller,

Rockford College, W. J. Risley, James Millikin University. (4) Address: How Mathematicians Work, Henry Blumberg, University of Illinois. (5) Business Meeting.

President Slaught sketched the history of mathematics teacher organizations in the United States leading up to the need and opportunities for service of this new Illinois Section. He made an appeal for the hearty coöperation of all college teachers of mathematics in Illinois.

Professor Comstock gave in detail the mathematics preparation of freshmen in Bradley Polytechnic Institute and some other Illinois colleges. He recommended better provision for students entering college with only one unit credit in high-school algebra, a more uniform standard for one unit in algebra, more attention to solid geometry in colleges, and less emphasis upon complicated symbolic manipulations. The general discussion of admission credits was lively and full of interest; all seemed to feel that algebra preparation was very unsatisfactory and numerous pleas for less formal work and more emphasis of general principles and methods were made.

By request, Dr. Lytle read his Ann Arbor address on the training of mathematics teachers; this address appears in full in the present number of the MONTHLY. The discussion put emphasis on the point of giving teachers more explicit training in the use of libraries, and on more independent preparation and forceful presentation of reports on special topics; Miss Miller offered a definite scheme for carrying out and grading such work.

Professor Blumberg described numerous methods of work, both humorous and serious, used by certain well-known mathematicians and concluded with a few generalizations on effective methods of work in the field of mathematics.

The old officers were reëlected for one more year; Chairman, J. A. FOBERG, Crane Junior College, Secretary-Treasurer, E. B. LYTLE, University of Illinois, members of Executive Committee, C. E. COMSTOCK, Bradley Polytechnic Institute, G. T. SELLEW, Knox College, L. S. SHIVELY, Mount Morris College. The question of the place and time of the next meeting was discussed and finally referred to the Executive Committee with power.

At the close of the program a group took luncheon together at the Illinois Union building.

ERNEST B. LYTLE, *Secretary-Treasurer.*

1820

First meeting of the Royal Astronomical Society, London, January 12; V. A. Puiseux, French geometer and analyst, born April 16; John Casey, Irish mathematician, born May 12; J. P. E. de Fauque de Jonquières, French mathematician and vice-admiral, born July 3; W. J. Rankine, Scotch engineer, born July 5; J. C. Houzeau de la Haye, Belgian bibliographer of astronomy, born October 7.

QUESTIONS AND DISCUSSIONS.

Edited by W. A. Hurwitz, Cornell University, Ithaca, N. Y.

A number of questions have been standing in this department for some time with no replies or with replies which still leave something to be desired in the way of completeness or finality. Several of these are reprinted below, with the intention of directing our readers' interest to them again, and stimulating replies. To those which have already received some attention short notes are appended, indicating the extent to which they still remain open for consideration.

QUESTIONS.

15. In the *Proceedings of the Royal Society of Edinburgh*, vol. 7, p. 144, in some mathematical notes by Professor P. G. Tait, it is stated:

"If $x^3 + y^3 = z^3$, then $(x^3 + z^3)^3y^3 + (x^3 - y^3)^3z^3 = (z^3 + y^3)^3x^3$.

"This furnishes an easy proof of the impossibility of finding two integers the sum of whose cubes is a cube."

How does this "easy proof" follow?

A partial reply to the above has been received showing that if $x^3 + y^3 = z^3$, then $(x^3 + z^3)^3y^3 + (x^3 - y^3)^3z^3 = (z^3 + y^3)^3x^3$. Can some one show how the "easy proof" then follows?

21. For the Diophantine equation

$$x^2 - y^3 = 17$$

there are known the following solutions:

$$\begin{array}{lrrrrrrrr} x = & 3, & 4, & 5, & 9, & 23, & 282, & 375, & 378661, \\ y = & -2, & -1, & 2, & 4, & 8, & 43, & 52, & 5234. \end{array}$$

One of our readers, who supplied the foregoing facts, desires to know the answers to the following questions: Are there other solutions of the given Diophantine equation? How may all the solutions of this equation be found by a systematic procedure?

In a reply to this question published in the Monthly for June, 1919, E. B. Escott showed how to obtain an infinite number of rational solutions of the equation, and in particular all the integral solutions contained in the above list. References to the literature given in connection with this reply indicate the existence of an infinite number of integral solutions. Thus one question explicitly raised regarding the equation still awaits an answer.

30. A certain Normal University wishes to offer thirty-five hours of college mathematics for the benefit of high-school teachers. What should these courses be in order that, primarily, they may be of the greatest value to high-school teachers of mathematics and, secondarily, that they may furnish stimulus for a more extended pursuit of the subject?

No reply to this question has been received. Some idea of conditions as they are may be obtained from the replies to another question in the Monthly for December, 1916, pp. 395–399.

34. Given the mixed integral and functional equation

$$\int_{x=0}^{x=h} f(x)dx = \frac{h}{6}\left[f(0) + 4f\left(\frac{h}{2}\right) + f(h)\right],$$

to determine the function $f(x)$. This equation is of rather fundamental practical value as it has to do with the most general solid whose volume is given by the prismatoid formula.

may violate all notion of logical precision. An annoying situation of a type less vicious than the cases enumerated by Professor Lovitt is brought about by a text-book which defines the principal value of an inverse function as the smallest positive value.

In the last discussion Professor Rees shows how the motion of a body acted on only by the force of gravity and a resistance proportional to velocity may be readily and easily studied by the use of differential equations and initial conditions in vector form. This paper affords a good instance of the economy, both in notation and actual work, resulting from the use of single vector relations in place of triplets of scalar relations.

I. Note on the Quadrature of the Parabola.

By Otto Dunkel, Washington University.

The article by Professor Moritz entitled "On the Quadrature of the Parabola" in the November issue of the Monthly has suggested to the writer to present another derivation of the same result, since, in addition to being fairly simple, this second method follows directly the classic process of defining an area as the common limit of an inferior and a superior sum. This development might be found easy enough to serve as an illustrative example in the presentation of the summation formula in the integral calculus.

Let the equation of the curve be $y = x^m$, $m =$ a positive integer, and suppose that it is desired to obtain the area between the curve, the x-axis and the ordinates at $x = a$ and $x = b$, where b is greater than a and both are positive. Divide the interval from a to b on the x-axis into n subintervals, equal or unequal, and upon them as bases erect two sets of rectangles, the one inscribed and the other circumscribed. The sums of the areas of these rectangles are respectively,

$$(1) \qquad I_n = \Sigma x_i^m(x_{i+1} - x_i), \qquad S_n = \Sigma x_{i+1}^m(x_{i+1} - x_i).$$

It will be shown that as n becomes infinite so that the length of the longest subinterval, δ, approaches zero, each of these sums approaches the same limit, which by the usual definition is the area desired. This limit will also be determined in the process.

It may easily be seen from a figure, especially when all the subintervals are equal, that

$$(2) \qquad \text{Limit } (S_n - I_n) = 0.$$

This also follows algebraically, for

$$S_n - I_n = \Sigma(x_{i+1}^m - x_i^m)(x_{i+1} - x_i) \leqq \delta\Sigma(x_{i+1}^m - x_i^m).$$

In the latter summation all the terms cancel except the first and last, and hence the difference, $S_n - I_n$, is less than or equal to $\delta(b^m - a^m)$. Thus it follows that (2) is true.

A quantity will now be determined which lies between I_n and S_n and is independent of n. It is clear that x_{i+1}^m, $x_{i+1}^{m-1}x_i$, $x_{i+1}^{m-2}x_i^2$, $\cdots$, x_i^m form a decreasing sequence of $m + 1$ positive terms and hence their arithmetic mean is greater than the smallest term x_i^m. Using this inequality it follows that

$$I_n < \Sigma(x_{i+1} - x_i)\,\frac{x_{i+1}^m + x_{i+1}^{m-1}x_i + x_{i+1}^{m-2}x_i^2 + \cdots + x_i^m}{m+1},$$

$$< \frac{1}{m+1}\,\Sigma(x_{i+1}^{m+1} - x_i^{m+1}) = \frac{b^{m+1}}{m+1} - \frac{a^{m+1}}{m+1}.$$

In a similar manner an inequality is found for S_n, and hence

$$0 < I_n < \frac{b^{m+1}}{m+1} - \frac{a^{m+1}}{m+1} < S_n. \tag{3}$$

By representing the three quantities I_n, $b^{m+1}/(m + 1) - a^{m+1}/(m + 1)$, S_n by points on a straight line and by considering the meaning of (2) and (3) as applied to these points, it will be obvious that the common limit of I_n and S_n is $b^{m+1}/(m + 1) - a^{m+1}/(m + 1)$. This then is the expression for the desired area. If the equation of the curve is $y = px^m$, it will be readily seen that the above result must be multiplied by p.

The same method, with a slight amount of extra manipulation, may be used for negative values of m and also for fractional values, excepting, however, the special case $m = - 1$.[1]

An elementary evaluation of the area of any segment of an ordinary parabola by means of special properties of the curve is given in the *Traité de Géométrie* by Rouché et Comberousse, 2d vol., p. 348 (8th ed., 1912). The properties here used are such as might be given in the ordinary text on analytics. A somewhat similar treatment occurs in the first volume of Goursat-Hedrick's *Mathematical Analysis*, p. 134, and is referred to as one of the processes used by Archimedes. Here the summation of a geometric series is employed, but this may be avoided and the proof simplified by comparing the areas of the interior triangles with certain corresponding exterior triangles. These two proofs are somewhat similar to the one employed by Professor Moritz.

II. Inverse Trigonometric Functions.

By W. V. Lovitt, Colorado College.

As I look over the available text-books on trigonometry the feeling grows upon me that they are hastily written and some topics inadequately treated. It is certain that many errors are present.

I have examined twenty-four different texts with special reference to their treatment of the inverse trigonometric functions. In five the treatment was so

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well to equation (2). For example, (2) is not true if arcsin 3/5 and arcsin 8/17 are angles whose terminal lines lie in the second quadrant.

Direction is given in some instances to prove the statement (1) by taking the tangent of both sides. Without some restrictions on the angles involved this does not prove the equality; else

$$60^\circ = 240^\circ$$

because it happens that

$$\tan 60^\circ = \tan 240^\circ.$$

Is it any longer a mystery why the student does not obtain a better grasp of his mathematics?

III. The Path of a Projectile when the Resistance Varies as the Velocity.

By E. L. Rees, University of Kentucky.

The very simple problem of finding the path (and its hodograph) of a projectile in a vacuum is treated in most elementary treatises on vector analysis, but the problem of finding the trajectory when the body moves in a resisting medium is usually not considered as the general problem does not lend itself readily to vector treatment. However, there is a special case of some interest which presents no difficulties. If we assume that the resistance varies as the velocity (which under certain conditions is approximately true for low velocities in the air) the vector differential equation of motion is of a very simple type and its solution differs in no essential way from that of a scalar differential equation of the same type. The equation of such a motion is

$$\frac{d^2\boldsymbol{r}}{dt^2} = \boldsymbol{g} - k\frac{d\boldsymbol{r}}{dt},$$

where $\boldsymbol{g}$ is the vector acceleration due to gravity, and k is a positive scalar const. This is a linear vector differential equation which may be integrated at once by applying the integrating factor e^{kt}.

Integrating we get

$$\frac{d\boldsymbol{r}}{dt} = \frac{\boldsymbol{g}}{k} + e^{-kt}\boldsymbol{C}_1$$

and

$$\boldsymbol{r} = \frac{\boldsymbol{g}t}{k} - \frac{e^{-kt}}{k}\boldsymbol{C}_1 + \boldsymbol{C}_2,$$

where $\boldsymbol{C}_1$ and $\boldsymbol{C}_2$ are the vector constants of integration.

Assuming $\boldsymbol{r} = 0$, $\boldsymbol{v} = \boldsymbol{v}_0$ when $t = 0$ we have

$$\frac{d\boldsymbol{r}}{dt} = \frac{\boldsymbol{g}}{k} + e^{-kt}(\boldsymbol{v}_0 - \boldsymbol{g}/k) \tag{1}$$

and

$$\boldsymbol{r} = \frac{\boldsymbol{g}t}{k} + \frac{1}{k}(1 - e^{-kt})(\boldsymbol{v}_0 - \boldsymbol{g}/k). \tag{2}$$

The first of these equations is the equation of the hodograph and the second is the vector equation of the trajectory.

The highest point reached by the projectile may be found by multiplying the first equation dotwise by $\mathbf{g}$ and setting $\mathbf{g}\cdot(d\mathbf{r}/dt) = 0$. Solving this equation for t we find $t = 1/k \log (1 - k\mathbf{v}_0\cdot\mathbf{g}^{-1})$, and this substituted in equations (1) and (2) gives the velocity and the position vector of the projectile at the highest point. Similarly, to find the time of flight and the range on the horizontal plane through the point of projection multiply the second equation by $\mathbf{g}\cdot$ and set $\mathbf{g}\cdot\mathbf{r} = 0$, solve for t, etc.

Since k is positive and t varies from zero to infinity e^{-kt} varies from 1 to 0 and the hodograph is seen to be the segment of a line which terminates at the tips of the vectors $\mathbf{v}_0$ and $\mathbf{g}/k$.

If the initial velocity is allowed to take all possible directions the hodographs will form a bundle of line segments extending from the tip of $\mathbf{g}/k$. If the initial speed is constant the segments will terminate in the surface of a sphere with center at the origin and radius v_0. If the angle of projection varies in a vertical plane and the initial speed is constant the segments will form a pencil one end of each segment resting on the tip of $\mathbf{g}/k$ (vertex of pencil) and the other on a circle of radius v_0. The vertex of the bundle or pencil corresponds to the limiting velocity which is the same for all initial velocities.

The cartesian equation of the trajectory is found from the vector equation in the usual way to be

$$y = \frac{g}{k^2}\log(1 - x/l) + \frac{mx}{l},$$

where

$$l = \frac{v_0 \cos\alpha}{k} \quad \text{and} \quad m = \frac{v_0 \sin\alpha}{k} + \frac{g}{k^2},$$

α being the angle of projection.

From this equation it is seen that the trajectory has an asymptote whose equation is $x = l$, which accords with the known fact that the trajectory of a body moving in any resisting medium has a vertical asymptote.

RECENT PUBLICATIONS.

REVIEWS.

Cajori's History of Mathematics.

A History of Mathematics. By Florian Cajori. New York, The Macmillan Company, 1919; pp. x + 514. Price \$4.

It is twenty-five years since Professor Cajori, then already known for his contributions in the field of mathematical history, published the first edition of the work under review. During this period the book has been reprinted six times, and the author and publisher have now united in giving to students a second edition, to a considerable extent rewritten and in certain chapters materially enlarged.

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The general plan of the first edition has been followed, the work being divided into three rather distinct parts, although not in any formal manner. The first part may be described as relating chiefly to the development of elementary mathematics and including the following main topics: the Babylonians, the Egyptians, the Greeks, the Romans, the Maya, the Chinese, the Japanese, the Hindus, the Arabs, and Europe during the Middle Ages. This part occupies 129 pages, or a little more than one fourth of the text. All things considered, this is a fair allotment of space.

The section relating to Europe during the sixteenth, seventeenth, and eighteenth centuries may be considered as the second part and may be described as concerned with the rise of higher mathematics. Naturally a large part of the work of these centuries was elementary, pertaining to the establishing of the present algebraic symbolism, to the discovery of the common theorems relating to equations, and to the better understanding of the nature of series; but the birth of higher mathematics, including modern higher geometry, the group theory, the general notion of functions, and the laying of scientific foundations, may also be said to have taken place in this period. To these centuries the author has assigned 148 pages, or somewhat more than 30 per cent of the text,—again a fair distribution of space.

The rest of the work is devoted to the period from 1800 to the present time, and includes 209 pages, or upwards of 43 per cent of the text. Roughly speaking, the space assigned to each of the three periods is in the ratio of 25 : 31 : 44, and for the purposes of students who will use this book the ratio is both significant and reasonable.

The first part of the work may be described as giving a racial view of the development of mathematics. This is natural and, for practical purposes, is necessary. Commerce had not yet come to the aid of scholars in the free transmission of thought, and racial characteristics predominated in science as they did in art, religion, and modes of government. In treating of Europe in the Middle Ages, however, Professor Cajori abandons the attempt to classify by races, tending apparently, and with some reason, to classify most of the scholars as churchmen and to consider Alcuin, Gerbert, Johannes Hispalensis, and Leonardo Fibonacci as belonging to the same branch of the human family tree.

In the second part of the work the tendency is to classify either by centuries or by periods which may be characterized by the influence of such men as Vieta, Descartes, Newton, and Euler. This is a natural and allowable treatment, although not the classification which some of those who give courses in the subject may prefer.

In the third part of the work the classification is by such subjects as synthetic geometry, analytic geometry, and algebra. Such a classification is, for the instructor in mathematics, the most helpful of all, and many readers will wish that Professor Cajori might have seen his way to extending it so as also to include the mathematical topics studied in the elementary school, the high school, and the junior college.

Taking the work as a whole, the classification of material is justifiable and the amount of information exceeds that to be found in any other general history of mathematics that has thus far appeared in English.

Even though one may read the book with a friendly eye it is probable that he will find himself giving expression to regret from time to time that Professor Cajori should have adopted certain forms of expression or have made certain definite assertions that are open to question. If a critic happens to have the audacity of youth, he will perhaps find himself giving a freedom to his pen that he will not approve in his later years. If he has himself yielded to the lure of authorship, he will remember Cowper's line,

None but an author knows an author's cares,

and will seek to be more charitable. When we consider that the first volume of Cantor's monumental work appeared forty years ago, and that for the last twenty years there has been maintained in the third series of the *Bibliotheca Mathematica* a department devoted to a severe criticism of that great treatise, we realize how easy it is to find fault with any book. Indeed, it often seems as if it requires more genius on the part of a critic to refrain from petty faultfinding than it does to record a hundred slips of the memory, of the judgment, or of the pen.

The reviewer now proposes to mention certain typical excellencies of Professor Cajori's work, and then to mention certain types of defects, leaving the details of these defects for others to find, or for such helpful criticism as may be acceptable when another edition is under consideration.

Among the features which will render the work helpful to students of the history of mathematics there may first be mentioned the considerable bibliography given in the footnotes. Such assistance is valuable not merely in supporting some particular assertion of the author's but in showing to the student that there is an extensive literature awaiting him in case he cares to pursue the study further. It is not necessary that the books referred to should all be easily accessible, and in this case some of them are not; so long as they are worthy they act as a stimulus to further study, and on this account the list that has been given might profitably have been much longer.

Another feature that seems to this reviewer very fortunate is that of quoting from the original source in various cases in which the language is particularly significant. For this purpose Professor Cajori has assumed on the part of his readers some knowledge of French, Latin, and German, and it will not often be the case that students are so ignorant of at least two of these languages as not to profit by the quotations given. There is, however, a stronger reason for commending the plan, namely, that the student is thereby encouraged to feel that he is prepared to go to the original sources to secure first-hand information. This reviewer would have advised even more, much more, of this style of quotation, placing the material in footnotes so as not to disturb the reader who can not use it, and not hesitating to include easy Italian; but this is merely a matter of personal preference; the significant thing is that the step has been taken.

A third commendable feature is that of revealing the human side of those scholars whose names are mentioned, thus showing the history of mathematics as the history of men. Plutarch was none the less of a historian because he was the world's greatest biographer. Professor Cajori has given a great amount of information about people, and in the main this information has been well chosen.

When it comes to a consideration of types of those features which may be classed with "things one would rather have left unsaid," perhaps the first general criticism in the minds of most readers will relate to the style in which the book is written. "It is style alone," said the elder Disraeli, "by which posterity will judge of a great work, for an author can have nothing truly his own but his style." While this assertion is open to the criticism that applies to most epigrams, no one can read the treatise under review without sympathizing with Voltaire's advice, "In every author let us distinguish the man from his works." Professor Cajori does not talk as he too often writes. His habit of writing so many sentences in a kind of inverted order will probably annoy most of his readers. While often an aid to emphasis, or a desirable variant to one's usual style, or a means of connecting two sentences euphonically, this order is used so frequently by Professor Cajori as to detract from the clearness of the expression and to become an unfortunate mannerism. Thus we have within a few lines of each other, the sentences, "A profound scholar . . . and a mathematician . . . was Pierre de Fermat;" "A great contribution to geometry was his *De maximis et minimis;*" and "A contemporary mathematician . . . was Blaise Pascal." Sometimes this unfortunate habit even leads to temporary confusion, as in the sentence beginning, "About forty years after Archimedes flourished Apollonius . . .," in which the reader must go to the end of the sentence to find which name serves as the subject in the phrase quoted. Occasionally the skein becomes even more tangled, as in the sentence, "Greater taste than for geometry was shown by the Hindus for trigonometry"—a sentence which can easily be parsed, but which is nevertheless unfortunately expressed. Such constructions are an author's own property; he has a legal right to his own style, and a moral right as well. Nevertheless one cannot read the book without frequently being conscious of the remark of Disraeli's quoted above. It is a safe assertion that Professor Cajori himself would be greatly surprised to know the number of times this peculiarity of style asserts itself. Somehow, too, this reviewer does not connect the man who wrote this book with a sentence like the following; "We abstain from introducing additional Greek opinion regarding Egyptian mathematics, *or from indulging in wild conjectures.*" The sentence is perfectly grammatical, but it lacks the dignity which characterizes the author.

Somewhat related to the matter of style is the frequent use of a man's initials when there seems to be nothing gained. This is seen, for example, in the sentence (p. 222) relating to Jean Bernoulli: "Johann admired the merits of G. W. Leibniz and L. Euler, but was blind to those of I. Newton." If there were any other Leibniz, Euler, or Newton whom the reader would naturally have in mind, the use of the initials, while unusual, might be understood; but to see such initials

with about half the names in the book is a matter of some annoyance. If the dates had been so frequently given with the names as to impress them on the mind, or to aid the reader to place certain men in proper chronological relation to others, there would have been a decided gain to counterbalance any annoyance arising from the appearance of the page; but one would as soon refer to W. Shakespeare and D. L. George as to I. Newton. One of the curiosities of this method of reference is seen in the case of P. Mersenne (p. 156) whose first name was Marin (p. 163), the P. evidently standing for Pater or Père.

It may be pardonable to mention also, in connection with the matter of style, the author's habit of referring to a writer whose biography has not already been given, or, indeed, to one who is quite unknown and who is never mentioned again. The student, who reads on page 32 what "H. G. Zeuthen finds," does not learn who H. G. Zeuthen is until he reads page 190; and when he reads (p. 44) the opinion of Venturi, he may search the book in vain to find who Venturi was or what his opinion may be worth. The same may be said of Thymaridas (pp. 59, 111), and a similar criticism may be made with respect to the Palatine Anthology, a work of which the student is quite certain to be ignorant. In this connection it may be suggested that it would have been better had "Thomas Finck, a native of Flensburg" (p. 151) been spoken of with reference to his nationality, since few readers will have the slightest idea in what country Flensburg is situated.

A second point of friendly criticism, for we should all be friendly in recognizing the amount of labor put upon this work, may properly relate to the statements of fact. Here the critic of the Eneström type will find more than this reviewer wishes were possible. A few of these statements will suffice to make clear the assertion that the book is not without its errors. That the Babylonian notation employed only two principles—"the additive and multiplicative" (p. 4) is itself contradicted on the same page in the statement that the subtractive principle was also employed. Indeed it should be said that the evidence of this fact is not confined by any means to the tablets found at Nippur. The frequent assertion that the Babylonians used sexagesimal fractions will be misunderstood; we have no evidence of such fractions before the Greeks introduced them into astronomy, even though the Babylonian use of sixty as a kind of radix in notation is well established, not merely by "two Babylonian tablets" but by many others. The fact that 30 sometimes meant $\frac{30}{60}$ does not illustrate the use of what the reader will understand by sexagesimal fractions, namely, degrees, minutes, seconds, thirds, and so on.

The date of Ahmes is, of course, still a matter of conjecture. When, however, it is given as "some time before 1700 B.C.," and when it next appears in the phrase "the incorrect formulæ of Ahmes of 3000 B.C.," and when it is next referred to as 3,000 years before Leonardo Fibonacci wrote his *Liber abaci* (1202 A.D.), it is safe to say that students will regret that certain assertions are made so positively.

The statement that "to Egypt Greece" is indebted for elementary geometry

(p. 15) is, of course, a matter of definition; but it will be misleading to the average reader until he has read five lines further on the page. The positive statement that Pythagoras sacrificed a hecatomb (p. 2) is not placed in doubt until the student has proceeded sixteen pages further in his reading. The statement that the method of exhaustion infers that the *circle* is exhausted (p. 23) is later corrected (p. 25) by the assertion that it is the "spaces between the polygons and circumferences" that are exhausted.

The statement that Plato did not acquire his taste for mathematics from Socrates (p. 25) is a rather strong one. In Dr. Jowett's delightful paraphrase of the *Theætetus* it is said that "he, Socrates, is a midwife, although this is a secret; he has inherited the art from his mother bold and bluff, and he ushers into light, not children, but the thoughts of men." Who knows how many thoughts of Plato were delivered by Socrates in this quasi-obstetric capacity?

That "Athens produced the greatest scientists and philosophers of antiquity" (p. 29) is a statement that will probably be criticized. As a matter of fact, it is always a cause for surprise that continental Greece produced so few scholars of this type in comparison with the islands of the Ægean Sea and with the Greek colonies.

Speaking of a space of 150 years, the statement that Sextus Julius Africanus was "an occupant of this long gap" sounds rather odd, although it is not probable that anyone will misinterpret it. That Pascal did not lead a "quiet and unaggressive life" (p. 163) will be an unpleasant surprise for most readers, and it would be interesting to know just what extended period of his life Professor Cajori had in mind in making the assertion.

Among the errors of fact there should be mentioned the subject of dates, already referred to in the case of Ahmes. It is not a matter of great importance to most readers whether Thales was born in 640 B.C. and died in 546 B.C., or whether the dates should be c. 640 and c. 542 respectively; and whether the eclipse which he prophesied took place in 585 B.C., or, as some writers assert, twenty-four years earlier. All that the general reader ordinarily cares for is an approximate date. Thus when the dates "580?–500? B.C." are given for Pythagoras, the purposes are sufficiently served. If, however, the dates for Pythagoras are queried, why should not the three dates for Thales also be given as doubtful? This is a work of reference and it is desirable that such statements of fact be accurate and that a doubt be expressed where one is generally supposed by good writers to exist. The same criticism applies in many cases throughout the book, including those relating to Anaxagoras, Archytas, Plato, Bede, Eratosthenes, Tabit ibn Korra, Hudde, and others.

As to the transliteration of Oriental names the author is no more at fault than most other historians. It is unfortunate that we have no board that can consider the whole matter with some semblance of authority and give us a standard system of diacritical marks and of the English equivalents of proper names as they appear in languages not using the Latin alphabet. If such forms as Ś, ṭ, ḥ, ī, and al- are to be used, as they are in this book, it is due to the reader

usual in English works, and also in the curious guise of C. S. Clavio (p. 184.) Mention should also be made of the genitive forms of certain proper names. Shall we expect Wallis' or Wallis's, where the name is not a plural form? There is authoritative sanction for either, but there is no sanction for using the two forms indiscriminately as is the case with various names throughout this work.

With respect to the index there should be mentioned the fact that it is unusually complete and that it has one decided improvement over most efforts of this kind. This improvement consists in giving the most important reference first, and in the use of subtitles under the heading entries, each of which is a great convenience. There are various omissions, such as Coss, Harun al-Rashid, Fine (O., but see Orontius), and Salviati, and there are a few misspelled names, like Durege for Durège, but on the whole there is no reasonable cause for complaint.

In closing this review it should be said once more that the purpose of the writer has not been to find fault with respect to small details. He has endeavored to call attention to the distinctively good features of the work and to state only the leading features that are likely to be subjects of adverse criticism. If the proper limits of a review of this kind had permitted, he would have been glad to discuss such questions as the Roman use of 120 as a diameter, the rise of the sexagesimal fraction, and the causes of the prominence of mathematics in various epochs and in various countries, and to commend specifically certain other features of value. He wishes, however, to add the statement that the book shows a wide reading of secondary material and a considerable study of original sources. It will, of course, have a place in every mathematical library of any importance and will prove a helpful work of reference, particularly with respect to the mathematics of the last two centuries.

David Eugene Smith.

Report of the President of the Carnegie Institution of Washington for the year ending October 31, 1919. Washington, D. C., 1919. 4to. 37 pp.

For the mathematician perhaps the most interesting items in this report by President R. S. Woodward, a charter member of the Association, are the paragraph concerning the Hooker telescope on Mount Wilson (pages 16–17) which "under repeated tests has proved efficient quite beyond the conservative theoretical predictions of attainable capacities," and the following paragraph (pages 18–19):

"The most elementary, the most essential, and hence the most widely used, if not esteemed, of the sciences is arithmetic. It is a fundamental requisite, in fact, of all exact knowledge. Ability to add, subtract, multiply, and divide affords probably the simplest test of capacity for correct thinking. Conversely, inability or indisposition to make use of these simple operations affords one of the surest tests of mental deficiency, as witnessed, for example, by numerous correspondents who are unable to or who refuse to apply these operations to the finances of the Institution. But the familiar science of arithmetic lies at the foundation also of a much larger and a far more complex structure called the theory of numbers. This theory has been cultivated by many of the most acute thinkers of ancient and modern times. It has more points of contact with quantitative knowledge in general than any other theory except the theory of the differential and

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gives a good example of the former class; a ball of the latter"); synectic (B. Williamson in *Encycl. Brit.* vol. 24, 1888, p. 72 "A function of a complex variable which is continuous one-valued, and has a derived function when the variable moves in a certain region of the plane is called by Cauchy Synectic in this region"); syntax; syntheme (Sylvester, 1844, *Coll. Math. Papers,* 1904, vol. 1, p. 91 "Let us agree to denote by the word syntheme any aggregate of combinations in which all the monads of a given system appear once and once only. . . . Let us begin with considering the case of duad synthemes"); synthetic; syntractrix and syntractory (G. Peacock, *Examples Diff. Calc.,* 1820 and G. Salmon, *Higher Plane Curves,* 1852); systatical (Jeake, *Arithmetic,* 1674, p. 662 "Three . . . is called a Systatical or Substantial Number, because all Sublimary Bodies consist of the three principal Substances, Sal, Sulphur, and Mercury") [obsolete]; syzygant; syzygetic and syzygy (Sylvester, *Cambr. and Dublin Math. Journal,* vol. 5, 1850, p. 276 "The members of any group of functions, more than two in number, whose nullity is implied in the relation of double contact, . . . must be in syzygy. Thus, *PQ, PQR, QR,* must form a syzygy").

It may be remarked that the editors of 'N.E.D.' have overlooked three words used in mathematical literature, namely: symptose, syntypic, and syrrhizoristic. The first two occur in the index to Cayley's *Coll. Math. Papers,* 1898, on p. 135. (The use of the term syntype, in natural history, *is* presented in 'N.E.D.'). Sylvester introduced the word syrrhizoristic (*Philosophical Trans.,* vol. 143, 1853, and *Coll. Math. Papers,* vol. 1, p. 585 "A syrrhizoristic series is a series of disconnected functions which serve to determine the effective intercalations of the real roots of two functions lying between any assigned limits").

While 'N.E.D.' lists several meanings of the word systatic, there is no reference to its use in mathematics. Have this word and asystatic (not in 'N.E.D.') ever been used as mathematical terms in English writings? They are familiar to the Frenchman (*Encyclopédie des sciences mathematique,* tome 2, volume 4, p. 224: "groupes systatiques et asystatiques"), to the German (Lie-Engel, *Transformationsgruppen,* Band 1, Leipzig, 1888, Kapitel 24: "Systatische und asystatische Transformationsgruppen," pp. 497–522), and to the Italian (L. Bianchi, *Lezioni sulla teoria dei gruppi continui,* Pisa, 1918, p. 185: "gruppi sistatici ed asistatici").

The Physical Society of London. *Report on the Theory of Gravitation.* By A. S. EDDINGTON, London, Fleetway Press, 1918. 8vo. 7 + 91 pp. Price, in paper, 6s. 3d.

Quotation from the Preface: "The relativity theory of gravitation in its complete form was published by Einstein in November, 1915. Whether the theory ultimately proves to be correct or not, it claims attention as one of the most beautiful examples of the power of general mathematical reasoning. The nearest parallel to it is found in the applications of the second law of thermo-dynamics, in which remarkable conclusions are deduced from a single principle without any inquiry into the mechanism of the phenomena; similarly, if the principle of equivalence is accepted, it is possible to stride over the difficulties due to ignorance of the nature of gravitation and arrive directly at physical results. Einstein's theory has been successful in explaining the celebrated astronomical discordance of the motion of the perihelion of Mercury, without introducing any arbitrary constant; there is no trace of forced argument about this prediction. It further leads to interesting conclusions with regard to the deflection of light by a gravitational field, and the displacement of spectral lines on the sun, which may be tested by experiment.

"The arrangement of this report is guided by the object of reaching the theory of these crucial phenomena as directly as possible. To make the treatment rather more elementary, use of the principle of least action and Hamiltonian methods has been avoided; and the brief account of these in Chapter VII is merely added for completeness. Similarly, the equations of electrodynamics are not used in the main part of the Report. Owing to the historical tradition, there is an undue tendency to connect the principle of relativity with the electrical theory of light and matter, and it seems well to emphasize its independence. The main difficulty of this subject is that it requires a special mathematical calculus, which, though not difficult to understand, needs time and practice to use with facility. In the older theory of relativity the somewhat forbidding vector products and vector operators constantly appear. Happily this can now be avoided altogether; but in its place we use the absolute differential calculus of Ricci and Levi-Civita."

Contents—I (Pages 1–13): The restricted principle of relativity; II (14–29): The relations of space, time, and force; III (30–40): The theories of tensors; IV (41–47): Einstein's law of gravitation; V (48–58): The crucial phenomena; VI (59–70): The gravitation of a continuous distribution of matter; VII (71–81): The principle of least action; VIII (82–91): The curvature of space and time.

NOTES.

The works of EVANGELISTA TORRICELLI, "edite in occasione del III centenario della nascita col concorso del Comune di Faenza da Gino Loria e Giuseppe Vassura," have been published in three volumes (about 1800 pages, Faenza, G. Montanari, 1919; price 60 lire). There is a valuable "Introduzione" (pages iii–xxxviii of the first volume) by Loria.

Vuibert (Paris) published in 1919 the first volume of a three-volume work by H. BROCARD and T. LEMOYNE entitled: *Courbes géométriques rémarquables (courbes spéciales) planes et gauches*. The volume contains 460 royal-octavo pages and is listed at 18 francs. It will soon be reviewed in this MONTHLY.

The Hydrographic Office, Washington, has recently published: *General Catalogue of Mariner's Charts and Books*, corrected to April 1, 1919 (293 pages). In Special Publication no. 60 of the U. S. Coast and Geodetic Survey, Mr. O. S. ADAMS makes *A Study of Map Projections in General* (24 pages). The author states that

"an attempt has been made to treat in simple form some of the fundamental ideas that underlie the subject of map projections in general. There has been no intention to develop any phase of the subject at any length, but merely to give briefly some suggestions under the different headings that, it is hoped, may be found helpful to those who wish to get an understanding of the subject."

The issue of *Nature* for November 6, 1919, was a "jubilee number" (84 pages), and contained about forty brief articles concerning progress in various phases of science. Sir Norman Lockyer, the founder of the journal, in November, 1869, wrote "Valedictory Memories," and H. Deslandres, director of the Astrophysical Observatory of Mendon, wrote the sketch of Sir Norman (of whom there is a fine portrait supplement) for the "Scientific Worthies" series. The article on "Science and the Church" is by Canon J. M. Wilson who states that he "was a fair mathematician" fifty years ago (he was a senior wrangler). Readers of *Euclid and his Modern Rivals* (1879, second edition, 1885) will recall that C. L. Dodgson (Lewis Carroll) and De Morgan found much to criticize in the Canon's

Elementary Geometry . . . following the Syllabus of Geometry prepared by the Geometrical Association.[1] His *Solid Geometry and Conic Sections* was familiar to a good many American students of mathematics of twenty-five years ago.

The *Journal für die reine und angewandte Mathematik* was founded by Crelle in 1826 with Gergonne's *Annales de mathématiques pures et appliquées* as a model. Success in the undertaking was partly assured through an arrangement he made with the Kultusministerium whereby he was allowed to add to the title of the *Journal* the words to be found even in volume 146, 1915: "Mit tätiger Beförderung hoher Königlich Preussischer Behörden." (It is a sign of the times that this legend does not appear in volume 149, 1919.) The "Beförderung" consisted on the one hand in issuing strong official recommendation of the *Journal* not only to universities, and institutions of a similar nature, but also, for example, to government boards and, through Prussian ambassadors, to foreign countries. On the other hand the "Beförderung" involved the purchase of a number of copies of the *Journal* which were distributed to various schools—a custom prevailing to very recent times. In this way Weierstrass, for example, while a gymnasium pupil, received inspiration by discovering an uncut copy of the *Journal* "mit den schönen Abhandlungen von Steiner, von denen auch ein Primaner etwas verstehen konnte."

ARTICLES IN CURRENT PERIODICALS.

ALUMNI BULLETIN, College of St. Thomas, St. Paul, Minn., volume 3, no. 1, February, 1919: "Rev. William Earnshaw Etzel," 14–15 [full page portrait, 14; died February 3, 1919; charter member of the Mathematical Association of America].

AMERICAN CATHOLIC QUARTERLY REVIEW, Philadelphia, volume 44, January, 1919: "Catholic church and the gentle science of numbers" by E. Von R. Wilson, 121–145.

AMERICAN MACHINIST, New York, volume 51, October 9, 1919: "The equation of the involute simplified" by N. Finkelstein, 693–694; "Cam design and construction" by F. De R. Furman, 695–698; "A little question in trigonometry" by K. H. Condit, 713–714.

AMERICAN STATISTICAL ASSOCIATION, Quarterly Publications, Boston, volume 16, September, 1919: "On functional relations for which the coefficient of correlation is zero" by H. L. Rietz, 472–476.

BULLETIN DES SCIENCES MATHÉMATIQUES, volume 54, May, 1919: Review by E. Picard of G. H. Halphen's *Oeuvres*, tome 2 (Paris, 1918), 105; "Mémoire relatif à l'étude des substitutions rationnelles à une variable" by G. Julia, 106–109; Review by G. Bigourdan of E. Belot's *L'origine des formes de la terre et des planètes* (Paris, 1918), 110–112; Review by R. le Vavasseur of F. X. Kugler's *Die Babylonische Mondrechnung* (Freiburg 1. B., 1900), *Sternkunde und Sterndienst in Babel-Assyriologische, astronomische und astralmytologische Untersuchungen* (Münster i. W., 1907–1913), and *Im Bannkreis Babels* (Münster i. W., 1910), 112–118; Review by E. Cahen of O. Hölder's *Die Arithmetik in Strenger Begründung* (Leipzig, 1914), 118–121; Review by P. du Plessis of H. von Sanden's *Praktische Analysis* (Leipzig, 1914), 121–124; "La série $\frac{1}{5}+\frac{1}{7}+\frac{1}{11}+\frac{1}{13}+\frac{1}{17}+\frac{1}{19}+\frac{1}{29}+\frac{1}{31}+\frac{1}{41}+\frac{1}{43}+\frac{1}{59}+\frac{1}{61}+\cdots$ où les dénominateurs sont 'nombres premiers jumeaux' est convergente ou finie" (suite et fin) by V. Brun, 124–128.—June: Review of E. J. Boudin's *Leçons de calcul des probabilites* faites à l'Université de Gand de 1846 à 1890, publieés avec des Notes et des additions par P. Mansevi (Paris, 1916), 129–133; Review by G. Loria of P. Duhem's *Système du monde*, tome 5 (Paris, 1917), 133–135; Review by E. Lebon of F. Frenet's *Recueil d'exercices sur le calcul infinitésimal* (seventh edition, Paris, 1917), 135–136; Review by E. Ouivet of A. Schoenflies's *Entwicklung der Mengenlehre und ihrer Anwendungen*, part 1 (Leipzig,

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Bulletin of the American Mathematical Society, volume 26, no. 2, November, 1919: "The twenty-sixth summer meeting of the American Mathematical Society" by E. J. Moulton, 49–66; "Form of the number of subgroups of prime power groups" by G. A. Miller, 66–72; "On the rectifiability of a twisted cubic" by T. Hayashi, 73–75; "Some generalizations of the satellite theory" by R. M. Winger, 75–79; Review by G. A. Miller of F. Cajori's *A History of Mathematics* (2d edition, New York, 1919), 79–85; Reviews by R. D. Carmichael of W. Ahrens's *Mathematische Spiele* (3d edition, Leipzig and Berlin, 1916) and of Por J. de Mendizabal Tamborrel's *Tratado Elemental de Goniometria* (2d edition, Mexico, 1917), 86; Notes, 87–93; New Publications, 94–96.

Contemporary Review, volume 116, December, 1919: "Einstein's theory of space and time" by A. S. Eddington, 639–643.

Educational Review, volume 58, October, 1919: "The new comedy of errors" by R. E. Moritz, 219–238 [Motto: "'Every absurdity has a champion to defend it, for Error is always talkative,' Goldsmith." First paragraph: "It is no uncommon experience for a teacher who stresses mental training and the formation of proper habits of thought to be reminded by such of his colleagues as have enjoyed the advantages of recent courses in pedagogy that he is educationally behind the times; that the doctrine of mental training is a superstition or an 'exploded myth'; that mental faculties can not be stimulated or strengthened, since modern psychology has shown that no such faculties exist; that modern educationists recognize the 'basis of content' as the only sound basis of education, and other similar catch phrases which permeate the educational literature of recent years."]—December, 1919: "The contribution of mathematics to world progress" by C. N. Moore, 413–419 [First paragraph: "One of the compensating features of a great war may be found in the light that it sheds on relative values. A nation under the stress of battle for its existence or its dearest principles seeks the most efficient way of doing things with a directness that is very rare in time of peace. Thus it came about during recent months that persons with varying degrees of mathematical knowledge were in great demand for war activities of the first moment. Some superficial observers have drawn from this fact the totally false conclusion that the applications of mathematics in the conduct of modern warfare are far more numerous and more important than its application to the tasks of peace. They have overlooked the real explanation, namely, that the tremendous urge of war necessity secured for mathematics a wide-spread recognition of its true importance in many technical and scientific undertakings."]

The Journal of Educational Psychology, volume 10, nos. 5–6, May–June, 1919: "The effects of special drill in arithmetic as measured by the Woody and the Courtis arithmetic tests" by J. E. Evans and Florence E. Knoche, 263–276.

The Monist, volume 29, no. 4, October 1919: "Indefinables and indemonstrables in mathematics and theology" by P. E. B. Jourdain, 547–559; "Lotze's theory of the subjectivity of time and space" by J. E. Turner, 579–600; "Professor Russell's infinite" by H. H. Williams, 616–619; "Infinity and the part-and-whole axiom" by H. M. Gordon, 619–629.

Nature, volume 104, no. 2608, October 23, 1919: Review by G. B. M[athews] of H. B. Hedrick's *Interpolation Tables* (Washington, 1918), 152.

La Nature, volume 46, September 20, 1919: "Comment déchiffrer les textes en langage secret. La cryptophotie" by N. Flamel, 188–192—September 27: "Une machine a tracer les courbes" by H. Volta [Quotation: "Dans un article consacré au Bathyrhéomètre de M. J. Delage nous avions indiqué le principe du curieux appareil appelé par Lord Kelvin le '*tide predicter*' et qui permet de réaliser mécaniquement les compositions d'un numbre quelconque de fonctions harmoniques. Mais cet instrument dont la principale application est la détermination de la marée d'un lieu, est susceptible d'autres emplois, car, légèrement modifié, il trace mathématiquement avec une grande régularité des courbes de natures très diverses. C'est la machine à décrire des courbes due à M. Rigge que nons allons étudier d'après *Scientific American*." The article in question was "Compound harmonic motion" by W. F. Rigge, *Scientific American Supplement*, February 9–16, 1918, vol. 85, pp. 88–91, 108–110.]

PROBLEMS AND SOLUTIONS.

Edited by B. F. Finkel and Otto Dunkel.

[Send all communications about problems and solutions to B. F. FINKEL, Springfield, Mo.]

PROBLEMS FOR SOLUTION.

2814. Proposed by NATHAN ALTSHILLER, University of Oklahoma.

The bisectors of the angles formed by the diagonals of an inscribed quadrilateral are: (1) parallel to the lines joining the midpoints of the arcs subtended by the opposite sides of the quadrilateral on its circumcircle; (2) parallel to the bisectors of the angles formed by any pair of opposite sides of the quadrilateral; (3) equally inclined to pairs of sides of the quadrilateral.

2815. Proposed by the late L. G. WELD.

A right circular cone is laid upon an inclined plane so that its element of contact makes a given angle with the slant line of the plane. Assuming that there is no slipping and that the rolling friction is negligible, find the time of oscillation of the cone.

2816. Proposed by W. H. ECHOLS, University of Virginia.

Find two points D and E on the sides AB and CB, respectively, of a triangle ABC such that $AD = DE = EC$.

Give a rule and compass construction.

2817. Proposed by W. D. CAIRNS, Oberlin College.

The normal probability curve is sometimes called the binomial curve from the correspondence of its ordinates to the terms of $(1 + 1)^k$. Find an expression for the "mean deviation" of these terms from the median for the case where $k = 2m$. Assume the scale division as unity.

2818. Proposed by S. A. COREY, Des Moines, Iowa.

What is the maximum error that could occur in computing the common logarithm of $(1 + x)$ by the formula:

$$\text{Log}_{10}(1 + x) = .144{,}764{,}827\frac{x}{1 + x} + .579{,}059{,}309\frac{x}{2 + x} + .072{,}151{,}17\frac{x^2}{1 + x},$$

where $0 \leqq x \leqq 6/10$?

2819. Proposed by B. F. FINKEL, Drury College.

Find the equation of the envelope of the system of circles inscribed in a triangle with a given base and a given vertical angle.

2820. Proposed by C. B. HALDEMAN, Ross, Ohio.

Given one angle and the radii of the inscribed and circumscribed circles, to construct the triangle geometrically.

2821. Proposed by FRANK IRWIN, University of California.

The quantities $x_1, x_2, \cdots x_n$ vary in such a way that their sum (or any other one of the elementary symmetric functions) remains constant; investigate the maxima and minima of the remaining elementary symmetric functions.

SOLUTIONS OF PROBLEMS.

272 (Mechanics) [1913, 64; 1919, 213]. Proposed by J. F. LAWRENCE, Stillwater, Okla.

A perfectly rough circular cylinder is fixed with its axis horizontal. A sphere is placed on it in a position of unstable equilibrium, and projected with a given velocity parallel to the axis of

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2814. Proposed by NATHAN ALTSHILLER, University of Oklahoma.

The bisectors of the angles formed by the diagonals of an inscribed quadrilateral are: (1) parallel to the lines joining the midpoints of the arcs subtended by the opposite sides of the quadrilateral on its circumcircle; (2) parallel to the bisectors of the angles formed by any pair of opposite sides of the quadrilateral; (3) equally inclined to pairs of sides of the quadrilateral.

2815. Proposed by the late L. G. WELD.

A right circular cone is laid upon an inclined plane so that its element of contact makes a given angle with the slant line of the plane. Assuming that there is no slipping and that the rolling friction is negligible, find the time of oscillation of the cone.

2816. Proposed by W. H. ECHOLS, University of Virginia.

Find two points D and E on the sides AB and CB, respectively, of a triangle ABC such that $AD = DE = EC$.

Give a rule and compass construction.

2817. Proposed by W. D. CAIRNS, Oberlin College.

The normal probability curve is sometimes called the binomial curve from the correspondence of its ordinates to the terms of $(1 + 1)^k$. Find an expression for the "mean deviation" of these terms from the median for the case where $k = 2m$. Assume the scale division as unity.

2818. Proposed by S. A. COREY, Des Moines, Iowa.

What is the maximum error that could occur in computing the common logarithm of $(1 + x)$ by the formula:

$$\mathrm{Log}_{10}(1 + x) = .144{,}764{,}827\frac{x}{1 + x} + .579{,}059{,}309\frac{x}{2 + x} + .072{,}151{,}17\frac{x^2}{1 + x},$$

where $0 \leqq x \leqq 6/10$?

2819. Proposed by B. F. FINKEL, Drury College.

Find the equation of the envelope of the system of circles inscribed in a triangle with a given base and a given vertical angle.

2820. Proposed by C. B. HALDEMAN, Ross, Ohio.

Given one angle and the radii of the inscribed and circumscribed circles, to construct the triangle geometrically.

2821. Proposed by FRANK IRWIN, University of California.

The quantities $x_1, x_2, \cdots x_n$ vary in such a way that their sum (or any other one of the elementary symmetric functions) remains constant; investigate the maxima and minima of the remaining elementary symmetric functions.

SOLUTIONS OF PROBLEMS.

272 (Mechanics) [1913, 64; 1919, 213]. Proposed by J. F. LAWRENCE, Stillwater, Okla.

A perfectly rough circular cylinder is fixed with its axis horizontal. A sphere is placed on it in a position of unstable equilibrium, and projected with a given velocity parallel to the axis of

277 (Mechanics) [1913, 196; 1919, 213]. Proposed by W. J. GREENSTREET, Burghfield Common, Berks., England.

Around a smooth fixed circular pulley is wound a massless inextensible string, and straight portions go to two free ends A and B to which masses are fastened. The mass at A is initially projected perpendicular to the string while the other is initially at rest. The length of the straight portion to the first mass is initially l and subsequently is r. Find the velocity of the second mass at that moment.

SOLUTION BY J. B. REYNOLDS, Lehigh University.

Let the masses initially at A and B be n and n', respectively. Let z be the distance n' has moved and r the distance from the pulley to n at any subsequent time; θ, the angle that r makes with its initial position l. If T is the tension in the string, the equations of motion are

$$T = -n(\ddot{r} - r\dot{\theta}^2), \qquad \frac{d}{dt}(r^2\dot{\theta}) = 0, \qquad T = n'\ddot{z}.$$

The second of these gives, $r^2\dot{\theta} = \text{const} = v_0 l$, v_0 being the initial velocity of projection. Again, since $r = z + \text{constant}$, $\ddot{z} = \ddot{r}$ and eliminating T between the first and third equations, we have

$$(n + n')\ddot{r} = nr\dot{\theta}^2 = \frac{nv_0^2l^2}{r^3},$$

or, since

$$\ddot{r} = \dot{r}\frac{d\dot{r}}{dr},$$

we get upon integrating,

$$(n + n')\dot{r}^2 = -\frac{n\,v_0^2l^2}{r^2} + nv_0^2 = nv_0^2\left(1 - \frac{l^2}{r^2}\right).$$

Therefore, since the velocity $\dot{z}$ of $n' = \dot{r}$ at any time, we have

$$\dot{z} = \frac{v_0}{r}\sqrt{\frac{n}{n + n'}}\sqrt{r^2 - l^2}.$$

2700 [1918, 215; 1919, 215, 365]. Proposed by the late ARTEMAS MARTIN.

In a factory 250 men are paid an average wage of \$15 each per week. The men are paid unequally, the wages being \$20, \$16, \$10, and \$8 per week respectively, for different classes of work. How many are employed at each rate of pay?

NOTE.—I am told that this question was set in a Civil Service examination paper to be worked by arithmetic. 2,896 answers have been found. Are there any more?

III. REMARKS BY H. S. UHLER, Yale University.

On page 366 of the October issue the interesting suggestion is made by Professor D. N. Lehmer that: "The actual number might be obtained by counting the number of lattice points in this quadrilateral." I have followed this suggestion by counting along lines parallel to the axis of ordinates. Let the sides of the trapezium the equations of which are $3X + 2Y - 625 = 0$, $X - Y = 0$, $875 - 4X - 2Y = 0$, and $Y = 0$ be denoted respectively by I, II, III, and IV. The coördinates of the vertices are (125, 125), ($145\frac{5}{6}$, $145\frac{5}{6}$), ($208\frac{1}{3}$, 0), and ($218\frac{3}{4}$, 0). The number of lattice points limited by sides I and II is $1 + 3 + 6 + 8 + 11 + 13 + 16 + 18 + 21 + 23 + 26 + 28 + 31 + 33 + 36 + 38 + 41 + 43 + 46 + 48 + 51 = 541$. The number limited by sides I and III is $2(52 + 51 + \cdots + 23 + 22) + 21 = 2315$. The number limited by sides IV and III is $20 + 18 + \cdots + 4 + 2 = 110$. The total number of lattice points within and on the periphery of the trapezium is 2966, which is identically the value already given [*1919*, 216]. The numbers of points on sides I, II, III, and IV are 42, 21, 0, and 10, respectively. The sum of the distinct points, 72, verifies my earlier remark [*1919*, 216] that: "The number of solutions involving one or more zeros is 72." I should like to know whether there is any general significance to the fact that the "error" 36 [$2966 - 2929\frac{11}{16}$] is exactly one-half of 72, *i.e.*, the total number of peripheral solutions.

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Therefore, since the velocity $\dot{z}$ of $n' = \dot{r}$ at any time, we have

$$\dot{z} = \frac{v_0}{r}\sqrt{\frac{n}{n + n'}}\sqrt{r^2 - l^2}.$$

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The number of such inequalities may be found from the coefficient of $c_{1i}c_{2j}c_{3k}c_{4l}$ in the iterated series. The first subscripts of the ϕ's in the first term of the squared factor admit 24 permutations, while the second term may be formed from it in four ways, each of the operators K_1K_2 retaining one of the ϕ's associated with it in the first term and in the opposite position. The 96 results so indicated reduce however to 48 by reason of the permutability of the terms of the squared factor. There are then 48 inequalities of precisely the same type as the one in question; but there are others of the same general type corresponding to any two permutations of the subscripts 1, 2, 3, 4, in the terms of the squared factor, making in all $\binom{24}{2}$ or 276 results, exclusive of the useless one $0 \geqq 0$ which is obtained when the same permutation is used twice. Of these, 24 are reducible inequalities of the type

$$(J_{15}'J_{26}''J_{37}'''J_{48}^{\text{iv}} - J_{15}'J_{26}''J_{38}'''J_{47}^{\text{iv}} - J_{15}'J_{26}''J_{47}'''J_{38}^{\text{iv}} + J_{15}'J_{26}''J_{48}'''J_{37}^{\text{iv}})_{\kappa_1\kappa_2\kappa_1\kappa_2}$$
$$= (J_{15}'J_{26}''{}_{\kappa_1\kappa_1}) \times (J_{37}'''J_{48}^{\text{iv}} - J_{38}'''J_{47}^{\text{iv}} - J_{47}'''J_{38}^{\text{iv}} + J_{48}'''J_{37}^{\text{iv}})_{\kappa_2\kappa_2} \geqq 0,$$

in which also each factor of the left member is separately $\geqq 0$. The other 204 are irreducible inequalities not isomorphic with the proposed formula, and falling into five classes which may be indicated by the permutation of first subscripts $(abcd)$ which would be present in the second term of

$$K_1(\phi_1\phi_2)K_2(\phi_3\phi_4) - K_1(\phi_a\phi_b)K_2(\phi_c\phi_d)$$

(second subscripts being omitted) for a typical case of each class. That is, 96 correspond to the permutation (1342), 48 to (1432), 24 to (3412), 24 to (3421), and 12 to (2143).

2761 [1919, 124]. Proposed by W. W. DENTON, University of Michigan.

Find the lengths of the side of an equilateral triangle whose vertices are at given distances a, b, c from a given point.

I. Solution by C. E. Mange, Junior, Washington University.

To construct the triangle.

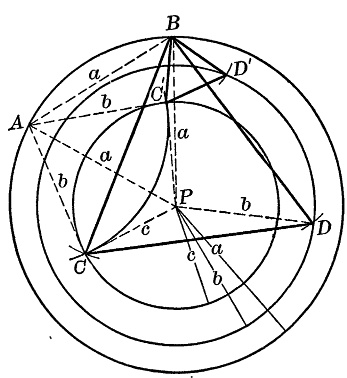
With given point, P, as a center, describe circles with radii a, b, and c, respectively. Assume $a > b > c$. Construct chord AB in circle (a) equal to a, and with A as a center describe an arc with radius b intersecting circle (c) in C and C'. With B as a center describe an arc of radius BC intersecting circle (b) in D.[1] Draw BC, CD, and DB. The triangle BCD is equilateral and fulfils the required conditions. (Similarly for the triangle $BC'D'$.)

Proof: Draw AC, AP, AB, BP, CP, and DP.

Now $AB = BP = PA$, by construction; therefore, $\triangle ABP$ is equilateral, and $\angle PBA = 60°$.

Since $\triangle ACB = \triangle BPD$, $\angle ABC = \angle PBD$; and hence $\angle ABC + \angle CBP = 60° = \angle CBD$. And since $CB = BD$, by construction, CBD is equilateral.

Area of



$$ACP = A = \sqrt{\left(\frac{a+b+c}{2}\right)\left(\frac{a+b-c}{2}\right)\left(\frac{a-b+c}{2}\right)\left(\frac{-a+b+c}{2}\right)},$$

$$\angle BAP = 60°, \quad \text{and} \quad \angle PAC = \text{arc} \sin \frac{2A}{ba}.$$

In triangle BAC,

$$BC = \sqrt{b^2 + a^2 - 2ba \cos\left[60° + \text{arc} \sin \frac{2A}{ba}\right]}$$
$$= \sqrt{b^2 + a^2 - 2ba\left[\frac{1}{2} \cdot \frac{b^2 + a^2 - c^2}{2ba} - \frac{\sqrt{3}}{2} \cdot \frac{2A}{ba}\right]}$$
$$= \sqrt{\frac{a^2 + b^2 + c^2}{2} + 2\sqrt{3}A}.$$
$$BC' = \sqrt{\frac{a^2 + b^2 + c^2}{2} - 2\sqrt{3}A}.$$

---

[1] D is that intersection which lies on the opposite side of BP from A, so that the four lines from B are in the order BA, BC, BP and BD. D' is the intersection determined in the same way when an arc of radius BC' is described.—Editors.

The number of such inequalities may be found from the coefficient of $c_{1i}c_{2j}c_{3k}c_{4l}$ in the iterated series. The first subscripts of the ϕ's in the first term of the squared factor admit 24 permutations, while the second term may be formed from it in four ways, each of the operators K_1K_2 retaining one of the ϕ's associated with it in the first term and in the opposite position. The 96 results so indicated reduce however to 48 by reason of the permutability of the terms of the squared factor. There are then 48 inequalities of precisely the same type as the one in question; but there are others of the same general type corresponding to any two permutations of the subscripts 1, 2, 3, 4, in the terms of the squared factor, making in all $\binom{24}{2}$ or 276 results, exclusive of the useless one $0 \geqq 0$ which is obtained when the same permutation is used twice. Of these, 24 are reducible inequalities of the type

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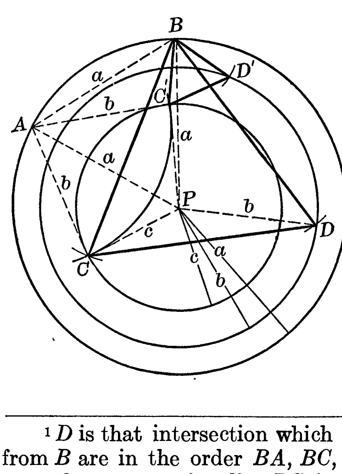

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II. Solution by the Proposer.

Consider first the case in which the plane of the desired triangle passes through the given point, and the problem is confined to two dimensions. Let P be the given point, $a \geqq b \geqq c \geqq 0$ the given distances, and e the length of the side of the required triangle. Let the angles between the lines drawn to the vertices be A, B, C, so that $A + B + C = 2\pi$ [2] and, therefore, one has

$$\cos^2 A + \cos^2 B + \cos^2 C = 1 + 2 \cos A \cos B \cos C.$$

From this and the law of cosines, $e^2 = a^2 + b^2 - 2ab \cos C$, etc., comes the equation,

$$e^4 - (a^2 + b^2 + c^2)e^2 + a^4 + b^4 + c^4 - a^2b^2 - b^2c^2 - c^2a^2 = 0,$$

and the values e^2,

$$e^2 = \tfrac{1}{2}(a^2 + b^2 + c^2 \pm \sqrt{\Delta_0}),$$

where

$$\Delta_0 = 3(a + b + c)(a + b - c)(c + a - b)(b + c - a).$$

The values of e^2 are real and unequal, if, and only if, the relation $b + c > a$ is satisfied; they are real and equal if, and only if, the relation $b + c = a$ is satisfied. Moreover, the values of e^2 are never negative; this may be shown by using the relation

$$a^4 + b^4 + c^4 \geqq a^2b^2 + b^2c^2 + c^2a^2.$$

Therefore, there are two, one, or no equilateral triangles having their vertices at the given distances a, b, c, from a given point lying in their plane, according as these distances themselves may be taken as the lengths of the sides of a non-degenerate triangle, a degenerate triangle, or no triangle, respectively.

It can be shown that the given point always lies outside or at the vertex of one found triangle, and that when a^2 is between $b^2 + bc + c^2$ and $(b + c)^2$, it lies outside both triangles. (It is, therefore, not true, as stated in G. R. Perkins's *Plane and Solid Geometry*, New York, 1860, p. 231, that the minus sign must be used before the radical in the above formula when the given point lies outside the triangle.)

In the general case, some variable parameter beside the given lengths must be introduced: let this be the distance p from the given point P to the plane of the desired triangle. The projections of a, b, c, on this plane are

$$(1) \qquad a' = \sqrt{a^2 - p^2}, \qquad b' = \sqrt{b^2 - p^2}, \qquad c' = \sqrt{c^2 - p^2}.$$

The length of the side of an equilateral triangle lying in this plane and having vertices at the distances a', b', c', from the foot of this perpendicular is the length sought, and may therefore be obtained by putting a', b', c', in place of a, b, c, viz.,

$$e^2 = \tfrac{1}{2}(a^2 + b^2 + c^2 - 3p^2 \pm \sqrt{\Delta_p}),$$

where

$$\Delta_p = 9p^4 - 6p^2(a^2 + b^2 + c^2) + \Delta_0 = (3p^2 - \alpha)(3p^2 - \beta),$$

$$\alpha = a^2 + b^2 + c^2 - \sqrt{2}\sqrt{(a^2 + b^2 + c^2)^2 - \Delta_0} = a^2 + b^2 + c^2 - \sqrt{2}\sqrt{(a^2 - b^2)^2 + (b^2 - c^2)^2 + (c^2 - a^2)^2},$$

and

$$\beta = a^2 + b^2 + c^2 + \sqrt{2}\sqrt{(a^2 - b^2)^2 + (b^2 - c^2)^2 + (c^2 - a^2)^2}.$$

These values of e^2 are real (for real values of p^2), if, and only if, p^2 lies in one of the intervals, $p^2 \leqq \tfrac{1}{3}\alpha$, $\tfrac{1}{3}\beta \leqq p^2$.

For the first interval, e^2 is never negative, but p^2 is negative except when the condition $b + c \geqq a$ is satisfied. For the second interval, both values of e^2 are always negative. These facts have been used in proving the following statements:

For each value of p subject to the condition $0 \leqq p^2 < \tfrac{1}{3}\alpha$, two equilateral triangles are found, provided the given distances a, b, c, may be taken as the lengths of the sides of a triangle.

(S) When the conditions $p^2 = \tfrac{1}{3}\alpha$ and $b + c \geqq a$ are both satisfied, just one triangle is found. In all other cases no triangle is found.

(D) One of the triangles found degenerates into a point if, and only if, the given distances are equal.

The cases in which the perpendicular from the given point to the triangle plane falls

(1) outside one triangle and inside the other;

---

[2] If the given point lies outside of the given triangle we shall have $B + C - A = 0$, but the equations of cosines still hold.—Editors.

(2) on a side of one triangle and outside the other, but (2) (S) at a vertex if there is only one solution; and

(3) outside both triangles, also (3) (S) outside, if there is only one solution: correspond respectively to the conditions

$$p^2 \lesseqgtr \frac{b^2c^2 - (b^2 + c^2 - a^2)^2}{2a^2 - (b^2 + c^2)} = f(a^*bc).$$

The case in which the perpendicular falls

(4) on the side of one triangle produced,

is a special case under case (1) and corresponds to the conditions

$$f(a^*bc) > p^2 = \frac{a^2c^2 - (a^2 + c^2 - b^2)^2}{2b^2 - (a^2 + c^2)} = f(b^*ac).$$

These formulas are indeterminate if, and only if, one of the triangles found is degenerate.

$f(a^*bc)$ and $f(b^*ac)$ are equal only when a and b are equal, and they then reduce to $\frac{1}{3}\alpha$; so that the property (2) (S) mentioned above may be regarded as the result of combining any two of the properties (2), (S), and (4).

Only the following eight combinations of the above properties are possible,

(1), (1) (4), (1) (D), (2), (2) (4) (S), (3), (3) (S), (S) (D).

The general conditions which are mentioned above have been derived by "projection" from the corresponding conditions when p is zero, that is, by making the substitutions (1). For example, the conditions that P falls on a side or a side produced are respectively $e = b + c$, $e = b - c$, which are equivalent to $a^2 = b^2 + bc + c^2$, $a^2 = b^2 - bc + c^2$. In the latter condition, when the relation, $a \geqq b \geqq c \geqq 0$, is to be preserved, care must be taken to interchange a and b. The value of a which makes P fall on a side produced is then determined by the equation $a^2 - ac + c^2 = b^2$. Substituting a' for a, etc., in this equation gives the corresponding condition for three dimensions, $p^2 = f(b^*ac)$.

III. Historical Notes by R. C. Archibald, Brown University.

This problem has been frequently discussed in books, pamphlets, and periodicals for more than a century. In 1803 L. N. M. Carnot gave indications of synthetic and analytic solutions of the following more general problem (*Géométrie de position*, Paris, 1803, pp. 381–382, 389–390): "Connoisant les trois angles d'un triangle, et les distances de leurs trois sommets à un point donné dans le même plan, trouver les trois côtés de ce triangle." Carnot points out that the method of solving this problem may be applied to solve the following example concerning the pyramid: "Connoissant tous les angles que font deux à deux les six arêtes d'une pyramide triangulaire, et les distances de ses quatre sommets à un point quelconque de l'espace, trouver toutes les dimensions de cette pyramide." Among many discussions of Carnot's problem in the plane, the following may be mentioned: By R. Götting, *Einen Punkt zu bestimmen, dessen Entfernung von drei gegebenen Punkten sich wie drei gegebene gerade Linien verhalten.* Progr. Torgau, 1888. 30 pp. + 1 plate—By Combier, "Note de géométrie," *Journal de mathématiques élémentaires*, vol. 3, 1879, pp. 120–126 (trigonometric discussion)—By A. H. Curtis, M. Jenkins, and J. McDowell, *Mathematical Questions with their Solutions from the 'Educational Times,'* vol. 44, 1886, p. 110; vol. 45, 1886, pp. 68–70.

The particular case for an equilateral triangle, with its vertices on three concentric circles of given radii, was considered in Gabriel Lamé's valuable *Examen des différentes méthodes employées pour résondre les problèmes de géométrie* (Paris, 1818, pp. 81–82), in G. Ritt's *Problèmes d'application de l'algèbre à la géométrie avec les solutions développées* (Paris, 1836, pp. 17–20), and in *Mathematical Questions with their Solutions from the 'Educational Times,'* (a) by W. S. McCay and W. S. Burnside, vol. 10, 1868, pp. 98–99; (b) by W. J. C. Miller, R. F. Davis, S. A. Renshaw, etc., vol. 26, 1876, pp. 24–28.

In *Nouvelles Annales de Mathématiques*, vol. 3, 1844, p. 376, Prouhet proposed the following problem: "Trois circonférences étant tracées sur un même plan, on propose de trouver sur ces circonférences, *en ne faisant usage que du compas*, trois points qui soient les sommets d'un triangle équilatéral." Solution by Breton (de Champ) is given in vol. 9, 1850, pp. 299–304.

Also solved by R. D. Bohannan, W. F. Cheney, Jr., William Herberg, H. Halperin, H. L. Olson, A. Pelletier, Elijah Swift, and C. C. Yen.

NOTES AND NEWS.

Edited by E. J. MOULTON, Northwestern University, Evanston, Ill.

Assistant professor W. E. MILNE, of the University of Oregon, has been promoted to a full professorship of mathematics.

Mr. F. C. KENT has been promoted to an assistant professorship of mathematics at the Oregon Agricultural College.

In the mathematics department of the University of Minnesota, Professor G. N. BAUER is on leave of absence for the winter quarter, Assistant Professor R. R. SHUMWAY has been appointed to an associate professorship and Mr. R. M. BARTON (*1919*, 420) has been promoted to an assistant professorship of mathematics.

At the University of Michigan, Associate Professor PETER FIELD has been promoted to a full professorship, Assistant Professor J. W. BRADSHAW has been promoted to an associate professorship, and Messrs. E. C. BLANKENSTEIN and O. J. PETERSON have been appointed instructors in mathematics.

Mr. V. G. GROVE, of Cornell University, has been appointed assistant professor of mathematics at Michigan Agricultural College.

Professor L. D. AMES has been made registrar at the University of Missouri.

Dr. H. E. WOLFE (*1920*, 90) has been appointed assistant professor of mathematics at Indiana University.

Dr. TOBIAS DANZIG is an instructor of mathematics in Johns Hopkins University.

Mr. L. H. RICE, of Tufts College, has been appointed instructor in mathematics at the Massachusetts Institute of Technology.

At the University of Maine, Mr. A. S. PRATT, of Brown University, has been appointed instructor in mathematics, and Mr. J. P. BALLANTINE, of Harvard University, instructor in mathematics and physics.

At Teachers College, Columbia University, Professor C. B. UPTON, of the department of mathematics, and secretary of the college since 1911, has been appointed provost.

Professor ALFRED BAKER, forty-four years a teacher of mathematics in the University of Toronto, retired on a Carnegie pension at the end of the academic year 1918–19. Professor A. T. DELURY succeeded him as head of the depart-

ment of mathematics. In the same department Lecturer I. R. POUNDER has been promoted to an assistant professorship.

At the University of Lille, J. CHAZY, professor of general mathematics (*1919*, 371), has been appointed professor of differential and integral calculus in place of the late Professor DEMARTRES (*1920*, 43).

Dr. J. FAIRON has been appointed professor of mathematics at the University of Liège.

Professor W. KILLING, of the University of Münster, has retired from active teaching.

Dr. F. SCHUR recently professor of mathematics at the University of Strassburg has been appointed to a professorship at the University of Breslau in place of the late Professor R. STURM (*1920*, 89); and Professor P. EPSTEIN, also recently at the University of Strassburg, has been appointed privatdozent at the University of Frankfurt a. M.

Dr. H. TIETZE and Dr. G. HESSENBERG have been appointed professors of mathematics at the University of Erlangen.

At the University of Freiburg i. Br., Dr. A. LOEWY has been appointed professor of mathematics, and Professor L. STICKELBERGER has retired from active teaching.

Dr. EMMY NOETHER, daughter of Professor MAX NOETHER, and Dr. SCHMEIDLER have been appointed privatdozents at the University of Göttingen.

Dr. E. CZUBER, professor of mathematics in the Technical High School at Vienna, has retired.

Actuary ARNFINN PALMSTROM has been appointed extraordinary professor of the science of insurance at the University of Christiania, with the special duty of conducting the insurance seminar at the university.

Professors E. L. A. MERLIN and L. M. M. STUYVAERT have been promoted to full professorships of mathematics at the University of Ghent.

Associate professor E. GAU has been appointed to the professorship of infinitesimal analysis at the University of Grenoble, succeeding Professor COLLET, who has retired from active teaching.

Professor E. DANIELE, of the University of Catania, has been appointed professor of rational mechanics at the University of Modena. The death is announced of Professor A. RICCO, director of the Observatory of Catania and vice-president of the International Astronomical Union.

Dr. R. C. MACLAURIN, president of the Massachusetts Institute of Technology since 1908, died January 15, 1920 in the fiftieth year of his age. He was born in Scotland and educated in New Zealand and at Cambridge University. He was professor of mathematics at the University of New Zealand 1898–1905, and dean of the faculty of law there from 1905 to 1907, when he was elected professor of mathematical physics in Columbia University.

Professor V. REINA, of the University of Rome, [*1920*, 45] died November 9, 1919, at the age of fifty-three years, and Professor E. MILLOSEWICH, of the Observatory del Collegio Romano, died December 5, 1919, at the age of seventy years. Professor Millosewich was secretary of the R. Accademia dei Lincei.

Professor E. H. BRUNS, director of the Observatory of the University of Leipzig, died September, 1919, at the age of seventy-one; and Dr. O. DANZER, privatdozent in the Technical High School at Vienna died March 26, 1919, as the result of an accident.

The Bordin prize (3000 francs) of the Academy of Sciences of the Institute of France is awarded in mathematics every two years for a memoir on a proposed problem. For 1919 the question was: "In the theory of integrals of total differentials of the third kind and of double integrals relative to an algebraic function of two independent variables, there has been proved the existence of certain integers, whose value is difficult to obtain and may depend on the arithmetic nature of the coefficients of the equation of the surface corresponding to the function. The Academy asks for a detailed study of these numbers in some important special cases." This prize was awarded to Professor SOLOMON LEFSCHETZ, of Kansas University, for a memoir entitled "Sur certains nombres invariants des variétés algébriques avec application aux variétés abéliennes." The principal results of the memoir were given in two notes published in tome 168 of the *Comptes Rendus* of the Academy: "Sur l'analyse situs des variétés algébriques," pages 672–674, March 31; and "Sur les variétés abéliennes," pages 758–761, April 14, 1919. There is a summary of the memoir by Emile Picard in the *Comptes Rendus*, December 22, 1919, pages 1200–1202, being his report for the committee upon the occasion of the formal award of the prize.

In 1882 there was established at Frankfurt a. M. a fund under the name of the Peter-Wilhelm-Müller foundation, with a capital of a million and a half marks, from the income of which, among other things a prize was to be given every third year for the best work in the following domains taken in order: art, poetry and music, philosophy and historical philology, mathematics and the natural sciences. Once this prize was awarded to a physicist by a jury in which a majority were mathematicians, and once it was awarded to a mathematician by a majority of physicists. In 1918 it was again the turn of mathematics and science, and a prize of 10,000 marks was divided equally between Professor D. HILBERT for the totality of his mathematical writings, and Professor A. EIN-

STEIN, for the great mathematical talent displayed in his development of relativity and gravitation theories.

Professor A. G. WEBSTER, of Clark University, has been elected honorary member of the Royal Institution of Great Britain.

Professor W. W. LANDIS, of Dickinson College, has returned to his academic work after serving as a Y. M. C. A. Secretary with the Italian army for over a year. He was accorded the honorary rank of major in the Italian army, and received several medals in recognition of the value of his work.

The mathematical library of the late Dr. R. A. HARRIS has been presented to Cornell University by Mrs. Harris. The collection comprises nearly three thousand volumes, including practically the complete literature on the theory of tides.

At the meeting of the Association of Teachers of Mathematics in New England at Cambridge on January 31, Professor J. L. COOLIDGE gave an address on "The annual meeting of the Mathematical Association of America."

At the meetings of the Chicago and Southwestern Sections of the American Mathematical Society in St. Louis, December 30–31, 1919, forty-four members of the Society were present and twenty-eight papers were read. Professor R. D. CARMICHAEL was elected Chairman, Professor C. N. MOORE member of the program committee, and Professor ARNOLD DRESDEN Secretary of the Chicago Section. This Section will meet in Chicago on April 9 and 10. A feature of the program is a symposium on "The field equations of electromagnetism and relativity" by Professors MAX MASON and A. C. LUNN.

In Section A of the American Association for the Advancement of Science Professor D. R. CURTISS was elected chairman and vice-president for the two years 1920–1921.

The following resolutions were passed by the Council of the American Mathematical Society at the meeting in New York on December 31, 1919:

1. "The Council regards the preparation and publication, in America, of a dictionary of mathematical terms as not only most desirable but also entirely feasible, provided that financial aid for the preparation of the manuscript can be secured.

2. "Impressed with possibilities for the more extensive development of pure and applied mathematics in America, and with the importance of such development to the nation, the Council records its conviction that there are undertakings whose active consideration would be highly desirable if adequate financial assistance might be regarded as available. Among such undertakings are: 1. The preparation and publication by societies or individuals of surveys, introductory

monographs, translations, memoirs, and treatises in important fields, including the history of mathematics; 2. The organization of research fellowships; 3. The preparation and publication of an encyclopædia of mathematics in English; 4. The preparation and publication of an annual critical survey, in English, of the mathematical literature of the world; 5. The preparation and publication of a biographical and bibliographical dictionary of mathematicians."

Nature announces that Girton College, Cambridge, has received a gift of £10,000, the capital and interest of which are to be applied during the next twenty years for the encouragement of scientific research by women in mathematical, physical and natural sciences.

The Mathematics Section of the Texas State Teachers' Association met in Houston, November 28, 1919. There were about one hundred teachers of mathematics in attendance. Dr. GOLDIE P. HORTON, instructor in pure mathematics at the University of Texas, was chairman for the year and presided at the meeting. The following papers were read: "Some extra text-book problems," by Mr. P. H. Underwood, Galveston; "Some definitions of mathematics," by Dr. P. M. Batchelder, University of Texas; "Limit proofs in geometry," by Prof. A. A. Bennett, University of Texas; "Compulsory mathematics in high school," by Miss Helen Carr, Orange; "Some desirable points in elementary texts," by Prof. G. C. Evans, Rice Institute. There was also discussion of these papers, of high school mathematics clubs, and of the work of the National Committee on Mathematical Requirements.

THE WORK OF THE NATIONAL COMMITTEE ON MATHEMATICAL REQUIREMENTS, January 13, 1920 [*1919*, 439–440; 462–463].

The preliminary report, made to the National Committee on Mathematical Requirements by a sub-committee, on "The Reorganization of the First Courses in Secondary Schools Mathematics" and issued as a basis for discussion on November 25, 1919, has received considerable attention during the short period since its publication. It has been discussed at various meetings of teachers organizations, both large and small, and reports on the results of such discussions have been received by the chairman of the National Committee.

In view of the fact, however, that the number of organizations actively cooperating with the National Committee has now reached 54 it is clear that this discussion has only just begun.

The National Committee at its last meeting in New York City, December 30 and 31, 1919, subjected the sub-committee report to a careful revision in the light of the suggestions, criticisms and comments received. The revised version of the report was published in February by the U. S. Bureau of Education (*Secondary School Circular*, No. 5, 12 pp.) as a preliminary report by the National Committee. *Comments, criticisms, suggestions for improvement, etc., are earnestly solicited from organizations, committees, and individuals.* They should be sent to the Chairman of the Committee (J. W. Young, Hanover, N. H.).

The response to such requests in the past has been most gratifying, and augurs well for the future. Only through coöperation on a large scale on the part of all interested in the problem of improving the teaching of mathematics can success be assured.

The mailing list to be furnished the U. S. Bureau of Education for the Reports of the National Committee is now in preparation. This list now includes all the members of the American Mathematical Society, the Mathematical Association of America, the Association of Teachers of Mathematics in New England, the Association of Teachers of Mathematics in the Middle States and Maryland, and the teachers of mathematics who are members of the Central Association of Science and Mathematics Teachers and many others. Requests have been sent to all the organizations coöperating with the National Committee for lists of their members. *Individuals desiring to be placed on the mailing list of the National Committee should send their names and addresses to the Chairman of the Committee without delay.*

A preliminary report on the principles to govern a revision of College Entrance Requirements was submitted by the National Committee to the Councils of the American Mathematical Society and of the Mathematical Association of America on December 30, 1919.

A report whose early publication may be expected is "The Doctrine of Formal Discipline and the Transfer of Training" (a critical examination of the literature by Vevia Blair).

Professor R. C. Archibald of Brown University has consented to prepare a report for the National Committee on the desirable professional training for teachers of mathematics in the United States and on courses primarily intended for prospective teachers.

The preparation of reports on Sources and Desirable Types of Problems, and on Elective Courses in Mathematics for Secondary Schools (their aims, content, organization, etc.) has begun, and exhaustive and authoritative investigations of the mathematical elements entering into the work of the various vocations, industries, and professions, are under consideration.

The appropriation of $16,000 by the General Education Board under which the National Committee is operating, was made to cover the expenses of the Committee for one year ending July 1, 1920. At the last meeting of the Committee, it was voted to request the General Education Board to give the Committee its support for another year ending July 1, 1921.

The Committee heartily approves the proposed organization of a National Council of Mathematics Teachers and has authorized its officers to attend the organization meeting of the Council in Cleveland on February 24th in connection with the meeting of the Department of Superintendence of the National Education Association.

The date and place of the next meeting of the National Committee have been tentatively set for Chicago, Illinois, on April 23rd and 24th, 1920.

THE AMERICAN MATHEMATICAL MONTHLY

OFFICIAL JOURNAL OF THE MATHEMATICAL ASSOCIATION OF AMERICA

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

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Fifth Summer Meeting of the Association, Chicago, September 6, 1920; Fifth Annual Meeting, ———, December, 1920. The following are Section meetings of the Association in 1920:

ILLINOIS, ———, ——

INDIANA, ———, ——

IOWA, Univ. of Iowa, Iowa City, May 1

KANSAS, State Agricultural College, Manhattan, April 3

KENTUCKY, Centre College, Danville, April 10

MARYLAND–DISTRICT OF COLUMBIA–VIRGINIA, Goucher College, Baltimore, Md., May

MINNESOTA, St. Catherine's College, St. Paul, May 29

MISSOURI, Kansas City, November 12–13

OHIO, Ohio State Univ., Columbus, April 2

ROCKY MOUNTAIN, Colorado College, Colorado Springs, April 2

EDITORIAL CORRESPONDENCE AND BOOKS FOR REVIEW should be addressed to the EDITOR-IN-CHIEF, R. C. ARCHIBALD, Brown University, Providence, R. I.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 27 King Street, Oberlin, Ohio.

THE DECEMBER MEETING OF THE MARYLAND-VIRGINIA-DISTRICT OF COLUMBIA SECTION.

The Maryland-Virginia-District of Columbia Section of the Mathematical Association of America met at George Washington University, Washington, D. C., morning and afternoon, December 6, 1919. Among those in attendance were the following members: H. L. Hodgkins, George Washington University; G. A. Bingley, C. C. Bramble, J. A. Bullard, G. R. Clements, G. H. Cresse, Alexander Dillingham, J. B. Eppes, P. E. Hemke and R. E. Root, U. S. Naval Academy; A. E. Landry and O. J. Ramler, Catholic University; Abraham Cohen and F. D. Murnaghan, Johns Hopkins University; Clara L. Bacon and Florence P. Lewis, Goucher College; Harry English, Washington Public Schools; H. M. Roeser, Bureau of Standards; W. E. Heal, Bureau of Plant Industry; J. J. Arnaud, A. A. Bennett and C. E. Norwood, Ordnance Department (Army); W. M. Hamilton, Nautical Almanac Office; O. S. Adams, H. G. Avers, Sarah Beall, W. D. Lambert and G. F. Winslow, Jr., Coast and Geodetic Survey.

The program contained the following papers:

(1) "Calculation of the date of Easter," Professor F. D. MURNAGHAN, Johns Hopkins University.

(2) "Precise leveling in the United States," Mr. H. G. AVERS, Coast and Geodetic Survey.

(3) "Proof that in a plane world infinite in extent, but finite in thickness, gravity would be a constant at any altitude," Mr. A. S. HAWKESWORTH, Naval Ordnance.

(4) "Geometrical proportion," Mr. W. E. HEAL, Bureau of Plant Industry.

(5) "Desirable changes in the mathematical courses in the high school," Mr. HARRY ENGLISH, Head of Mathematics Department, Washington High Schools.

(6) "Report of the Summer Meeting of the Association at Ann Arbor, Mich.," Dr. G. H. CRESSE, U. S. Naval Academy, and Professor FLORENCE P. LEWIS, Goucher College.

(7) "Internal constitution of the earth," Mr. W. D. LAMBERT, Coast and Geodetic Survey.

(8) "A simple instrument for the inversion and the mechanical calculation of trigonometric functions," Dr. T. DANZIG, Johns Hopkins University.

Interesting general discussions followed the reading of the various papers.

OSCAR S. ADAMS, *Secretary.*

THE THIRD ANNUAL MEETING OF THE MISSOURI SECTION.

The third annual meeting of the Missouri Section of the Mathematical Association of America, postponed from 1918 on account of the war, was held in St. Louis on the afternoons of Monday and Tuesday, December 29–30, 1919. The meeting would normally have been held in Columbia on the Saturday following Thanksgiving Day, in conjunction with the meeting of the Southwestern Section of the American Mathematical Society, but the time and place of meeting were changed so that these two sections might meet with the Chicago Section of the American Mathematical Society during convocation week of the American Association for the Advancement of Science. The meeting next year is scheduled to be held in Kansas City, and the meeting two years hence, which was to have been held in St. Louis, will take place in Columbia.

The Monday session, presided over by Professor William H. Roever, Chairman of the Section, was for the reading of papers and for the annual business meeting. The Tuesday session was a joint meeting with the Chicago and Southwestern Sections of the American Mathematical Society and Section A of the American Association for the Advancement of Science, and was presided over by Professor O. D. Kellogg, Chairman of Section A and Vice-Chairman of the Missouri Section. On Tuesday evening at the American Hotel Annex the four mathematical sections already mentioned held a joint dinner at which forty-eight persons were present.

At the Monday meeting there was an attendance of twenty-two, including the following members of the Mathematical Association of America: Charles Ammerman, McKinley High School, St. Louis; G. D. Birkhoff, Harvard University; J. A. Caparo, University of Notre Dame; Alfred Davis, Soldan High School, St. Louis; E. L. Dodd, University of Texas; Otto Dunkel, Washington University; H. J. Ettlinger, University of Texas; E. R. Hedrick, University of Missouri; Louise H. Huff, St. Louis; A. R. Nauer, St. Louis; P. R. Rider, Washington University; Percival Robertson, The Principia, St. Louis; W. H. Roever, Washington University; A. J. Schwartz, Cleveland High School, St. Louis; H. E. Slaught, University of Chicago; G. W. Smith, University of Kentucky; E. H. Taylor, Eastern Illinois State Normal School, Charleston, Ill.; W. H. Zeigel, State Normal School, Kirksville, Mo.

At the business meeting the report of the Committee appointed to consider the Preliminary Report on the Reorganization of the First Courses in Secondary School Mathematics by the National Committee on Mathematical Requirements was adopted.

It was voted that the Section urge the National Committee on Mathematica Requirements to take positive steps toward having speakers in behalf of mathematics address general rather than mathematical meetings.

The Section went on record as urging that a minimum of at least one year of

high school mathematics be required for graduation by all high schools in the state of Missouri.

Professor R. A. WELLS of Park College was elected Chairman and Mr. W. A. LUBY, of the Kansas City Polytechnic Institute, Vice-Chairman for the coming year. It was voted that the office of Secretary-Treasurer be considered permanent, and the present officer was reëlected.

It was voted that a committee be appointed by the new chairman to act on the reports of the National Committee on Mathematical Requirements. This committee has been appointed as follows: Professor E. R. Hedrick, University of Missouri, chairman; Professor W. H. Zeigel, State Normal School, Kirksville; Professor R. R. Fleet, William Jewell College; Miss Zoe Ferguson, Central High School and Junior College, St. Joseph; Mr. Alfred Davis, Soldan High School, St. Louis; Mr. Percival Robertson, The Principia, St. Louis.

At the joint meeting on Tuesday the following addresses were given:

I. "Recent advances in dynamics."[1] Address of the retiring Vice-President of Section A of the A. A. A. S., PROFESSOR G. D. BIRKHOFF.

II. "Some recent developments in the calculus of variations." Address of the retiring chairman of the Chicago Section of the American Mathematical Society, PROFESSOR G. A. BLISS, University of Chicago.

III. "A suggestion for the utilization of atmospheric molecular energy." MR. H. H. PLATT, Philadelphia.

At the regular session of the Missouri Section the following seven papers were read:

(1) Opening address as President of the Association, PROFESSOR H. E. SLAUGHT, University of Chicago.

(2) "The determination of logarithmic formulas," PROFESSOR E. R. HEDRICK, University of Missouri.

(3) "A Simple Treatment of Fourier's Series," PROFESSOR LOUIS INGOLD, University of Missouri, and MR. T. W. JACKSON, Jamestown College, N. D.

(4) "An Elementary Method of Quadrature," PROFESSOR OTTO DUNKEL, Washington University.

(5) "Plans of the National Committee on Mathematical Requirements," MR. CHARLES AMMERMAN, McKinley High School, St. Louis.

(6) "Preliminary report of the National Committee on Mathematical Requirements," PROFESSORS HEDRICK, ZEIGEL, and FLEET, MISS ZOE FERGUSON and MR. ALFRED DAVIS.

(7) "Geometric treatment of certain optical problems," (Illustrated by lantern views and models.) PROFESSOR WM. H. ROEVER, Washington University, chairman of the Section.

In the absence of the authors the paper by Professor Ingold and Mr. Jackson was read by Professor Hedrick. Abstracts of the papers, except (5) and (6), follow below, the numbers corresponding to the numbers in the list of titles above:

[1] Published in *Science*, January 16, 1920, pp. 51–55.

The series to be actually summed is such that the sum is obvious and the whole process involves only elementary algebra.

The method can be extended to negative values of m, $m = -1$ being excluded, and also to fractional values. Compare *1920*, 116–117.

(7) The object of Professor Roever's paper was to illustrate a purely geometric method of treatment by its application to the solution of some problems in geometric optics. The problems treated in this way were certain problems on brilliant points which have already been treated analytically by the author in *Annals of Mathematics*, second series, vol. 3, April, 1902; *Transactions of the American Mathematical Society*, vol. 9, July, 1908; and AMERICAN MATHEMATICAL MONTHLY, vol. 20, December, 1913. An example of this geometric method is given in an article entitled: "Geometric Description of the Halo on the Dome of the St. Louis Cathedral," which will soon appear in a scientific number of the *Washington University Studies*.

PAUL R. RIDER, *Secretary-Treasurer*.

ENVELOPE ROSETTES.[1]

By WILLIAM F. RIGGE, Creighton University, Omaha, Neb.

One way of drawing a cardioid is to make a pen start in phase 90° from the center of a disk and move along a radius with simple harmonic motion, while the disk revolves with a uniform angular speed of the same period. If now, instead of a *simple* harmonic movement with the equation $\rho = 1 - \cos\theta$, the amplitude being unity, we give the pen a *double* harmonic motion, and write the equation

$$\rho = (1 - \cos\theta) + (1 - \cos m\theta),$$

in which m differs from unity by some small aliquot fraction; we shall then get a series of harmonic curves which have one variable parameter, and which must therefore have a common envelope. The problem before us is to find this envelope.

The Inner Envelope a Cardioid.—The method of procedure, according to the textbooks, is to differentiate the above equation by regarding the variable parameter m as the only variable in it, and then to eliminate the parameter between these two equations. This will give us

$$\sin m\theta = 0,$$

[1] Readers of this article are reminded of earlier articles by Professor Rigge in this MONTHLY ("Concerning a new method of tracing cardioids," *1919*, 21–32; "Cuspidal rosettes," *1919*, 332–340) in which the discussion, with special reference to possibilities of his machine for tracing curves, is along similar lines. We have already referred (*1920*, 132) to the interesting illustrated account of this machine in the *Scientific American Supplement*, 1918, February 9 and 16 (partly reproduced in *La Nature*, 1919, September 27) —EDITOR.

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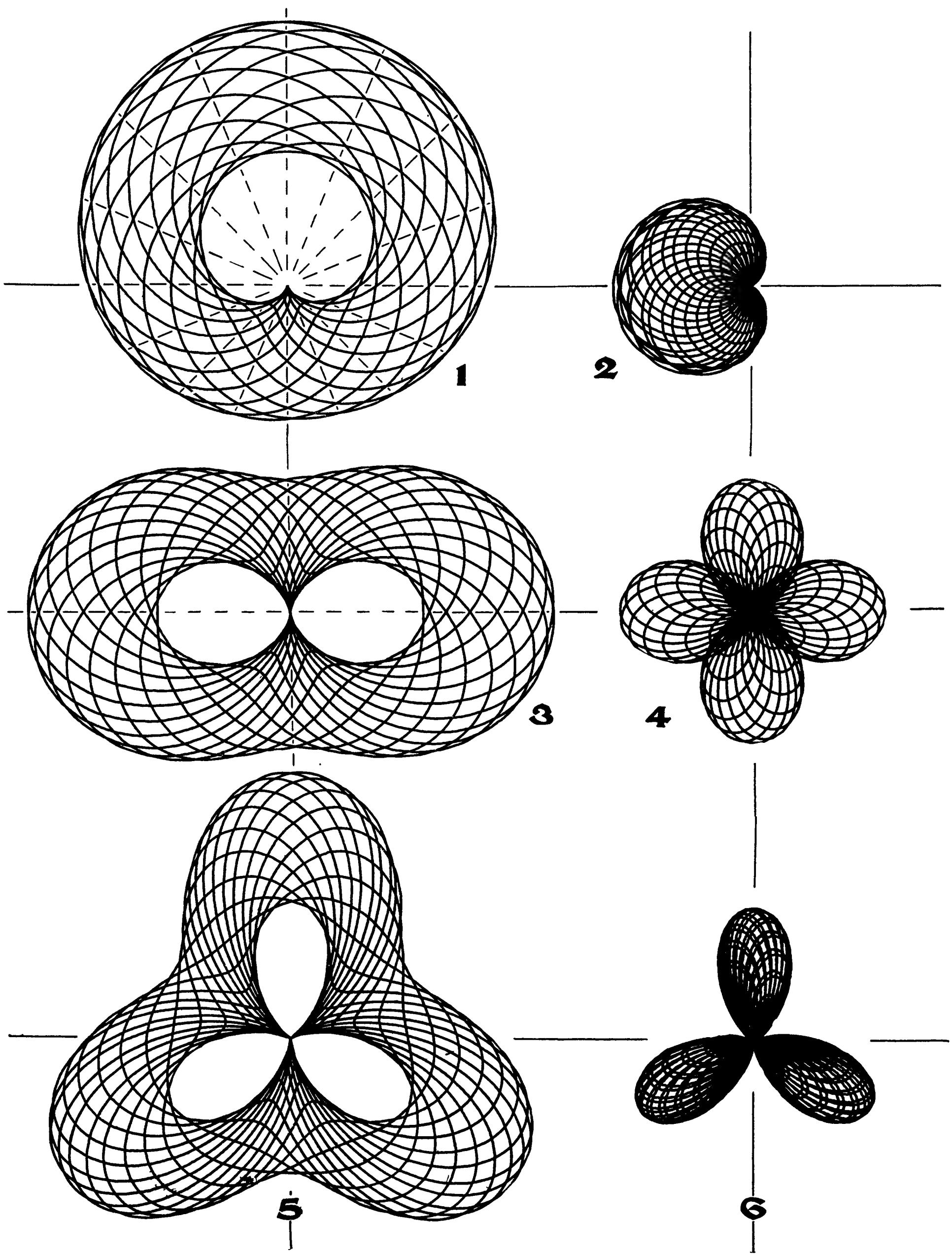
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and then

$$\pm 1 = \rho - 2 + \cos\theta,$$

so that the equation of the envelope becomes[2]

$$\rho = (1 - \cos\theta) + 2$$

and

$$\rho = 1 - \cos\theta.$$

There is therefore an inner envelope which is a cardioid, and an outer one in which the radius vector of this cardioid is increased by 2.

Fig. 1 has been drawn to represent these curves. This figure, like all those in this article, has the same position on the page that it had on the disk at the moment when it was completed and the drawing pen had returned to its initial position. If we can imagine the disk, which turns in a clockwise direction, to be now alone arrested, the pen will keep on moving up and down along the vertical line of the page through the cusp, that is, along what is generally denoted as the Y axis in figures, but which we may call here the mechanical axis of Y. The mathematical axis of $+ Y$ which is used in the equations just given and which convention directs to run always upward, runs to the right in this Fig. 1, so that the figure must be turned 90° in an anticlockwise direction in order to have it oriented in the usual way. The reason for this departure from the customary mathematical practice was that, by presenting the mechanical aspect of the figures, the changes that come over them when the initial phase or position of the pen or the rotation frequency of the disk is altered, may be seen to better advantage. The mathematical axes must therefore be rotated to suit each figure in particular. This will present no great difficulty.

The motion of the pen in Fig. 1 was the resultant of two simple harmonic movements both of the same amplitude, one with a unit period and the other with a period m, 15/16 or 16/15 as long, while the disk had a period of either component. In practice component A had a wheel with 32 cogs which made 15 revolutions while component B with 30 cogs made 16, the disk in the meantime with a 30- or 32-cog wheel making 16 or 15 turns. In Fig. 1 a 32-cog wheel with 15 revolutions was used on the disk. A radius (through the cusp) may be seen to cut the compound curve in 15 points. Had a 30-cog wheel with 16 revolutions been employed, there would have been 16 such intersections.

The pen was placed at the center of the disk (at the cusp) when both of its components were in phase 90°. When set in motion the pen started to draw a

[2] It is rather questionable to call m a variable parameter since only one value of m is considered at a time. There is a single curve with several lobes, giving the appearance of so many different curves all tangent to their envelope. Perhaps we might speak of it as the *envelope of the lobes*. It is true that this envelope happens to be the same for all values of m, at least for all rational values of m, and that its equation can be obtained by the usual process if we make m a variable parameter, but the envelope of a family of curves might be quite different from the "envelope" of the lobes of any one of them, and in the case of the curves represented by the given equation it might be difficult to apply the theory of envelopes to the variation of the given curve produced by a continuous variation of m.—EDITOR.

cardioid twice the size of the inner envelope, but this at once, although gradually, changed into a curve that became more and more curtate as the pen receded farther from the center at each revolution, until, in the middle of its compound period, when $\cos\theta + \cos m\theta$ was equal to zero, it momentarily drew the arc of a circle with the radius 2. After this the lobes of the curve repeated themselves in inverse order, while their axes kept on swinging in the same direction.

A study of Fig. 1 shows that the points of intersection of the lobes are arranged in radial lines at equal angular intervals, and that the points of tangency of the curve with the two envelopes are also spaced equiangularly.

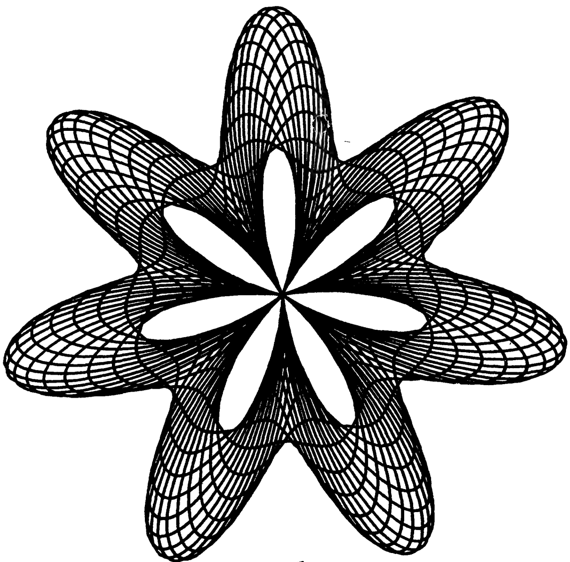
The Outer Envelope a Cardioid.—There is a second way of drawing an envelope that is a cardioid. In the first case we placed the pen at the center of the disk when the phase of each of its two harmonic components was 90°. Now let us make the phase 0° at the center. The first component A, if used alone, will then trace the circle $\rho = \sin\theta$, and the two together will trace $\rho = \sin\theta + \sin m\theta$. Proceeding as before, we find the envelope

$$\pm 1 = \rho - \sin\theta$$

or

$$\rho = 1 + \sin\theta$$

and $\rho = -1 + \sin\theta$ or $(1 + \sin\theta) - 2$, which are identical, or rather coincident, the first being traced as usual by the positive extremity of ρ, say by point P, and the other by a point P' at the constant distance of 2 from P in the negative direction of ρ. There is then practically only one envelope, which we may in the mechanical sense call an outer one. The mathematical axis of $+Y$ now runs to the left in Fig. 2. We may note that the equation of the second envelope of Fig. 2 has the constant -2 as opposed to $+2$ in the second one of Fig. 1.



7

Envelope Rosettes.—Generalizing the above results by using $n\theta$ in place of θ, we may apply the same principles to rosettes. Thus if we take $n=2$, there are two complete compound cycles of the pen to one of the disk, that is, the components A and B turn twice 15 and 16 times while the disk makes as before its usual 15 or 16 revolutions. We then have a rosette in the inner envelope of Fig. 3 and in the outer one of Fig. 4, using the latter expression in the mechanical sense. In the mathematical sense, however, there are two envelopes in Fig. 4, not coincident, but lying at

right angles to one another. The direction of the mathematical axes has also undergone a change. Their position in these and subsequent figures may readily be deduced from the respective equations, and will for that reason no longer be referred to.

When $n = 3$ we see that the usual inner envelope is a rosette in Fig. 5, and that the outer one in Fig. 6 is an equal one. Fig. 7 shows a septifolium as an inner envelope. The outer one was not drawn because it would have been almost totally black, as Fig. 6 leads us to suspect, on account of the great number of its close and overlapping lines.

From a mathematical point of view all these seven figures present envelopes that are rosettes. When the common phase of the two components A and B is made 90° at the center of the disk, as in Figs. 1, 3, 5, 7, we have two envelopes, an inner one which is a rosette, and an outer one in which the radius vector of the inner one is increased by 2. When the common phase is 0° at the center, as in Figs. 2, 4, 6, there are also two envelopes, both being equal rosettes. They are coincident when n is odd, but crossed equiangularly when n is even. From a mechanical standpoint, the first class of figures may be said to have rosettes as their inner envelopes and the second to have corresponding equal rosettes as their outer envelopes, the number of lobes being doubled in the latter case when n is even.

The Ratio of the Periods of the Components, or the Value of m.—In all the cases presented m was taken as 15/16 or 16/15, the harmonic component A making $15n$ revolutions and B $16n$, while the disk rotated 15 or 16 times. The number of revolutions of the disk, 15 or 16, must be $1/n$ that of one of the components. We may select either except when n is a factor of the one used, for then, as soon as the pen has run through one complete compound cycle, it will begin to retrace the curve already drawn, so that the figure will present a disappointing appearance of incompleteness, since it will have only one nth as many lines as it ought to have. For this reason the disk had to make 15 turns for $n = 2$ and 16 for $n = 3$. For $n = 7$, 15 were made, but 16 would have done equally well. For $n = 5$ (not shown) they had to be 16.

The Starting Phases of the Components.—When the phases of the components A and B were 90° and the pen was set down at the center of the disk, the inner envelopes it traced were the rosettes shown in Figs. 1, 3, 5, 7. The identical figures, only turned at right angles, were drawn when the pen was started in phase 0° on the mechanical Y axis at the distance $+ 2$ from the center, that is, at the upper end of a lobe. The reason is that in a quarter of a turn of the disk one of the components A or B advances exactly one or n quarters of a period also and the other only one-fourth of 1/15 or 1/16 more or less. This difference is insensible in practice when $m - 1$ is very small. In like manner the identical "outer-envelope" rosettes, turned at right angles, resulted, Figs. 2, 4, 6, when the pen was started on the Y axis at the distance $+ 2$ in phase 90° instead of at the center in phase 0°. From this it follows that the pen may be started in any equal phase α of its components and set down on the mechanical Y axis at the

distance 2 sin α from the center in the last case and 2 cos α in the first, in order to draw the same respective rosette, which will then be turned through the angle α on the disk. For this reason we might call the rosettes in Figs. 1, 3, 5, 7, "cosine" rosettes and those in Figs. 2, 4, 6, "sine" rosettes.

Transition Envelopes.—The principle just stated may be applied to show the transition from the (mechanical) outer- to the inner-envelope cardioid. Thus Figs. 8–12 are intermediate between Figs. 2 and 1. In all of these seven figures the starting point was at the center of the disk, but the phases were taken at 15° intervals from 0° to 90°. In Fig. 2 the phases of the pen at the start were 0°. In Fig. 8 the phases were 15°, in Fig. 9, 30°, in Fig. 10, 45°, in Fig. 11, 60°, in Fig. 12 75°, and finally in Fig. 1, 90°. The transition may thus be readily followed, and the axis of the envelope seen to swing round with uniform speed.

This identical series of seven transition envelopes might have been obtained by keeping the starting phases of the components at 90° and setting down the pen on the mechanical Y axis at the distance of twice the sines of 0°, 15°, $\cdots$ 90° from the center. In this case the axes of the envelopes would have remained stationary on the mechanical X axis. When the pen is set beyond the distance 2 sin 90°, the envelopes become curtate. Their inner faces will be the outer ones in Figs. 2, 8–12, 1, while their outer ones will tend to become more circular.

Unequal Starting Phases of the Components.—Instead of starting the pen with its components in equal phases, phase differences of any magnitude may be used. By studying the usual generation of Fig. 1 as given before, the initial position of the pen on the mechanical Y axis may so readily be deduced from the position it has there corresponding to the given phase difference, that numerical exemplifications are not necessary. The application to rosettes in general is also sufficiently obvious.

Finally Fig. 13 shows a transition envelope for an even value of n intermediate between Figs. 3 and 4.

THE COLLEGE AS A TRAINING SCHOOL FOR HIGH SCHOOL TEACHERS.[1]

By ERNEST B. LYTLE, University of Illinois.

This audience is no doubt familiar with the wonderful growth of high schools in the United States. In 1906 there were 52,394 pupils enrolled in the high schools of Illinois; in 1916 there were 102,870 enrolled, an increase of 96 per cent in ten years in Illinois alone. (*Illinois High Schools*, L. W. Smith, p. 9.) In the light of the recent marvellous growth in the number and community importance of our high schools, it is not necessary to argue here the great significance

[1] Address read before the Ann Arbor meeting of the Mathematical Association of America, Sept. 5, 1919; also before the Illinois Section of the Association, Nov. 22, 1919.

distance 2 sin α from the center in the last case and 2 cos α in the first, in order to draw the same respective rosette, which will then be turned through the angle α on the disk. For this reason we might call the rosettes in Figs. 1, 3, 5, 7, "cosine" rosettes and those in Figs. 2, 4, 6, "sine" rosettes.

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of training high school teachers. Colleges must see the problem and attack it vigorously. In training able teachers our colleges and universities not only render the high schools and their communities a great service but they help themselves by improving the preparation of the students coming to them for further education.

Our particular present interest is to discuss the part mathematicians are now taking in teacher preparation and to suggest ways and means of increasing their service in this direction. Good teaching requires (1) knowledge of subject matter and (2) a sympathetic understanding of students. There is no upper limit to the amount of scholarship desirable, the only question is how much scholarship is possible under given conditions. The few people who believe one can know too much to teach well are looking in the wrong place for the trouble; it is not too much scholarship but rather too little sympathetic understanding of students which makes many failures in teaching. However true it may be that some great scholars are poor teachers because they have permitted themselves to grow unsympathetic and impatient with immature minds, yet sympathetic understanding of the difficulties of less mature persons is by no means incompatible with high scholarship; we all know many master teachers who possess both scholarship and sympathy for the learner.

While recognizing no upper bound to knowledge of subject-matter desired in a teacher, it is still pertinent to inquire, "How much mathematics is it reasonable to require of prospective teachers of mathematics under present conditions?" Today our colleges are quite generally requiring from 20 hours (one hour a day for two school years) to 30 hours (one hour a day for three years) in mathematics to obtain their official recommendation to teach in our high schools. In many cases prospective teachers elect more than the minimum requirement from the regular advanced courses in mathematics. The usual sequence of courses through the calculus is quite uniformly required; but beyond this there seems to be little absolute requirement and considerable variation in practice. Most colleges making any pretense at high school teacher training offer a special Teachers' Course which considers the values, methods, subject matter, texts, reform movements, and best literature of secondary mathematics. The University of Michigan, in 1893, was the first college to offer a course on the teaching of algebra and geometry and by 1912 there were 34 American colleges offering such courses (Bibliography 2, p. 6). In 1916 there were 40 out of 100 selected colleges and universities offering some such teachers' course (Bibliography 8, p. 395). When offered such a course is required, or very strongly advised, for recommendation to high school mathematics positions. Most students elect a course in the theory of equations and determinants as an advanced course in algebra; not quite so uniformly they elect a course in projective or modern geometry. You are no doubt familiar with the discussions in the MONTHLY on the value, methods and content of such an advanced course in geometry.[1] As brought out in these discussions the value and importance of such a course in modern geometry is not

[1] AM. MATH. MONTHLY, Jan., 1914, p. 30, 31, 32; Feb., 1914, p. 63; Apr., 1918, p. 159.

yet fully appreciated. History shows how for a period the world neglected synthetic methods in its enthusiasm over the newly discovered analytic geometry, but later had to be brought back to a realization of the great beauty, power, and simplicity of synthetic geometry methods; and so also teachers-to-be after being introduced to the wonderful analytic geometry, in a later course need to be impressed with the value of a combination of the two methods.

Another important course is a unifying course generally offered under some such title as "Fundamental Concepts" or "Synoptic Course." Teachers particularly need to have perspectives, foundations, and essential conceptions impressed upon them. Teachers in high schools have spoken to me with considerable enthusiasm on the value of such a course. Young's book on *Fundamental Concepts of Algebra and Geometry* has ably made a start in this direction but in my own opinion we mathematicians are not yet appreciating and emphasizing this type of work as strongly as it deserves. Mitchell found only 8 such courses in the 100 institutions which he investigated (Bibliography 8, p. 396).

Teachers need work in the History of Elementary Mathematics since historical remarks in the class-room arouse interest; and teachers familiar with the historical background of their subject are not so easily misled by radical would-be reformers. History courses were offered in 24 of Mitchell's 100 institutions; 9 others devoted part of their teachers' course to history, making 33 institutions giving some formal teaching in the history of mathematics (Bibliography 8, p. 396). Columbia University and the University of Michigan offer graduate courses in the History of Mathematics.

These mathematics courses offered particularly for students preparing to teach—"Teachers' Course," "Fundamental Concepts" or "Synoptic Course," and "History of Elementary Mathematics"—are usually taken in the senior, occasionally in the junior year, and with few exceptions require as prerequisites courses at least through the calculus. Sometimes these courses are offered in the Schools of Education but generally by instructors specially trained in pure mathematics. Teachers College of Columbia University offers the widest variety of courses on the teaching of mathematics; their 1919–1920 special bulletin offers 11 different courses not counting practice teaching; the 1917–18 catalog of the University of Chicago School of Education offers four such courses and the University of Illinois gives three special teachers' courses.

One type of valuable teacher training seems to be nowhere explicitly offered, that is, the oral presentation of papers on special topics worked up in the libraries. Teachers certainly need training in the use of libraries, acquaintance with books other than text-books, as well as practice in the lucid and convincing presentation of well-organized material collected from the best sources our libraries afford. No doubt many colleges give some of this type of training implicitly in connection with other courses or in clubs. We raise the question whether this important type of training is not being neglected by being allowed to remain implicit? Is not a seminar for undergraduates aimed explicitly to train in the use of libraries and oral presentation worthy of emphasis? Is it not probable that high school teachers

thus definitely introduced to the riches of our libraries would show a greater interest in mathematics as a science and make greater efforts to increase their knowledge of the subject by more individual reading and study?

Besides the mathematics courses specially considered students preparing to teach frequently elect even up to a total of 40 hours from courses in pure mathematics such as differential equations, theory of functions of both real and complex variables, differential geometry, solid analytic geometry, higher algebra, limits and series. They sometimes elect applied courses in statistics, astronomy, physics and mechanics.

When so many colleges require twenty or more hours in mathematics before they will recommend graduates for high school positions, why is it that we still find so many of much less training teaching mathematics today? One great cause is the practice, which administrators believe is a necessity, of assigning to a teacher well prepared to teach one or two subjects classes in a third or fourth and sometimes a fifth subject. In our high schools we find many teaching mathematics as a third or fourth subject who make no pretense of either special training or interest in mathematics. One suggestion for meeting this so called necessity is to increase the number of junior, township and community high schools since these types of organizations make possible earlier and more complete departmentalization and make places for more special teachers who can devote their time to one or at most two departments of instruction.

Up to this point, with the possible exception of the "Teachers' Course," we have considered only that part of teacher training which aims at knowledge of subject matter. But mathematics in our high schools is a means of educating boys and girls; student psychology is the second and a very important side of teacher training. "Professional Training" is the term usually employed to characterize that part of teacher training which aims to give knowledge of adolescent interests, motives, and learning processes. A teacher may be saturated with mathematical knowledge and yet find it impossible to induce lively youngsters to increase their knowledge of the subject. The fluent character and great variations in adolescent mental life make teaching an arduous task. Too many pass by this side of teacher training with the dogmatism, "Oh, teachers are born, not made," or "Actual experience is the only way of learning how purposefully to influence students." Is it not as reasonable to say that physicians and surgeons must get all their knowledge of the human body from practice? Doctors are required to spend long periods in the study of human physiology and anatomy with all the generalizations from the experiences of previous successful doctors before they are permitted by law to practice medicine themselves. Why should it be thought unreasonable to expect teachers to study the experience of successful teachers and get their generalizations and suggestions before they are permitted to undertake the important work of directing the mental life of our young people? While granting all possible importance to natural abilities yet in no work at all comparable with education in importance do we rely entirely on methods of trial and error in experience with no preliminary training in theory. Notwithstanding

some objectors the requirement of some professional training for teachers is already a fact. The North Central Association of Colleges and Secondary Schools, including over one thousand high schools, now requires that "the minimum professional training of teachers of any academic subject shall be at least eleven semester hours in education" (Bibliography 7). The Illinois school law now requires an examination in educational psychology, and the principles and methods of teaching for a state high school teacher's certificate (*1919 Illinois School Law*, Circular 138, p. 6).

Schools and Colleges of Education are meeting this demand by developing courses in educational psychology, principles of secondary education, theory of teaching, special methods in particular subjects, observation and practice teaching, history of education, and educational sociology. While we who are workers in an old, exact and well-developed science may have felt that some of the past work in education has been unscientific in method and contains hasty and unwarranted generalizations, yet is not blundering always common in a new field? Education is very very much younger than mathematics and yet educational investigators of really scientific ideals have already appeared. Cannot we mathematicians be constructive rather than merely destructive critics; can we not be sympathetic and suggestive in our attitude toward this comparatively new, important and complex field of investigation? A body of educational facts and guiding principles is slowly being accumulated to help the teacher. The wide use of mental tests during the World War is a tribute to the painstaking workers in education. We should therefore require prospective teachers to take good courses in educational psychology, technique and theory of teaching, history of education, and, in case they have never taught, some practice teaching as their work in Professional Training.

The work in practice teaching is being constantly improved in quality and there is a noticeable tendency to raise the standards of scholarship of supervisors of practice teaching in our demonstration high schools. For example, at the University of Illinois where a new high school is about to be opened, the ideal set for the supervisor of mathematics work is one with a Ph.D. degree in both mathematics and education, if such cannot be found then there must be a Ph.D. degree in mathematics and as much training in education as possible. Likewise in each department they plan to require a doctor's degree in the field to be supervised together with specialization in education.

The Wisconsin plan of "directed teaching" in contrast to practice teaching deserves special mention. This plan is fully described in the *Eighteenth Year Book of the National Society for the Study of Education*. Limited time makes possible here only a hasty sketch of this unique plan. Each class is at all times in charge of an expert regular staff teacher; from one to three college seniors in training are assigned to a particular class; they become students in this class and participate in all class activities by preparing lessons and reciting just as regular students do, and this participation gives them the exact student viewpoint; at the same time these seniors discuss methods of organization and technique with the

expert teacher in charge to get the teacher's viewpoint; by consistent unfailing excellence a senior may win the right to assist the staff teacher and finally becomes the class leader for short or even long periods, but always under the direction of the staff expert; preparation through participation is the key note of this plan. The purpose of this special mention of the Wisconsin directed teaching scheme is its suggestions. Why cannot this plan be adapted to teacher training in our college classes as now organized? Why not assign one prospective teacher to each expert member of our present college faculties; by participation, assisting and personal contact with this expert teacher cannot the student get the best possible teacher training? Would not a plan of associating inexperienced beginners with older experts for careful direction and help be much better than our present plan of putting assistants in sole charge of sections to sink or swin unaided? Can we get practice teaching in college classes without waiting for the formation of practice schools which are often long delayed by the budget difficulties? It is not suggested that experimental schools be abolished or discouraged but rather that teacher training possibilities may be extended to college classes as at present organized.

Besides furnishing college students with first class teacher training our colleges and universities should also help teachers already in service. College instructors should coöperate heartily in high school teacher associations. The annual University of Illinois High School Conference has been a very potent factor in teacher contact and mutual inspiration; it has given high school teachers the points of view and advice of more expert mathematicians and it has given college instructors more sympathetic understanding of the high school teachers particular problems. The teachers' book-shelf should also receive the careful attention of college men. There is need for books in English such as Klein's *Elementarmathematik vom höheren Standpunkte aus*. High school teachers need more books like Young's *Fundamental Concepts of Algebra and Geometry*, Carson's *Mathematical Education*, Nunn's *Teaching of Algebra*, Whitehead's *Introduction to Mathematics* and *Organization of Thought*, Keyser's *Human Worth of Rigorous Thinking*, *Monographs on Mathematics* edited by Young, and specially little contributions similar to Teubner's "Mathematische Bibliothek" on topics like *Der Begriff der Zahl in seiner logischen und historischen Entwicklung*, *Der Pythagoreische Lehrsatz*, *Konstruktionen in begrenzter Ebenen*, etc. The writing or translating of books which will find places on the secondary teachers' book shelf would be a distinct service in raising standards of scholarship among teachers of mathematics.

In order that the important work of secondary teacher training should not be neglected each college department of mathematics, specially the larger ones, should assign to one member of its staff the problem of elementary teacher training. This instructor should visit high schools as often as possible and keep in touch with actual high school conditions; he should take as his field of research the problem of devising ways and means of improving instruction in mathematics.

Summing up the suggestions as points for your discussion, we college teachers

can help high school teacher training (1) by emphasizing the desirability of the greatest possible scholarship in mathematics, (2) by developing strong courses in theory of equations, advanced geometry, fundamental concepts and history, (3) by emphasizing in all our courses perspectives and big ideas, (4) by developing the use of libraries, knowledge of best literature and ability to prepare and forcefully present papers on mathematical topics, (5) by making one member of the staff responsible for investigation in the field of teacher training, (6) by encouraging professional training specially in adolescent psychology, (7) by taking constructive and not destructive attitudes toward scholarly investigation in education, (8) by demanding that supervisors of mathematics teacher training work have a Ph.D. or equivalent in mathematics as well as special training in education, (9) by studying the Wisconsin directed teaching plan with the purpose of modifying it to apply to college classes, and (10) by writing and translating books or articles which appeal to the present interests and attainments of high school teachers.

American standards of mathematics teacher preparation, both in scholarship and professional training, are below those of some other countries, specially those of France and Germany, and we are still far from the ideal set up by our American Commission on Mathematics Teaching (Bibliography 2, pp. 13–14). However American standards are slowly but surely rising; let us help this upward movement.

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QUESTIONS AND DISCUSSIONS.

Edited by W. A. Hurwitz, Cornell University, Ithaca, N. Y.

DISCUSSIONS.

In our last number, Professor Rees gave an instance of the use of vectors in connection with a problem in dynamics. In the first discussion below he indicates how the vector notation may be used to attack several questions in elementary geometry. It cannot be said that the proofs are much shorter than the usual ones; but the elegance of the vector formulation is obvious.

In an article in this department last October, devoted to the application of the theory of probability to various questions connected with the game of "craps," Mr. B. H. Brown gave the following theorem: *In any series of games where the probability of winning is constantly p, the average number of games won, up to and including the first lost game, is the reciprocal of the probability of losing.* He used this theorem to determine the average number of "rolls" required to complete a game. His proof, although relatively simple, involved the summation of the power series for $(1 - p)^{-2}$. In the second discussion this month Miss Charlotte Dickson gives an immediately obvious proof of the theorem. While the wording is phrased with reference to Mr. Brown's particular application, the work clearly holds for the general theorem.

Professor Frumveller contributes an "experience" article on the teaching of logarithms, suggested by the paper of Professor McClenon in the September Monthly. Like many articles on methods of teaching, its value lies not so much in the presentation of new pedagogical ideas as in recommendations concerning the distribution of emphasis. All teachers who are not committed to pure formalism will agree that equivalence of the logarithmic and exponential relationships, $y = \log_b x$, $x = b^y$ must be presented as the very soul of the theory of logarithms. With regard to Professor Frumveller's discussion of the difficulties associated with the negative characteristic, it is fair to state that none of the objections rightfully made to the notation -3.93817 can be extended to the notation $\bar{3}.93817$, which may legitimately be understood to mean $-3 + 0.93817$. Professor Frumveller closes his discussion by answering the query, for this kind of problem, when the wrong method can give the right result. The answer involves an elementary notion of number-theory.

Mr. C. N. Schmall in the closing discussion gives an illustration of the study of geometric questions by analytic methods. Sometimes the view is expressed that Cartesian coördinates should be introduced into the course in geometry in the secondary schools—after proofs of theorems on parallels and perpendiculars have made this logically possible—and used for certain demonstrations in lieu of the Euclidean tools. If such a procedure is to be seriously considered, it is desirable to produce instances in which analytic proofs are preferable. Can our readers supply a few such cases?

$$\begin{cases} a\boldsymbol{A} + a'\boldsymbol{A}' = b\boldsymbol{B} + b'\boldsymbol{B}' = c\boldsymbol{C} + c'\boldsymbol{C}', \\ \quad a + a' \quad = \quad b + b' \quad = \quad c + c'. \end{cases}$$

Calling the intersections of AB, BC, and CA with the corresponding sides of the other triangle, D, E, and F respectively, and proceeding in the usual way we find[1]

$$(a - b)\boldsymbol{D} = a\boldsymbol{A} - b\boldsymbol{B},$$

$$(b - c)\boldsymbol{E} = b\boldsymbol{B} - c\boldsymbol{C},$$

$$(c - a)\boldsymbol{F} = c\boldsymbol{C} - a\boldsymbol{A}.$$

Adding we have

$$(a - b)\boldsymbol{D} + (b - c)\boldsymbol{E} + (c - a)\boldsymbol{F} = 0.$$

Since the sum of the scalar coefficients of this equation is zero the theorem is proved.

It will be noted that the triangles may be in the same or different planes. Thus the proof is perfectly general.

CEVA'S THEOREM. *If in the triangle ABC the points A', B', and C' are taken on the sides BC, CA, and AB respectively, in such a way that*

$$(AC'/C'B) \cdot (BA'/A'C) \cdot (CB'/B'A) = 1,$$

then the lines AA', BB', and CC' are concurrent.

Proof. Calling the first and second ratios s/r and t/s respectively, the third ratio must be r/t by the hypothesis of the theorem. We have then

$$(s + t)\boldsymbol{A}' = s\boldsymbol{B} + t\boldsymbol{C},$$

$$(t + r)\boldsymbol{B}' = t\boldsymbol{C} + r\boldsymbol{A},$$

$$(r + s)\boldsymbol{C}' = r\boldsymbol{A} + s\boldsymbol{B}.$$

Adding $r\boldsymbol{A}$ to the first equation, $s\boldsymbol{B}$ to the second, and $t\boldsymbol{C}$ to the third, we see that the second members of the resulting equations are equal. Hence

$$(s + t)\boldsymbol{A}' + r\boldsymbol{A} = (t + r)\boldsymbol{B}' + s\boldsymbol{B} = (r + s)\boldsymbol{C}' + t\boldsymbol{C}.$$

The scalar coefficients of these equations satisfy the condition of the theorem, hence the proof is complete.

II. ON A THEOREM IN THE THEORY OF PROBABILITIES.

BY CHARLOTTE DICKSON, American Telephone and Telegraph Co., New York.

In this MONTHLY for October, 1919, Mr. Bancroft H. Brown gives a method for determining the "average number of rolls needed to decide a game of craps."

[1] We assume here that $a - b$, $b - c$, and $c - a$ are not zero. If any of these differences vanish then some of the corresponding sides are parallel, *i.e.*, intersect at infinity. Our justification for extending the proof to include the points at infinity lies in the fact that the points of intersection remain collinear when one or more of them recede to infinity.

A solution of the same problem by a somewhat simpler method might be of interest.

Let X be the average number of "rolls" needed further to decide a "point." Now we know that it is necessary to make at least one roll; and the probability of this, being certainty, is expressed by unity. If the "point" is not decided at the first roll (the probability of which is p or $1 - q$), the average number of rolls needed is still X. Therefore we arrive at the following equality,

$$X = 1 \times 1 + (1 - q)X$$

from which we find that $X = 1/q$.

This solution and that suggested by Mr. Brown are given by Louis Bachelier in his *Calcul des Probabilités,* volume 1, Paris, 1912, page 11.

III. Concerning the Teaching of Logarithms.

By A. F. Frumveller, Marquette University.

In this Monthly for September, 1919, Professor McClenon brought up the question of introducing logarithms by the historic method of geometric progression; he mentions as an alternative, the complete abandonment of all theory, and the laying down of mechanical rules of thumb for the student until the use of the tables has become automatic. The first or historic method has, it seems, been tried out successfully at the Hyde Park High School, Chicago; Mr. Josef Nyberg explained his proceedure in the October number of this Monthly for 1918, p. 337, but on reading his explanation it would seem as if the lack of directness of this approach would render it unsatisfactory in the hands of teachers less skillful and inspiring than Mr. Nyberg himself.

The late S. A. T. C. arrangement gave the present writer a splendid opportunity of trying out on a large scale the teaching of logarithms to students of average ability; and it so happened that the few days spent in this study afforded the only agreeable interlude in an experience that to most of us was like a nightmare. Our method was this; an equation like $y = x^2$ is written down; we define 2 to be the logarithm of y to the base x, and write $\log_x y = 2$ as the statement of this definition. This we call "translating the exponential equation to the logarithmic form," and the student is drilled at once in making translations of this kind by the dozens, till he can do it automatically; numerical cases like $9 = 3^2$ are next handled orally, with all possible modifications; *e.g.*, if 3 is the log of 8, what base am I using? etc. Many algebras are well supplied with such oral exercises; after a lively quiz of this sort, the students, *all* of them, had the idea of a logarithm firmly fixed in their minds. The rules for operating with logarithms gave no difficulty, since by definition a logarithm is merely another name for an exponent; fractional and negative exponents were then brought in and *translated* into logarithmic form, using always an exponential equation $y = x^{\pm n}$ as a starting point, and then employing numbers in place of y, x, and n, till the idea took root.

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(1) $\dfrac{6)3.93817 - 6}{0.65636 - 1}$, (2) $\dfrac{6)57.93817 - 60}{9.65636 - 10}$,

$n = (4.5329)/10 = 0.45329$; $n = (4532900000)/10{,}000{,}000{,}000 = 0.45329$;

(3) $\dfrac{6)7.93817 - 10}{1.32303 - 1.66666}$, (4) $\dfrac{6)2.93817 - 5.00000}{0.48969 - 0.83333}$,

$n = (2.104)/(4.641) = 0.4533$; $n = (3.088)/(6.813) = 0.4533$;

(5) $\dfrac{6)-2.06183}{-0.34364}$, (6) $\dfrac{6)-3.93817}{-0.65636}$,

$n = 1/(2.206) = 0.4533$; $m = 1/(4.5329) = 0.2206$.

The first two are the standard methods, but all are correct, except the last; here by a *mere accident*, the digits of the quotient turn out to be right, though the negative sign throws the number into the denominator; had we taken the seventh root, not even the digits would have been found, much less the proper sign!

After studying these results, the class voted unanimously that the use of negative characteristics was absurd and misleading; every one of these boys could use logarithms easily and correctly from that time forward. As an experiment in teaching logarithms, complete success was obtained, so that the writer sees no need of ever changing this method.

As a matter of curiosity, the question of settling under what circumstances the digits of the mantissa (in case no. 6 above) turn out correct was looked into; the reader may be interested in the answer. Let us use 0_k as a symbol for k zeros adjacent to one another in a given decimal number N; and let

$$\begin{aligned}\log N^x &= \log\,(0.0_k n_1 \cdots n_\lambda)^x \\ &= x[\log\,(n_1 \cdots n_\lambda) - \log\,(10^{k+\lambda})] \\ &= x[(\lambda - 1)\,.s_1 \cdots s_5 - (k + \lambda)] \text{ as correctly written.}\end{aligned}$$

Written *wrongly*, by subtracting the integral parts of these numbers while leaving the decimal part untouched, we get

$$\log\,(N^x) = x[-\,(k + 1)\cdot(s_1 \cdots s_5)].$$

Now let $x = 1/y$; then if $|k + 1| \equiv (\lambda - 1)$, (mod y), *i.e.*, if the quotients $(k + 1)/y$, $(\lambda - 1)/y$, have the same remainders, the mantissa $(s_1' \cdots s_5')$ in both cases will be the same.

IV. Relating to the Analytical Geometry of the Circle.

By Charles N. Schmall, New York.

E. H. Askwith in his *Analytical Geometry of the Conic Sections* (London, A. & C. Black, 1908), p. 78, § 90, says: "We could by analysis prove all the geometrical properties of the circle. It must not, however, be supposed that

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these properties are in all cases more easily proved by analysis than by pure geometry. Sometimes the methods of analysis are short and simple; but there are cases where they are complicated and inferior to the methods of pure geometry." He then proceeds to give a case illustrating the latter part of his statement, taking an example in which the use of analysis offers no advantage. He proves analytically that angles in the same segment of a circle are equal.

As an illustration of the alternative case of Mr. Askwith's remarks, I submit the following discussion, by analysis, of the relation between two circles in a plane. This example may prove interesting to students, as an instance where analytical methods are useful.

Let R and r be the radii of the two circles, and d the distance between their centers. Take the center of the first circle as the origin of rectangular coördinates, and the line through the centers as the x-axis.

Then the equations of the two circles are

$$x^2 + y^2 = R^2, \tag{1}$$

$$(x - d)^2 + y^2 = r^2. \tag{2}$$

Solving (1) and (2) for x and y, we get

$$x = \frac{R^2 - r^2 + d^2}{2d}, \tag{3}$$

$$\begin{aligned} y &= \pm \frac{1}{2d}\sqrt{4R^2d^2 - (R^2 - r^2 + d^2)^2} \\ &= \pm \frac{1}{2d}\sqrt{(2Rd + R^2 - r^2 + d^2)(2Rd - R^2 + r^2 - d^2)} \\ &= \pm \frac{1}{2d}\sqrt{\{(R + d)^2 - r^2\}\{r^2 - (R - d)^2\}} \\ &= \pm \frac{1}{2d}\sqrt{(R + d + r)(R + d - r)(r + R - d)(r - R + d)} \end{aligned}$$

or,

$$y = \pm \frac{1}{2d}\sqrt{(R + r + d)(R - r + d)(R + r - d)(r - R + d)}. \tag{4}$$

An examination of (3) and (4) shows that:

1. The abscissa x is always real.

2. If the continued product under the radical sign in (4) is *positive,* then the two points of intersection of the circles are *real.* These two points have the same abscissa, and two ordinates equal, but of opposite signs. Hence, *when two circles intersect, their common chord is bisected perpendicularly by the line through their centers.*

3. Since the first factor under the radical sign is always *positive,* the two values of y will be *real* if (a) the three other factors are *positive,* or (b) one of them is

Therefore, by (9) and (11), the circumferences will not meet when the difference of the radii is greater than the distance between the centers, *i.e.*, in that case one circle lies wholly within the other. Also, by (10), the circles will not meet when the sum of the radii is less than the distance between the centers; in which case, the circles are externally apart from each other.

RECENT PUBLICATIONS.

REVIEWS.

*The Philosophy of Mr. B*rtr*nd R*ss*l, with an appendix of leading passages from certain other works.* By P. E. B. Jourdain. London, G. Allen & Unwin, 1919. 96 pages. Price 3s. 6d.

This whimsical little book will be enjoyed by logicians who have not lost all their sense of humor, and by humorists who have not lost all their sense of logic. For others the humor may seem rather heavy and labored, and the logic a little more so. There is much quoting of the immortal works of Lewis Carroll,—indeed the Red Queen furnishes the motto for the book, "Even a joke should have some meaning,"—and many mildly amusing anecdotes are given. The following quotation from the chapter on the use of the Identity in Logic will give an idea of the spirit of the book:

"Mr. Austin Chamberlin, according to the *Times* of March 27th, 1909, professed to deduce the conclusion that it is not right that women should have votes from the premisses that 'man is man' and 'woman is woman.' This method requires that one should have made up his mind about the conclusions before discovering the premisses by what, no doubt, Jevons would call an 'inverse or inductive method.' Thus the method is of use only in speeches and in giving good advice. Mr. Austin Chamberlin afterwards rather destroyed one's belief in the truth of his premisses by putting limits to the validity of the principle of identity. In the course of the Debate on the Budget of 1909 he maintained against Mr. Lloyd George, that a joke was a joke except when it was an untruth; Mr. Lloyd George, apparently, being of the plausible opinion that a joke is a joke under all circumstances."

D. N. Lehmer

A short course in college mathematics comprising thirty-six lessons in Algebra, Coördinate Methods, and Plane Trigonometry. By R. E. Moritz. New York, Macmillan, 1919. 12mo. 9 + 236 pp. Price \$2.00.

Extracts from the Preface: "There has been an increasing demand in recent years for shorter courses in mathematics, based on the assumption, which our experience during the Great War has to some extent verified, that the ordinary processes of education may be greatly accelerated. . . .

"This little text is based on the supposition that such condensation and acceleration is possible. It was first prepared and printed for use in the mathematics classes of the Army and Navy Students Training Corps. . . .

"The book contains but thirty-six lessons, of which eighteen are given to the subject of trigonometry and the other eighteen to topics in algebra, to graphs, and to coördinate methods.

Yet it is hoped that no essential principle of elementary trigonometry has been omitted, and that this subject has been treated with a completeness sufficient for the needs both of the engineer and the student of the more advanced branches of mathematics, such as analytical geometry and the calculus."

Contents: Chapter I, Algebra: Factoring; radicals, fractional and negative exponents; imaginary quantities; quadratic equations; applied problems in quadratic equations; some problems in gunnery; review, 1–43. II, Graphic methods: Coördinates and simple graphs; related graphs; straight line graphs; simultaneous straight line graphs; curve plotting; maxima and minima; areas; the straight line and circle; graphic solution of equations; review,44–109. III, Trigonometric functions: The general angle and its measures; the trigonometric or circular functions; reductions to the first quadrant; functions of an acute angle; trigonometric graphs; solution of right triangles; arithmetic solution of oblique triangles; functions of the sum and difference of two angles; inverse trigonometric functions and trigonometric equations; review, 110–178. IV, Logarithms and their use: Exponents and logarithms; logarithmic computation; application of logarithms to mensuration of plane figures; logarithmic and exponential curves, 179–203. V, Logarithmic solution of triangles: Oblique triangles, cases I, II, III, IV; miscellaneous problems involving triangles; review, 204–232. Index, 233–236.

The Mystery of Space. A Study of the Hyperspace Movement in the Light of the Evolution of New Psychic Faculties and An Inquiry into the Genesis and Essential Nature of Space. By R. T. BROWNE. New York, Dutton, 1919. 8vo. 18 + 395 pp. Price $4.00.

Preface, first paragraph: "Mathematics is the biometer of intellectual evolution. Hence, the determination of the *status quo* of the intellect at any time can be accomplished most satisfactorily by applying to it the rigorous measure of the mathematical method. The intellect has but one true divining rod and that is mathematics. By day and by night it points the way unerringly, so long as it leads through materiality; but, falteringly, blindly, fatally, when that way veers into the territory of vitality and spirituality."

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NOTES.

In *General Mathematics,* [for first year in a high school] (Boston, Ginn, 1919. 12mo. 16 + 488 pp. $1.48), by R. SCHORLING and W. D. REEVE, members of the Association, "the material purposes to present such simple and significant principles of algebra, geometry, trigonometry, practical drawing, and statistics, along with a few elementary notions of other mathematical subjects, the whole involving numerous and rigorous applications of arithmetic as the average man (more accurately the modal man) is likely to remember and to use."

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ARTICLES IN CURRENT PERIODICALS.

Acta Mathematica, volume 42, no. 3, 1919: "Gaston Darboux (1842–1917)" by D. Hilbert, 269–273 ["Traduction du discours prononcé le 12 mai 1917 à la séance publique annuelle de l'Académie des sciences de Goettingue (*Nachrichten der K. Gesellschaft der Wissenschaften zu Göttingen*, 1917, p. 71)"]; "Darboux's Anteil an der Geometrie" by L. P. Eisenhart, 275–284 ["Uebersetzung der am 6. September 1917 vor der vereinigten Sitzung der 'American Mathematical Society' und der 'Mathematical Association of America' in Cleveland gehaltenen Vorlesung (*Bulletin of the American Mathematical Society*, volume 24, 1918, p. 227)"].

American Journal of Mathematics, volume 41, no. 4, October, 1919: "The ten nodes of the rational sextic and of the Cayley symmetroid" by A. B. Coble, 243–265; "Functions of matrices" by H. B. Phillips, 266–278; "On the Lüroth quartic curve" by F. Morley, 279–282; "On the order of a restricted system of equations" by F. F. Decker, 283–298; "On the Lie-Riemann-Helmholz-Hilbert problem of the foundations of geometry" by R. L. Moore, 299–319.

Bulletin of the American Mathematical Society, volume 26, no. 3, December, 1919: "Note on convergence tests for series and on Stieltjes integration by parts" by R. D. Carmichael, 97–102; "Note on a physical interpretation of Stieltjes integrals" by R. D. Carmichael, 102–105; "A derivation of the equation of the normal probability curve" by W. D. Cairns, 105–108; "Bôcher's boundary problems for differential equations" [review of M. Bôcher's *Leçons sur les Méthodes de Sturm dans la Théorie des Equations Différentielles Linéaires et leurs Développements Modernes*, . . . edited by G. Julia (Paris, 1917)] by R. G. D. Richardson, 108–124; "Dickson's *History of the Theory of Numbers*" [review of Vol. I, Washington, 1919] by D. N. Lehmer, 125–132; "The calculus of probability" [review of G. Castelnuovo's *Calcolo delle Probabilità* (Milano-Roma-Napoli, 1919)] by R. D. Carmichael, 132–135; "Corrections," 136–137; "Notes," 136–141; "New publications," 142–144.

Bulletin of the Calcutta Mathematical Society, Volume 9, no. 2, March, 1919: "The late Sir Gooroodas Banerjee" by S. K. Banerji, i–v and portrait frontispiece ["By the death of Sir Gooroodas Banerjee, which melancholy event took place on the 2d of December, 1918, the Calcutta Mathematical Society has lost not only its senior Vice-President but one who took the deepest interest in all its proceedings, one whose great learning, splendid endowments of head and heart, deep piety, purity of character and loftiness of aims and principles were always object-lessons to its members. Manifold as were the activities of Sir Gooroodas Banerjee, his life is one of considerable interest to the mathematician. Sir Gooroodas began his life as a teacher of mathematics and ended it as an author of mathematical text-books. Whether as a lawyer, a judge, or an educationist, Sir Gooroodas owed, in no inconsiderable degree, his sound judgment, his accurate logic and his strong common sense to his high attainments as a mathematician.

"Sir Gooroodas was born on the 27th of January, 1844, in Narikeldanga, in the suburbs of Calcutta. . . . Many years ago he wrote a book on the Elements of Arithmetic and Algebra. For a long time he could hardly find time and leisure to pursue this favourite subject of his and it was only after his retirement from the Bench that he could renew his mathematical activities. . . . [He] brought out in 1906 his *Elementary Geometry* adapted to modern methods. This book is a small volume of 142 pages and comprises within its limits the substance of the first six books of Euclid and almost all the elementary propositions of Solid Geometry. The most attractive feature of the book is that a beginner can learn with ease and within a short time all the important elementary truths of Geometry. This book has already passed through several editions and can be seen in the hands of almost all our school-going boys. . . . Quite recently Sir Gooroodas formed an elaborate plan of bringing out a series of mathematical text-books in our own vernacular Bengali under the name *Saral Ganit* and published three parts dealing respectively with Arithmetic, Algebra, and Geometry based on modern methods. All three volumes are characterized by the same remarkable simplicity and conciseness which form the distinguishing features of his English treatise on Elementary Geometry"]; "On the figures of equilibrium of two rotating masses of fluid for the exponential potential e^{-kr}/r," Part 1 by A. Datta, 59–70; "Fourier's series and its influence on some of the developments of mathematical analysis" by A. C. Bose, 71–84 ["In the *Bulletin of the Calcutta Mathematical Society* (vol. 7, pp. 33–48), an account has been given of the Life and Work of Fourier. The present paper endeavours to state briefly how some of the concepts of modern mathematics of a far-reaching character, arose out of Fourier's Analysis, justifying P. E. B. Jourdain's remark that 'if it is safe to trace back to any single man the origin of those conceptions with which pure mathematical analysis has been chiefly occupied during the nineteenth century and up to the present time, we must, I think, trace it back to Jean Baptiste Joseph Fourier

(1768–1830).' . . . He died full of honours and to the infinite regret of his colleagues who had come to appreciate his eminence as a physicist and a mathematician. As Arago said 'his life had been so varied, so laborious, so gloriously interlaced with the greatest events of the most memorable epochs of French History, that the scientific discoveries of the illustrious Secretary [of the Academy of Sciences] had nothing to dread from the incompetence of the panegyrist.'

"We bow then to the greatness of Jean Baptiste Joseph Fourier and we greet him, in the 20th century, with the eloquent words of Sir William Rowan Hamilton, another great Mathematician whom Fourier charmed and inspired in writing of fluctuating functions:

"Fourier with solemn and profound delight,
Joy born of awe, but kindling momently
To an intense and thrilling ecstasy,
I gaze upon thy glory and grow bright:
As if irradiate with beholden light;
As if the immortal that remains of thee
Attune me to thy spirit's harmony,
Breathing serene resolve and tranquil might,
Revealed appear thy silent thoughts of youth,
As if to consciousness, and all that view
Prophetic, of the heritage of truth
To thy majestic years of manhood due:
Darkness and error fleeing far away,
And the pure mind enthroned in perfect day."];

"On the numerical calculation of the roots of the equations $P_n^m(\mu) = 0$ and

$$\frac{d}{d\mu} P_n^m(\mu) = 0$$

regarded as equations in n" by B. Pal, 85–95; "On some new theorems in the geometry of masses" by S. Dhar, 96–107; "On the electric resistance of a conducting spheroid with given electrodes" by S. Kar, 109–114—Volume 10, no. 1, June, 1919: "On surface waves and tidal waves near a promontory" by S. Banerji, 1–10; "On the potentials of uniform and heterogeneous elliptic cylinders at an external point" by N. Sen, 11–27; "Notes on inversion" by T. Bhattacharyya, 29–34; "On the use of Ritz's method for finding the vibration-frequencies of heterogeneous strings and membranes" by N. K. Majumdar, 35–42; "On the steady motion of a viscous fluid due to the rotation of two rigid bodies about arbitrary axes" by B. Dutt, 43–61; "Obituary Notices," 63 [1. The late Mr. Chandrashekkar Sircar; 2. The late Principal Ramendrasundar Trivedi]—No. 2, September: "New methods in the geometry of a plane arc. II—Cyclic points and normals" by S. Mukhopadhyaya, 65–72; "Origin of the Indian cyclic method for the solution of $Nx^2 + 1 = y^2$" by P. C. Sen-Gupta, 73–80 [First and last paragraphs: "The object of the present paper is to discuss the probable origin of the 'Cyclic Method' (*Chakrabala*) for the solution of $Nx^2 + 1 = y^2$ in rational integers as given in Bhaskara's *Vijaganita*. Two hypotheses have been advanced as regards its origin: first that the method has an ultimate Greek source and secondly that it is purely Indian. I shall first discuss the former view and shall next show that it is untenable in the light of the reasons which, I trust, are put forth herein for the first time. . . . It is thus seen that to arrive at the Indian cyclic rule it is not at all necessary to determine an approximation to $\sqrt{N}$ either by the Archimedian method or by any other method. It is further evident that the rules are immediate deductions from the lemma of Brahmagupta and the sole credit of finding a method for the solution of $Nx^2 + 1 = y^2$ belongs to him"]; "On the motion of an ellipsoid of revolution in a viscous fluid in the light of Prof. Oseen's objection to Stokes's treatment of the case of the sphere" by B. Pal, 81–94 ["The motion of a sphere in a viscous fluid has been investigated by various writers, including Stokes, Profs. Whitehead, Oseen, Lamb, Burgess, the results obtained being more or less satisfactory according to the degree of approximation to which the differential equations are satisfied.

"In the present paper, I propose (1) to obtain the solution of the problem of the motion of translation of an ellipsoid of revolution of small ellipticity in a viscous fluid, the method adopted being similar to that of Prof. Lamb for treating the corresponding problem in the case of the sphere, and (2) to show how the results obtained by me although different in some respects from those given by Oberbeck, the only important writer who investigated the ellipsoidal problem before me, are free from any objection similar to that pointed out by Prof. Oseen in Stokes's treatment of the spherical problem."]; "On a class of ellipsoidal harmonics and a method of

solving the wave equation in ellipsoidal coordinates" by S. Banerji, 95–104; "Some cases of tidal oscillations in canals of variable section" by S. Dasgupta, 105–116; "The stress-equations of equilibrium" by S. Basu, 117–121; Review by S. M. of Miller, Blichfeldt, and Dickson's *Theory and Applications of Finite Groups* (New York, 1916), 123–124.

Colorado College Publications, science series, volume 12, no. 3, 1907: "On the transformation of algebraic equations by Erland Samuel Bring (1786)," translated and annotated by F. Cajori, 63–90—No. 7, 1910; "A history of the arithmetical methods of approximation to the roots of numerical equations of one unknown quantity" by F. Cajori, 171–287—No. 15, November, 1919; "On non-ruled octic surfaces whose plane sections are elliptic" by C. H. Sisam, 639–655.

Creighton-Courier, Creighton University, Omaha, Nebraska, volume 8, No. 12, February 7, 1920: "Einstein's theory of gravitation" by W. F. Rigge, 2–3.

Educational Review, volume 59, January, 1920: "Mathematics as a study" by G. S. Painter, 19–40. [First paragraph: "It is a certainty that educational wisdom will not die with this generation. Never in our history perhaps have such perplexity and indecision prevailed as now relative to the ends and methods of education. Nothing is any longer simple or fixt; everything is in a state of flux. The parvenu educational iconoclast is abroad in the land, and with opprobrious assurance hesitates not to lay unholy hands upon our most sacred educational traditions and institutions. No science or subject of investigation is any longer of value in and of itself, but only in so far as it contributes to something else. The implicit motive back of all this attitude is one of vulgar utility which is regarded as the sole and only reason for the existence of anything. We are, then, somewhat in a state of educational anarchy in which, with characteristic unwisdom, the blind are leading the blind with the conventional result of all getting into the ditch."]

L'Enseignement Mathématique, volume 20, no. 5, October, 1919: "Sur une transformation élémentaire et sur quelques intégrales définies et indéfinies" by C. Cailler, 317–337; "Sur l'intégrale $n!\int_0^h \frac{h^n e^{-(hx/1-h)}}{1-h}\,dh$" by F. Vaney and M. Paschoud, 338–346; "Extension de la notion de Jacobien" by M. Stuyvaert, 347–354; "Sur la représentation proportionnelle en matière électorale" by G. Pólya, 355–379 [First paragraph: "Dans plusieurs périodiques non mathématiques,[1] j'ai essayé de mettre en contact l'analyse mathématique avec l'énorme diversité des opinions émises sur la question de la représentation proportionnelle en matière électorale. La partie la plus intéressante de la recherche est, me semble-t-il: trouver, dans une littérature de controverse qui s'éloigne beaucoup de l'exposition et des sujets mathématiques habituels, des principes tangibles, des faits susceptibles d'une explication exacte et les 'mettre en équation.' Dans les travaux cités j'ai énoncé plusieurs résultats mathématiques. Je les ai vérifiés expérimentalement par des exemples, j'ai tâché de les rapprocher du bon sens sans l'aide des formules, mais j'ai dû omettre les démonstrations. Dans les lignes suivantes, je donnerai l'analyse exacte, une analyse très élémentaire d'ailleurs, mais qui ne sera peut-être pas dépourvue d'un certain intérêt pour quelques lecteurs"]; "Mélanges et correspondance" ["Au sujet de la publication de mon article sur 'Les origines d'un problème inédit de E. Torricelli' (*L'Enseignement Mathématique*, vol. 20, pp. 245–268), je dois signaler que M. Michele Cipolla, professeur à l'Université de Catane, vient de faire paraître une important étude sur le même problème: Michele Cipolla, "I triangoli di Fermat e un problema di Torricelli," *Atti dell'Accademia Gioenia di scienze naturali in Catania*, serie 5, vol. 10, memoria XI . . ." Emile Turrière.]; "Chronique;" "Notes et documents;" Review by C. A. Laisant of G. Bouligand's *Cours de géométrie analytique* (Paris, 1919), 390–391; Review by A. Buhl of P. Boutroux's *Les principes de l'analyse mathématiques*, tome 2 (Paris, 1919), 391–392; Review by E. Turrière of F. C. Clapier's *Sur les surfaces minima ou élassoides* (Paris, 1919), 392–393; Review by A. Buhl of G. H. Halphen's *Œuvres*, and R. de Montessus de Ballore's *Introduction à la théorie des courbes gauches algébriques* (Paris, 1918), 393–397; "Bulletin Bibliographique."

Journal of Educational Research, University of Illinois, volume 1, no. 1, January, 1920: "Hurdles, a series of calibrated objective tests in first year algebra" by M. A. Dalman, 47–62.

Journal of the Indian Mathematical Society, volume 11, no. 5, October, 1919: "A problem of diophantine approximation" by G. H. Hardy, 162–166; "A general theorem

[1] *Schweiz. Zentralblatt für Staats- und Gemeindeverwaltung*, 1919, no. 1; *Journal de statistique suisse*, 1918, no. 4; *Wissen und Leben*, January and February, 1919. *Zeitschrift fur die gesamte Staatswissenschaft* (sous presse).

relating to the cartesian oval" by A. C. L. Wilkinson, 167–172; "Multiplication of infinite integrals" by K. B. Madhava, 173–180; "A proof of Bertrand's postulate" by S. Ramanujan, 181–182; "Astronomical notes" by T. P. Bhaskara Sastri, 183–184; Problems and solutions, 185–200; "Mathematics from periodicals" by V. B. Naik, i–v.

Mathematical Gazette, volume 9, October, 1919: "The graphical treatment of differential equations" by S. Brodetsky, 377–382 (to be continued) [*First two paragraphs:* "The subject of this paper is one that presented itself in a piece of work of a practical nature. The development of aeroplane flight naturally suggested the investigation of the motion of a body in a resisting medium. In general this problem was too difficult for solution. One therefore looked for legitimate means of simplifying it. One simple type of motion allied somewhat to areoplane motion was that of a plane lamina moving in air under the earth's attraction. Even here difficulties arose in the solution of the resulting equations of motion; yet further simplification was not permissible, for care had to be taken that the simplified problem did bear some relation to the actual problem presented by nature. It was ultimately found that the simplest possible form of the problem gave rise to the differential equation

$$\frac{dy}{dx} = -\frac{x}{y} - (x^2 + y^2)^{1/2},$$

or, putting $(x^2 + y^2)^{1/2} = r$,

$$\frac{dr}{dx} = -y.$$

The disconcerting feature about this equation was the fact that it was found to be quite insoluble by any of the 'standard forms' dealt with in books on differential equations. Several mathematicians tried to discover a 'transformation' or an 'integrating factor,' but without success. Yet a solution had to be found somehow.

"It then occurred to the writer that where analysis had failed, geometry might succeed. The solution of a differential equation of the first order is of course represented geometrically by a family of curves. The ordinary treatment of differential equations consists in seeking for an analytical representation of these curves. Since, then, the analytical formula was apparently unobtainable, might not the curves themselves be 'graphable'? The result was a complete success"]; "Coördinate geometry in schools" by W. J. Dobbs, 383–388 (to be continued); Reviews of F. S. Woods and F. H. Bailey's *Analytic Geometry and Calculus* (Boston, 1918), H. B. Phillips's *Differential and Integral Calculus* (New York, 1916–1917), H. Hancock's *Theory of Maxima and Minima* (Boston, 1917), H. Bateman's *Differential Equations* (London, 1918), and *Annuaire pour l'an* 1919, (Bureau des Longitudes, Paris, 1919) 389–391 [*Last paragraph of last mentioned review:* "The essays which are always a notable feature in the Annuaire, are this year by MM. Appell and Hamy. The former finds a congenial topic in 'The figures of relative equilibrium of a rotating homogeneous liquid' (60 pp.). After an historical sketch of the theory up to the work of the late MM. Liapounoff and Poincaré, he deals with the problem of stability and the phenomena at the point of bifurcation. The value of the paper is greatly enhanced by a full bibliography. M. Hamy treats of the inference of the real diameters of minor planets and satellites from a study of the interference fringes (27 pp.). He believes that, with the more powerful instrumental opportunities at our disposal in the 100 in. reflector at Mount Wilson, it will be quite possible to determine the angular-diameters of 1st magnitude stars. The excellent table of contents runs to 69 pp.].

Messenger of Mathematics, volume 48, no. 11, March, 1919: "The dissection of rectilineal figures "(conclusion) by W. H. Macaulay, 161–165; "On a Diophantine problem" (second paper) by H. Holden, 166–176—No. 12, April: "On a Diophantine problem" (conclusion) by H. Holden, 177–179; "Sur quelques intégrales definies" by S. P. Shensen, 179–184; "On n-poled cassinoids" by H. Hilton, 184–192 [The polar equation of the curve is $r^{2n} - 2r^nc^n \cos n\theta + c^{2n} = a^{2n}$].

Nature, volume 104, November 20, 1919: "Percussion figures in isotropic solids" by J. W. French, 312–314—November 27: "Gravitation and light" by O. J. Lodge, 334 [*The note:* "It may or may not have been noticed that the refractivity $(\mu - 1)$ at any point, required to produce the Einstein deflection, is the squared ratio of the velocity of free fall from infinity to the velocity of light."]; "A new astronomical model" by A. L. Cortie, 343–344. [*First paragraph:* "The illustrious scholar Gerbert (A.D. 940–1003), afterwards Pope under the name of Sylvester II., was apparently the first of the schoolmen who illustrated his theoretical lessons on astronomy by the use of globes, which he constructed with his own hands. About the year A.D. 1700 George

Graham invented a machine to show the movements of the earth and planets about the sun, a copy of which was made by Charles Boyle, the Earl of Orrery. Hence the name of an apparatus very useful for illustrating lessons in astronomy, although Sir John Herschel did call orreries 'very childish toys.' But surely the difficulty in teaching astronomy is to make the young pupil think in three dimensions. What are we going to do when the relativists would have us imagine phenomena in four dimensions?" *Last paragraph:* "Dr. Wilson is to be heartily congratulated on having produced such a valuable, workable astronomical model. So many science masters—excellent men!—have desired to acquire it that he has felt justified in putting it upon the market and getting it made in quantities. The price is 22*l.* net, carriage paid to any part of the United Kingdom. All communications regarding the model should be addressed to Dr. Wilson himself at 43 Fellows Road, London, N.W. 3."].—December 4: "Gravitation and light" by O. J. Lodge, 354; "The displacement of light rays passing near the sun" by A. Anderson, 354; "Einstein's relativity theory of gravitation" by E. Cunningham, 354–356; [Note], 360 ["The *Times* of November 28 contains an article from Prof. Einstein on his generalized principle of relativity. Prof. Einstein remarks at the beginning of the article: "After the lamentable breach in the former international relations existing among men of science, it is with joy and gratefulness that I accept this opportunity of communication with English astronomers and physicists. It was in accordance with the high and proud tradition of English science that English scientific men should have given their time and labour, and that English institutions should have provided the material means, to test a theory that had been completed and published in the country of their enemies in the midst of war.' After a brief account of the general nature of the theory, which does not add anything to what has been summarised by Prof. Eddington in his report to the Physical Society, Prof. Einstein concludes: 'The great attraction of the theory is its logical consistency. If any deduction from it should prove untenable, it must be given up. A modification of it seems impossible without destruction of the whole. No one must think that Newton's great creation can be overthrown in any real sense by this or any other theory. His clear and wide ideas will for ever retain their significance as the foundation on which our modern conceptions of physics have been built. . . . By an application of the theory of relativity to the taste of readers, to-day in Germany I am called a German man of science, and in England I am represented as a Swiss Jew. If I come to be regarded as a *bête noire*, the descriptions will be reversed.' Prof. Eddington, in the *Contemporary Review*, quotes from Newton's *Opticks:*—'Query 1. Do not bodies act upon light at a distance, and by their action bend its rays?'"]; [Einstein theory at the anniversary meeting of the Royal Society], 361–362.—December 11: "Gravitation and light" by O. J. Lodge, 372 ["Jupiter ought just to show the Einstein deflection, for if it pass between two stars a couple of diameters of the planet apart, their temporary relative displacement will be a 'third' of arc, the sixtieth of a second; and this could be measured with a heliometer"]; "The deflection of light during a solar eclipse" by A. S. Eddington and A. C. D. Crommelin, 372–373; "Einstein's relativity theory of gravitation. II—The nature of the theory" by E. Cunningham, 374–376—December 18: Review by S. Brodetsky of Maria M. Roberts and Julia T. Colpitts's *Analytic Geometry* (New York, 1918), P. Goyen's *Elementary Mensuration, constructive plane geometry, and numerical Trigonometry* (London, 1919) and J. B. Shaw's *Lectures on the Philosophy of Mathematics* (Chicago, 1918), 390–391; "Deflection of light during a solar eclipse" by W. H. Dines, L. F. Richardson, and A. Anderson, 393–394. "Einstein's relativity theory of gravitation. III.—The crucial phenomena" by E. Cunningham, 394–395.—January 22, 1920: "Gravitation and light" by J. Larmor, 530; "The Einstein theory and spectral displacement," two notes by H. F. Moulton and A. C. D. Crommelin, 532—January 29; "The works of Toricelli" by J. L. E. D., 557–558; "The deflection of light during a solar eclipse" by A. Anderson, 563; "Displacement of spectral lines" by R. W. Lawson, 565 ["In view of the discussion in *Nature* and elsewhere on this subject, the following extract from a recent letter of Prof. Einstein may be of interest: 'Zwei junge Physiker in Bonn haben nun die Rot-Verschiebung der Spektral-Linien bei der Sonne so gut wie sicher nachgewiesen und die Gründe des bisherigen Misslingens aufgeklärt.' I have heard no details, but doubtless an account of this work will be available before long"]—February 5: "The predicted shift of the Fraunhofer lines," by J. Rice and A. S. Eddington, 598–599; "The straight path" by A. A. Robb, 599; "Mathematics in the United States" by G. B. M[athews], 601—February 12: "Euclid, Newton, and Einstein" by W. G., 627–630; "The theory of relativity" by A. C. D. Crommelin, 631–632.

Philosophical Magazine, volume 38, November, 1919: "The Bessel-Clifford function and its applications" by G. Greenhill, 501–528.

eight under the heading "Algebra"; thirteen under "Theory of numbers"; one hundred and twenty under "Analysis"; twenty-nine under "Groups"; seventy-six under "Geometry"; twelve under "Applied mathematics."]—Volume 21, no. 1, January, 1920: "The strain of a gravitating sphere of variable density and elasticity" by L. M. Hoskins, 1–43; "The geometry of hermitian forms" by J. L. Coolidge, 44–51; "Certain types of involutorial space transformations" by F. R. Sharpe and V. Snyder, 52–78.

AMERICAN DOCTORAL DISSERTATIONS.

E. F. SIMONDS, "Invariants of differential configurations in the plane," *Transactions of the American Mathematical Society*, 1918, volume 19, pp. 222–250 (Columbia, 1917).

UNDERGRADUATE MATHEMATICS CLUBS.

EDITED BY U. G. MITCHELL, University of Kansas, Lawrence.

CLUB TOPICS.

Although much has already been printed concerning the abacus and its uses,[1] we believe that our readers will find the following article by Professor Leavens decidedly interesting and helpful since it gives an independent discussion based upon the personal impressions and first-hand information of a westerner who has come into contact with the present-day use of the abacus in the far east.

17. THE CHINESE SUAN P'AN.

By DICKSON H. LEAVENS, College of Yale in China.

The Chinese suan p'an[2] or abacus is familiar to many from an occasional sight of it on a laundryman's table, but it is perhaps usually regarded either as a device full of the mystery of the East and beyond the grasp of the Occidental, or as an instrument fit only for the ignorant "Celestial" and beneath the notice of one who has studied arithmetic.

A little investigation, however, will show one that it is not only perfectly

[1] Two of the best discussions in English are probably C. G. Knott's article, cited below, and Leslie's *Philosophy of Arithmetic* (Edinburgh, 1820), pp. 15–100. Leslie gives, in great detail, examples of the representation of numbers in different scales of notation and of operations by means of them. From his discussion one can readily see how certain theorems on divisibility of numbers and even the summation of special infinite descending series may be inferred from the use of the abacus.

Some excellent illustrations and references to the literature of the abacus can be found in Smith and Mikami's *History of Japanese Mathematics* cited below. Other readily accessible sources of information are the descriptions given in current histories of mathematics and articles in encyclopedias under the titles "Abacus" and "Calculation."

[2] In this MONTHLY (1919, 256), it is noted that this word, with various spellings, appears in the *New English Dictionary*, but that "the Chinese word Soroban . . . . is not given." Soroban, however, is not Chinese, but Japanese, being the Japanese pronunciation of the same characters, which are used in the written language of both countries.

The same note (following the dictionary) translates suan p'an wrongly as reckoning *board*. The Chinese character for board is romanized pan, hence the error; but it is quite a different character from the one here used, which means a *plate* or *tray;* the pronunciation is also different, p in the most used system of romanization representing practically our b, while p' is similar to our p.

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simple, but that it is also not to be despised as an aid in arithmetical work when an adding machine is not available.

The instrument consists simply of a rectangular frame, usually with a back so that it forms a kind of tray, containing a number of rods, about 1/8 inch in diameter, made of wood, bone, or metal, set parallel to each other and to the shorter dimension of the frame. On each rod are strung seven wooden beads, the upper two separated from the lower five by a "bridge" which runs the whole length of the frame. In the Chinese suan p'an, the beads are roughly in the shape of a 60° equatorial zone section of a sphere; in the Japanese soroban, they are in the form of a double cone, with a rather sharp angle where the bases join, and it is claimed that this sharp edge makes for quicker manipulation than the rounded Chinese form.

The most usual size of the instrument is about seven by fourteen inches, and contains thirteen rods or columns, with beads about 7/8 inch in diameter. They may also be obtained with more columns, or with smaller beads, down to about 3/8 inch in diameter; but these are too small for convenience, and the standard size, or a little smaller, is the most practicable. The cost is from fifty cents to a few dollars, depending on the size, and on the quality of the wood.

Each rod represents one column, arranged in decreasing powers of 10 from left to right, as in our system of notation. No decimal point is marked on the ordinary instrument, the user locating it mentally for the problem he is working on. Banks sometimes have the columns labelled, to facilitate computations in the exchange of money. The westerner will find it convenient to tie strings between the columns, separating them into groups of threes; coloring the beads differently in groups of three columns, as is done with the keys of adding machines, might also be a help.

In each column, each of the lower five beads represents 1 unit, and each of the upper two represents 5 units. Properly, only four beads below the bridge, and one above are needed to set up 9 in each column; the Japanese soroban does omit one of the upper beads, thus allowing only 10 to be set up. The Chinese form permits the setting up of 15 in each column; this is never done in addition, as 10 is at once carried to the next column on the left, but in long division it is a convenience to have the extra capacity in setting up the steps of the problem.

It is in this use of one upper bead to represent 5 units that the suan p'an is far superior as a practical device to the abacus of ten beads, often sold as a children's toy or for use in primary arithmetic. On the suan p'an, the eye can take in at a glance the number in a group of beads, never more than four; while on the ten bead form, it is often necessary to stop and count. The arrangement in vertical columns is also a preparation for the decimal notation, much better than the horizontal rows of the American form. Still another advantage is that the beads slide easily on the wooden rods, while in the American form, the thin wires are easily bent, and the beads do not slide so smoothly.

Before beginning a problem, the lower beads are slid down by tilting the instrument, and the upper ones are pushed up with a sweep of the hand, leaving

none adjacent to the bridge. Numbers are set up by moving the beads towards the bridge. The figure below represents the number 15387652 set up, the decimal point to be provided according to the conditions of the problem, and there still being five blank columns to the left.

```
        0 0 0 0 0 0 0 0 0 0 0 0 0
        0 0 0 0 0 0   0         0

                    0   0 0 0 0
Bridge  -------------------------
                  0   0 0 0 0   0
                      0 0 0     0
                      0 0
        0 0 0 0 0   0         0
        0 0 0 0 0 0 0       0 0
        0 0 0 0 0 0 0     0 0 0 0
        0 0 0 0 0 0 0 0 0 0 0 0 0
        0 0 0 0 0 0 0 0 0 0 0 0 0

                  1 5 3 8 7 6 5 2
```

The addition of 1 or 2 to this number is done simply by moving up 1 or 2 beads in the unit column. If it is desired to add 3, moving up 3 beads would make 5 beads below the bridge, and we do not wish to use the 5th bead. But adding 3 is the same as adding 5 and subtracting 2, so we bring down one of the upper beads to the bridge, and push down, or as we might say, "erase" the two lower ones. Similarly to add 4, we add 5 and erase 1, etc. To add 8 to 2 (or to any number greater than 2) we add 1 in the next column to the left, and erase 2, etc. Note that we do not actually get 5 beads below, and then change them for 1 above, etc.; we foresee that carrying is necessary and do it as we add. It follows that the same digit may be added in different ways, according to what digit it is to be added to. For instance to add 7 to 0, 1, or 2, we add 5 and 2; to add 7 to 3, 4, 8, or 9, we subtract 3, and add 1 (*i.e.* 10) in the next column to the left; to add 7 to 5, 6 or 7, we add 2, subtract 5, and add 1 (*i.e.* 10) in the next column to the left. Chinese books of instruction give tables of all possible combinations of two digits, showing just when to add the original number, and when to add 5 or 10, and subtract the complementary part. For a child learning to use the instrument, the learning of these tables is perhaps a necessity; but an adult need never see such tables, and with a little practice will soon learn the combinations, so that the fingers make them automatically with little help from the brain.

The addition of numbers of several digits is not done vertically by columns, as in our system, but horizontally by the complete numbers. The first number is set up, then the digits of the second are added, one by one, working from left to right. Then the third number is added, and so on. The working from left

to right is a distinct advantage, as it gives the digits in the order in which they are written or spoken. This enables one person to use the instrument while another calls off the numbers to him, if necessary; although it is of course more often used by one man alone.

For a fuller description of the method of addition, and also for subtraction, multiplication, division, and extraction of roots, the reader is referred to the full and clear explanations given by Knott,[1] and by Smith and Mikami.[2]

Since the principle of the instrument is perfectly simple, it requires only a little practice to develop fair accuracy and speed in the use of it, at least for addition. It is much simpler to learn than typewriting, but as in that, the best way is by regular daily practice. Fifteen minutes daily will in a short time familiarize one with it, so that the carrying becomes practically automatic. This is one of the features of the instrument that recommends it strongly to the writer. There is none of the mental tension of remembering partial totals. If interrupted, you can stop at any point (provided only you keep track of where you leave off) and take it up again. There is of course the objection that if you make a mistake, you must go back to the beginning, whereas in pencil addition, you have some column totals already written down. But even there, you may have no record of the amounts carried, and so have to start afresh. In any addition, the best check is, probably, to add again. The writer has found the abacus very useful in accounting work, which occupies a good deal of his time. As the columns have to have a pencil total anyway, he first adds up with pencil, and then again by abacus. This gives a very good check, as there is almost no chance of making the same mistake by the two different methods.

Another very useful place for the suan p'an is in adding numbers that are not written in columns. The westerner is trained from childhood to add in columns, and it may not be possible for him to become proficient enough on the abacus to save much time there, although as remarked above the work is made easier. But cross addition of large numbers, or the addition of items picked out here and there in an account book, or of numbers on separate slips, as a pile of checks from the bank, is very difficult for the ordinary person, unless he copies them down in a column. It is here that the abacus saves much time, for the left hand can run along from number to number, or turn over the pile of slips, while the right hand adds. The addition should of course be repeated as a check, preferably taking the numbers in the opposite order. There are cases of this kind where a written list is needed, but when only the total is required the abacus is much quicker.

The writer has used the instrument chiefly for addition, and is doubtful whether the abacus methods of multiplication and division have much advantage

[1] C. G. Knott, "The abacus in its historic and scientific aspects," *Transactions of the Asiatic Society of Japan*, Yokohama, vol. xiv, 1886. This is reprinted in an abridged form, under the title "The calculating machine of the East: the abacus," in Horsburgh, *Modern Instruments and Methods of Calculation, A Handbook of the Napier Tercentenary*, London, Bell, [1911], pp. 136–154.

[2] D. E. Smith and Y. Mikami, *A History of Japanese Mathematics*, Chicago, Open Court, 1914, Chapter III.

PI MU EPSILON FRATERNITY, Syracuse University, Syracuse, N. Y. [*1918*, 271–273].

During the year 1918–19 there were forty-one members, of whom sixteen were faculty and graduates and twenty-five were undergraduates. The officers were: Director, Professor John L. Jones; vice-director, Professor Louis Lindsey; secretary, Gertrude Reynolds '19; treasurer, Donald F. Sears '20; librarian, Agnes Wilcox '20; executive committee, the above officers and Roy Horst '19, Helen De Long '19, Ora M. Tanner '19; scholarship committee, Professors Floyd F. Decker and William H. Metzler, Roy Horst '19, Ethel M. Hicks '19 and Lona Preston '19.

December 2, 1918: Outline of plan for the year. Discussion of mathematical magazines.

January 6, 1919: Report of the scholarship committee and election of new members.

January 27: Initiation of new members. "Rerating of regent's papers" by Professor Lindsey.

February 17: "The method of least squares" by Joseph Atwell '19; "An application of the binomial theorem" by William Start '19.

March 10: "Normals to conics" by Ora Tanner '19, Cornelia Tyler '19 and Bertha Adams '19.

March 31: "The planimeter and how to integrate by mechanical means" by Professor Street.

April 28: "Teaching graphs in high school" by Professor Lindsey.

May 12: Election of officers. Informal talks by the faculty and seniors.

May 14: Annual picnic.

May 26: Special meeting to vote on the establishment of a chapter at Ohio State University. (Note: A new chapter of Pi Mu Epsilon has been established at Ohio State University.)

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL AND OTTO DUNKEL.

Send all communications about Problems to B. F. FINKEL, Springfield, Mo.

PROBLEMS FOR SOLUTION.

2822. Proposed by A. M. HARDING, University of Arkansas.

Show that the sum of the series

$$1 + 3\cdot 2 + 5\cdot 2^2 + 7\cdot 2^3 + \cdots + (2n - 1)2^{n-1}$$

to n terms is $3 - 2^n + (n - 1)2^{n+1}$.

2823. Proposed by S. A. COREY, Des Moines, Iowa.

Let TQ and PR be diameters of a circle with center O. Bisect TO at X and draw PQ. On PQ erect the perpendicular XW and on PR, the perpendicular QV. Prove that $OX\cdot PV = PW\cdot PQ$.

2824. Proposed by G. Y. SOSNOW, Newark, N. J.

If n_1, n_2, n_3, n_4 be the lengths of the four normals and t_1, t_2, t_3, the lengths of the three tangents drawn from any point to the semi-cubical parabola, $ay^2 = x^3$, then will $27n_1n_2n_3n_4 = at_1t_2t_3$. [From *Mathematical Tripos Examination*, Cambridge, England.]

2825. Proposed by the late L. G. WELD.

A ball, having a coefficient of resilience α, strikes a rigid plane surface, inclined at an angle θ from the horizontal, after falling through a height h. What is the distance from the first to the second point of impact with the plane?

2826. Proposed by ALBERT A. BENNETT, University of Texas.

As a standard form for a square non-singular symmetric matrix under certain transformations, may be taken the form in which only the elements in the secondary diagonal are different from zero, and each of these is equal to unity. Analogously, as a standard form for a square non-singular skew-symmetric matrix (and hence incidentally of even order), may be taken the form in which only the elements of the secondary diagonal are different from zero, while the half of these which are towards the upper right-hand corner are each minus one, and the remaining half towards the lower left-hand corner, are each plus one. Denote both of these standard matrices by N.

Give simple parallel proofs that if M be given as non-singular and symmetric or non-symmetric as the case may be, a matrix P exists such that, with the usual notation

$$M = PNP'.$$

2827. Proposed by B. F. FINKEL, Drury College.

Find the equation of the envelope of the system of circles inscribed in a triangle having a given base and a given altitude.

2828. Proposed by T. M. BLAKSLEE, Ames, Iowa.

On page 72 of R. B. Hayward's *The Algebra of Coplanar Vectors and Trigonometry* occurs the sentence: "It will be a good exercise for the student to show that cos $(90°/7) = \frac{1}{2}\sqrt{x_1}$, where x_1 is the greatest root of the equation,

$$x^3 - 7x^2 + 14x - 7 = 0."$$

(1) Do not merely verify but deduce the equation and find x_1. (2) Deduce the x-equation (x_1, x_2, x_3, x_4, the roots) such that its greatest root x_1 gives $\cos(90°/9) = \cos 10° = \frac{1}{2}\sqrt{x_1}$. (3) Of what angles are $\frac{1}{2}\sqrt{x_1}, \cdots \frac{1}{2}\sqrt{x_4}$, in (2), the cosines? Develop a method of writing out at once $\cos(nv)$ in terms of powers of $\cos v$ if these are given for $(n-1)v$ and $(n-2)v$. The same for $\sin(nv)$. (4) Use the results of (2) and (3) to find the number of degrees in a radian. Hence, find π from radian instead of radian from π as is usual.

SOLUTIONS OF PROBLEMS.

411 (Algebra) [1914, 121; 1919, 268, 459]. Proposed by V. M. SPUNAR, Chicago, Ill.

Determine $x_1, x_2, x_3 \cdots x_p$ from the equations:

$$\begin{array}{lllll}
x_1 + x_2 & + x_3 & + \cdots + x_p & = a_0, \\
b_1x_1 + b_2x_2 & + b_3x_3 & + \cdots + b_px_p & = a_1, \\
b_1^2x_1 + b_2^2x_2 & + b_3^2x_3 & + \cdots + b_p^2x_p & = a_2, \\
\cdot\;\cdot\;\cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot\;\cdot & \cdot\;\cdot \\
b_1^{p-1}x_1 + b_2^{p-1}x_2 & + b_3^{p-1}x_3 & + \cdots + b_p^{p-1}x_p & = a_{p-1}.
\end{array}$$

A solution has been sent in by P. J. DA CUNHA, University of Lisbon, Portugal, in which the analysis is very much the same as that previously printed, but he also considers the case when some of the b's are equal. For example,

if $p = 3$ and $b_1 = b_2 \gtrless b_3$, we must have

$$\begin{vmatrix} 1 & 1 & a_0 \\ b_1 & b_3 & a_1 \\ b_1^2 & b_3^2 & a_2 \end{vmatrix} = 0.$$

If this condition is satisfied, the expressions for x_1 and x_2 in the general solution take indeterminate forms, and there will be an infinite number of solutions given by

$$x_1 + x_2 = \frac{a_0b_3 - a_1}{b_3 - b_1}, \qquad x_3 = \frac{a_0b_1 - a_1}{b_1 - b_3}.$$

463A (Geometry) [1915, 161; 1919, 414]. Proposed by B. J. BROWN, Kansas City.

If μ and ν are the parameters of the two confocal conicoids through any point on the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

show that $\mu+\nu+a^2+c^2=0$, along a central circular section.

SOLUTION BY WILLIAM HOOVER, Columbus, Ohio.

The parameters being μ and ν, the conicoids confocal with

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \tag{1}$$

are

$$\frac{x^2}{a^2+\mu}+\frac{y^2}{b^2+\mu}+\frac{z^2}{c^2+\mu}=1 \tag{2}$$

and

$$\frac{x^2}{a^2+\nu}+\frac{y^2}{b^2+\nu}+\frac{z^2}{c^2+\nu}=1. \tag{3}$$

(1) − (2) gives

$$\frac{x^2}{a^2(a^2+\mu)}+\frac{y^2}{b^2(b^2+\mu)}+\frac{z^2}{c^2(c^2+\mu)}=0 \tag{4}$$

and (1) − (3),

$$\frac{x^2}{a^2(a^2+\nu)}+\frac{y^2}{b^2(b^2+\nu)}+\frac{z^2}{c^2(c^2+\nu)}=0. \tag{5}$$

The equation of the central circular sections of (1) is

$$c^2(a^2-b^2)x^2-a^2(b^2-c^2)z^2=0, \tag{6}$$

$a^2>b^2>c^2>0$. Since (4), (5), and (6) must be consistent for values of x, y, z, not all zero,

$$\begin{vmatrix} \dfrac{1}{a^2(a^2+\mu)}, & \dfrac{1}{b^2(b^2+\mu)}, & \dfrac{1}{c^2(c^2+\mu)} \\ \dfrac{1}{a^2(a^2+\nu)}, & \dfrac{1}{b^2(b^2+\nu)}, & \dfrac{1}{c^2(c^2+\nu)} \\ c^2(a^2-b^2), & 0 \quad , & -a^2(b^2-c^2) \end{vmatrix}=0.$$

After removing the factor $\mu-\nu \neq 0$, and certain other factors which are not zero by the conditions above, this determinant reduces to

$$a^2+c^2+\mu+\nu=0.$$

499 (Geometry) [1916, 341; 1919, 414]. Proposed by NATHAN ALTSHILLER, University of Oklahoma.

Find the surfaces all the plane sections of which are circles.

SOLUTION BY W. D. CAIRNS, Oberlin College.

With any position of three mutually perpendicular axes, for a fixed value of x the equations of the surfaces must be of the form $l_1(y^2+z^2, y, z)=0$, where l_1 designates a linear form, the coefficient of (y^2+z^2) not vanishing; similarly, for $y=$ const. and for $z=$ const. they must take the forms $l_2(x^2+z^2, x, z)=0$ and $l_3(x^2+y^2, x, y)=0$, respectively. It may be shown by theorems relating to the equivalence of polynomials that this is possible only if the equations of the surfaces are of the linear form:

$$l(x^2+y^2+z^2, x, y, z)=0,$$

in which the coefficients are constants; that this is sufficient is a well known fact. Hence, the class of surfaces is composed of all spheres.

Also discussed by WILLIAM HOOVER.

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If μ and ν are the parameters of the two confocal conicoids through any point on the ellipsoid

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Also discussed by William Hoover.

supplemental to the angle ABC or $C'B'A'$, B' being the middle point of AC; hence T lies on the circle through A', B', C', that is to say, on the nine-point circle of the triangle ABC."

In *The Mathematician*, vol. 2, pp. 289–290, July 1847, Fenwick discussed the nine-point conic[1] which may be defined as follows: Given a triangle ABC and a point P in its plane, a conic can be drawn through the following nine points: (1) The middle points of the sides of the triangle; (2) the middle points of the lines joining P to the vertices of the triangle; (3) the points where these last named lines cut the sides of the triangle. Fenwick proved that this conic is the locus of the center of the conics passing through the four points A, B, C, and P; when P is the orthocenter of the triangle the conic is a rectangular hyperbola. Brianchon and Poncelet proved (*Annales de mathématiques pures et appliquées* (Gergonne), tome 11, pp. 205, 213, January, 1821) that every rectangular hyperbola through A, B, and C must also pass through the orthocenter and that the locus of the centers of such hyperbolas is the nine-point circle of the triangle. The asymptotes for any one of the hyperbolas are the pair of Wallace lines corresponding to the ends of a certain diameter of the circumscribed circle. (Steiner, *l.c.*, cf. also Weill, *Journal de mathématiques spéciales*, 1884, p. 16.) When this diameter passes through the center of the inscribed circle we have that the corresponding Wallace lines intersect at the point of contact of the nine-point and inscribed circles (Neuberg, *Mathesis*, 1893, p. 86). The equilateral hyperbola with this point as center has been called the *hyperbola of Feuerbach* (Neuberg, *l.c.*) since the theorem that the nine-point circle of a triangle is tangent to the inscribed and the three escribed circle was enunciated and proved by Feuerbach in a publication of 1822. I have not seen a reference to the three similar hyperbolas corresponding to diameters through the centers of the escribed circles.

Corresponding to the diameter through the orthocenter (that is, Euler's line) we have the *hyperbola of Jeřábek* (Jeřábek, *Mathesis*, 1888, pp. 81–84), and to the diameter through the Lemoine point,[2] we have the hyperbola called by Neuberg (*Journal de mathématiques spéciales*, 1886, p. 73) *hyperbola of Kiepert* (*Nouvelles annales de mathématique*, 1869, pp. 40–42.).

A more general form of Dr. Sousley's problem was given by Weill (*Journal de mathématiques spéciales*, 1884, p. 13): If m and m' are the Wallace lines of any two points M and M' on the circumcircle, then the angles between m and m' are equal to one or other of the angles under which the line segment MM' is seen from a point of the circumcircle.

Also solved by P. J. DA CUNHA, A. PELLETIER, and the Proposer.

NOTES AND NEWS.

EDITED BY E. J. MOULTON, Northwestern University, Evanston, Ill.

Dr. W. H. WILSON, of the Massachusetts Institute of Technology, has been instructor in mathematics at the University of Iowa since last September.

At the University of Oklahoma Associate Professor E. P. R. DUVAL resigned, his resignation to become effective at the end of the first semester, and is engaged in fruit growing at Springdale, Arkansas. Mr. E. E. COWAN has been appointed instructor in mathematics.

At Yale University Assistant Professor W. R. LONGLEY has been promoted to a full professorship, and Instructor J. K. WHITTEMORE to an assistant professorship of mathematics.

[1] This name seems to be due to Beltrami, 1863, "Intorno alle coniche dei nove punti e ad alcune questioni che ne dipendone," *Memorie dell'accademia delle scienze dell' Istituto di Bologna*, serie 2, vol. 2 (1862), pp. 361–395; also *Opere matematiche di Eugenio Beltrami*, vol. 1, 1902, pp. 45–72.

[2] This is the point of intersection of the symmedians of a triangle, that is, the lines through the vertices of the triangle symmetric to the medians through those vertices, with respect to the corresponding angle bisectors.

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At Yale University Assistant Professor W. R. Longley has been promoted to a full professorship, and Instructor J. K. Whittemore to an assistant professorship of mathematics.

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[1] This name seems to be due to Beltrami, 1863, "Intorno alle coniche dei nove punti e ad alcune questioni che ne dipendone," *Memorie dell'accademia delle scienze dell' Istituto di Bologna*, serie 2, vol. 2 (1862), pp. 361–395; also *Opere matematiche di Eugenio Beltrami*, vol. 1, 1902, pp. 45–72.

[2] This is the point of intersection of the symmedians of a triangle, that is, the lines through the vertices of the triangle symmetric to the medians through those vertices, with respect to the corresponding angle bisectors.

Assistant Professor A. E. BABBITT, of the University of Nebraska, has resigned his position in the mathematics department, and has accepted the position as secretary and actuary with the Lamar Insurance Company of Jackson, Miss.

At the University of Texas are the following part-time instructors in mathematics this year in the school of pure mathematics: Messrs. J. E. BURNAM, H. H. HAMMER, CLAUDE BAILEY, and C. H. ROBERSON; Mr. C. M. CLEVELAND was appointed instructor in mathematics in the school of applied mathematics.

At the University of Kansas, Assistant Professor SOLOMON LEFSCHETZ has been advanced to an associate professorship.

Mr. P. C. PORTER is dean of the college and professor of mathematics in Baylor College for Women, Belton, Texas.

P.ERRE ROBINSON, now a graduate student at the University of Chicago, has accepted an instructorship in mathematics at the University of Chicago High School.

Dr. I. A. BARNETT, now Benjamin Peirce instructor at Harvard University, will go to University of Illinois as instructor in mathematics next year.

Mr. C. C. WYLIE who, aside from war work with the Technical Staff of the Ordnance Department, has for six years been an assistant in the U. S. Naval Observatory, has taken a position in the department of astronomy in the University of Illinois.

The official title of Professor E. R. HEDRICK, of the University of Missouri, has been changed from professor of mathematics to professor of mathematics and of the teaching of mathematics.

At Washington University, Miss JESSICA M. YOUNG, instructor in Astronomy at Northwestern University during the first semester, has been instructor in mathematics and astronomy since the beginning of the second semester; Mrs. PEARL C. MILLER has been an assistant in the mathematics department since the middle of last December.

Professor E. W. BROWN, of Yale University, whose term of office as a representative of the American Mathematical Society on the Division of Physical Sciences in the National Research Council would have expired this month, resigned in December. The Council of the Society appointed as his successor Professor OSWALD VEBLEN of Princeton University.

The government of Servia has created a university of Lioubliana (formerly Laibach) and Professors J. PLEMELJ, of the University of Czernowitz (Roumania), and R. ZOUPANTCHITCH (French form of Suppantschitch), honorary docent of the polytechnic school at Vienna, have been appointed professors of mathematics.

At the University of Clermont Dr. G. GIRAUD has been appointed chargé de cours for differential and integral calculus in place of Professor A. C. E. PELLET who has retired from active teaching. At the University of Poitiers Dr. RENÉ GARNIER has been appointed chargé de cours for rational and applied mechanics in place of Professor M. FRÉCHET now at the University of Strassburg (*1919*, 371).

The following paragraph is from *Science*, February 13: "Boston University has concluded an arrangement for an exchange of professorships in mathematics for the college year 1920–1921 with Tsing Hua College, Peking, China. Professor R. E. Bruce, chairman of the department in Boston University, will exchange with Professor A. H. HEINZ, of Tsing Hua. Professor HEINZ, head of the department of mathematics, is a graduate of the University of Missouri and has been at Tsing Hua nine years. This college is under the control of the Chinese government and was founded with part of the returned Boxer Indemnity. Professor Bruce will sail from the Pacific coast in April. Professor Heinz will reach this country in time to begin his work at Boston University at the opening of the college in September."

Dr. ADOLF HURWITZ, professor of mathematics in the federal polytechnic school at Zurich, since 1892, died in November 1919, aged sixty years.

Nature for December 25, 1919, notes:

"The last day of this year marks the bicentenary of the death of JOHN FLAMSTEED, first Astronomer Royal of England, and the rector of the parish of Burstow, Surrey, where he is buried, uncommemorated, we understand, by an monument. Flamsteed was born four years after Newton, and was a native of Derbyshire, being the son of a well-to-do maltster. Though prevented by illness from attending a university, he was devoted to mathematical studies, and in 1671 sent a paper to the Royal Society. Three years later he published his 'Ephemerides,' a copy of which, being presented to Charles II. by Sir Jonas Moore, led to Flamsteed being appointed on March 4, 1675, 'our Astronomical Observer' at a salary of 100£ per annum, his duty being 'forthwith to apply himself with the most exact care and diligence to the rectifying the tables of the motions of the heavens and the places of the fixed stars, so as to find out the so much desired longitude of places for the perfecting the art of navigation.' The observatory at Greenwich, constructed partly of brick from old Tilbury Fort and of timber and lead from the Tower of London, was designed by Wren and built at a cost of 520£, the money being derived from the sale of spoilt gun-powder. The struggles and disputes, the dogged perseverance, and the memorable achievements of Flamsteed have their place in the history of astronomy, but it may safely be said that never has king or Government made a better investment than when Greenwich was built and Flamsteed made passing rich on 100£ a year."

We have already noted (*1920*, 45) the formation of a committee, or "Provisional International Mathematical Union" charged with the duty of consulting mathematicians and circulating statutes in different countries with a view to organizing an International Mathematical Union. C. J. de la VALLÉE POUSSIN was appointed the president of this committee, W. H. YOUNG the vice-president, and LAMB, PICARD, and VOLTERRA honorary presidents; there were four "secretaries" representing Belgium, France, Roumania and Italy. The Italian secretary, Professor Reina, died last November (*1920*, 143). In December, 1919, five months after the Brussels Congress, American mathematicians learned for the

first time of the appointment of such a provisional committee. On December 30 the Council of the American Mathematical Society appointed Professor L. E. DICKSON chairman of a committee to deal with communications coming from the provisional union. It has developed, however, that at an earlier date the union (with its secretaries representing Belgium, France and Roumania) called for an International Mathematical Congress (so-called), at the University of Strasbourg, the same to commence September 22, 1920. According to information recently received it would appear as if English mathematicians were with Americans equally ignored in the decision with regard to the desirability of a Congress at the time stated.

The following reports of Summer Sessions in 1920 have been received.

University of Chicago, First Term, June 1–July 28; Second Term, July 29–September 3. By Professor E. H. MOORE: Hermitian matrices of positive type in general analysis, 4 hours, Limits and Series, 4 hours, first term only. By Professor G. A. BLISS: Functions of a complex variable, 4 hours; Integral calculus, 5 hours. By Professor L. E. DICKSON: Theory of matrices and bilinear and quadratic forms, 4 hours; Theory of numbers, 4 hours. By Professor M. W. HASKELL: Projective geometry, 4 hours; Topics in analytic geometry, 5 hours. By Professor J. W. A. YOUNG: Theory of equations, 4 hours; Differential calculus, 5 hours. By Professor E. W. CHITTENDEN: Differential equations, 4 hours; College Algebra, 5 hours. By Professor W. D. MACMILLAN: Celestial mechanics II, 4 hours; Descriptive astronomy, 5 hours. Courses in plane analytic geometry and trigonometry will also be given.

University of Colorado, First term, June 14–July 21; second term, July 22–August 28. By Professor ABRAHAM COHEN, Johns Hopkins University: Teachers course in mathematics; Differential equations. By Professor G. H. LIGHT: Review course in mathematics; Calculus of variations. By Dr. GUY W. SMITH, University of Kentucky: Trigonometry; Plane analytical geometry; Differential calculus. The following advanced courses will be given by Professors COHEN and LIGHT: Theory of algebraic equations; Definite integrals; Theory of a complex variable; Elliptic integrals and functions; An introductory course in analysis; Differential geometry.

Columbia University, July 6–August 13. Undergraduate courses in elementary algebra, plane geometry, logarithms and trigonometry, solid geometry, college algebra, analytical geometry and calculus will be given by various members of the staff. Graduate courses are offered as follows: By Professor E. KASNER: Fundamental concepts of modern mathematics; Geometric transformations and groups. By Professor W. B. FITE: Differential equations. By Dr. G. E. PFEIFFER: Introduction to higher algebra. In the School of Education, Teachers College, several courses will be given by Professor UPTON and Mr. BRECKENRIDGE for teachers of mathematics in secondary schools. Professor SMITH will lecture on the history of mathematics and give a practicum in the teaching of mathematics.

Harvard University, July 6–August 14. By Professor C. L. BOUTON: Trigonometry; Analytic geometry. By Professor G. D. BIRKHOFF: Differential calculus; Integral calculus. These courses will be accepted as half-courses towards the degrees of A.B., A.A., and S.B.

University of Illinois, June 21–August 14, By Professor A. R. CRATHORNE: Mathematical theory of statistics; Differential equations. By Professor A. J. KEMPNER: Functions of a complex variable; Advanced algebra. By Dr. E. B. LYTLE: Teachers' course; Plane trigonometry. By Dr. J. R. KLINE: Differential calculus. By Mr. W. E. EDINGTON: Integral calculus; College algebra. By Mr. PETTIT: Trigonometry; College algebra.

University of Iowa, First Term, June 16–July 27. By Professor H. L. RIETZ: Teachers' course; Solid geometry. By Dr. J. W. CAMPBELL: Trigonometry; Analytic geometry; Differential equations. By Dr. W. H. WILSON: College algebra; Calculus; Reading course. By Mr. R. E. GLEASON: Solid geometry. Second Term, July 28–August 31. By Professor J. F. REILLY: Analytic geometry; Calculus; Reading course. By Mr. R. E. GLEASON: College algebra; Reading course. The courses are five hours per week in class through the sessions. When transformed into hours of the ordinary academic year the credit for each course taken during the first session is two semester hours and for the second session five-sixths of this amount.

University of Kansas, June 10–July 31. By Professor C. H. ASHTON: Mechanics, 3 semester hrs. credit; Algebra, 3 hrs. By Professor S. LEFSCHETZ: Theory of numbers, 3 hrs.; Analytic geometry, 2 hrs. By Professor J. J. WHEELER: Calculus, 3 hrs.; Solid geometry, 3 hrs. Second session, July 22–August 19. By Professor E. B. STOUFFER: Mathematical theory of investments, 2 hrs.; Trigonometry, 2 hrs.

University of Michigan, June 28–August 20. By Professor BEMAN: Differential equations, 2 hours. By Professor W. B. FORD: Advanced algebra; Advanced calculus, 2 hours; Theory of potential, 2 hours. By Professor P. FIELD: Vector analysis, 2 hours. By Professor L. C. KARPINSKI: History of mathematics. By Professor J. W. Bradshaw: Projective geometry, 2 hours. By Professor T. R. RUNNING: Graphical methods, 2 hours. By Professor W. B. CARVER: Introduction to the mathematical theory of interest, 2 hours. By Dr. R. B. ROBBINS: Mathematical theory of statistics, 2 hours.

Northwestern University, June 28–August 20. By Professor D. R. CURTISS: College algebra, Differential calculus. By Professor C. E. WILDER: Trigonometry; Analytical geometry. Each course bears credit of 3 semester hours.

University of Oklahoma, June 9–August 3. By Professor REAVES: Analytic geometry, 5 hrs.; Advanced analytic geometry, 3 hrs. By Professor GOSSARD: Intermediate algebra, 3 hrs.; The teaching of secondary mathematics, 2 hrs.; Integral Calculus, 3 hrs. By Professor N. ALTSHILLER-COURT: College, algebra, 3 hrs.; Plane trigonometry, 3 hrs. Projective geometry, 3 hrs. By Professor E. D. MEACHAM: Differential calculus, 3 hrs.; Solid analytic geometry, 3 hrs.

University of Pennsylvania, July 5–August 14. Courses in Plane geometry;

Differential calculus; Integral calculus; Solid geometry; Plane trigonometry; Intermediate algebra; College algebra; Analytic geometry; Projective geometry; Theory of statistics; will be offered by the following staff: Professor G. H. HALLETT, Assistant Professor H. H. MITCHELL, Assistant Professor S. P. SHUGERT and Mr. THOMAS.

Queen's University, July 5–August 13. Freshman and Sophomore Mathematics will be given by the following staff: Professor J. MATHESON; Professor D. BUCHANAN; Professor C. F. GUMMER; Messrs. N. MILLER, H. VAN PATTER and F. M. WOOD; and Miss GRACE H. JEFFREY.

THE WORK OF THE NATIONAL COMMITTEE ON MATHEMATICAL REQUIREMENTS, MARCH 1920. (Cf. *1920,* 145–146.)

At the last meeting of the General Education Board in New York on February 28th, the sum of $25,000, was appropriated for the use of the National Committee on Mathematical Requirements to continue its work for the year beginning July 1, 1920.

The preliminary report on "The Reorganization of the First Courses in Secondary School Mathematics" published for the Committee by the U. S. Bureau of Education has been distributed widely. Copies of the report have gone to all the state departments of education, to all county and district superintendents in the United States and to all city superintendents in cities and towns of over 2500 population. It has been sent to all the normal schools in the country, to some 1500 libraries and to almost 300 periodicals and newspapers. In addition it has been sent to about 4500 individuals, the names and addresses of which were furnished the Bureau of Education by the National Committee. This list of individuals consists chiefly of teachers of mathematics and principals of schools throughout the country. Additions to this mailing list to secure copies of the future reports of the Committee can still be made. Individuals interested in securing these reports should send their names and addresses to the Chairman of the Committee (J. W. Young, Hanover, N. H.).

A subcommittee consisting of Professor C. N. Moore of the University of Cincinnati, Mr. W. F. Downey of Boston and Miss Eula Weeks of St. Louis has been appointed to prepare a report for the Committee on Elective Courses in Mathematics for Secondary Schools. Any material or suggestions for this report may be sent directly to any member of the subcommittee.

The recent work of the National Committee had a place on the program of the organization meeting of the National Council of Teachers of Mathematics held in Cleveland on February 24, in connection with the meeting of the Department of Superintendence of the National Education Association. The meeting for the organization of the National Council was enthusiastically attended. A constitution was adopted and officers and an executive committee elected. Mr. J. A. Foberg, of the National Committee on Mathematical Requirements, was elected Secretary-Treasurer of the National Council.

The American Mathematical Monthly

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CONTENTS

Fifth Summer Meeting of the Association, Chicago, September 6, 1920; Fifth Annual Meeting, ———, December, 1920. The following are Section meetings of the Association in 1920:

IOWA, Univ. of Iowa, Iowa City, May 1
KANSAS, State Agricultural College, Manhattan, April 3
KENTUCKY, Centre College, Danville, April 17
MARYLAND–DISTRICT OF COLUMBIA–VIRGINIA, Goucher College, Baltimore, Md., May
MINNESOTA, St. Catherine's College, St. Paul, May 29
MISSOURI, Kansas City, November 12–13
OHIO, Ohio State Univ., Columbus, April 2
ROCKY MOUNTAIN, Colorado College, Colorado Springs, April 2

EDITORIAL CORRESPONDENCE AND BOOKS FOR REVIEW should be addressed to the EDITOR-IN-CHIEF, R. C. ARCHIBALD, Brown University, Providence, R. I.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 27 King Street, Oberlin, Ohio.

A MECHANISM FOR THE SOLUTION OF AN EQUATION OF THE *N*TH DEGREE.

By A. L. CANDY, University of Nebraska.

W. Peddie[1] read a paper on this subject before the Fifth International Congress of Mathematicians held in Cambridge, England, in 1912. This paper is found in the *Proceedings* of this Congress, Vol. I, pp. 399–402. Professor Peddie's communication also contains a picture of his mechanism. From this picture, as well as from his description, it is quite evident that this instrument can not be manufactured by any one except an expert mechanic, who has access to a shop well equipped for working in metal. The crude instrument, that I shall describe and explain herein, was made by myself. A similar one can be made by any one who has a moderate amount of mechanical ingenuity and a few simple tools, out of materials that can be procured almost anywhere. Both the design and the proof that I here submit, seem to me to be simpler, although the underlying principle is virtually the same as that used by Professor Peddie.

Description of the Mechanism. This mechanism, shown closed in Fig. 1, and open in Fig. 2, consists of a main bar about 32 inches long, to which are hinged three arms each about 8 inches long, the distances between the hinges being equal. A lighter connecting bar is attached to the free ends of the arms in such a manner that these arms always turn through the same angle. On the main bar, and also each of the arms, are beveled cleats along which grooved slides move freely. Each of these slides on the main bar carries an eye-headed screw, like those used to fasten the hanging cords to picture frames. These screws are placed so that when the instrument is closed and these slides are at their zero points, as shown in Fig. 1, the eyes are in line with the pins of the hinges. Each slide on the arms carries a small drum that is held firm by means of a milled nut. To each of the drums is attached a small, flexible, inelastic cord, which passes through the eye carried by the adjacent slide on the main bar, and is fastened to the next slide below on the main bar, the lower end of the last cord being made fast to the main bar. The first slide is held in place by means of a small iron pin inserted in holes in the main bar. A graduated circular scale is placed under the first arm, from which the roots of the equation are read. The scales for reading the positions of the slides are marked off on the left side of the main bar. The instrument may be used (1) in a vertical position, as shown in the figures, so that the lengthening of any string by unwinding will cause some of the slides to move downwards by their own weight; or (2) lying on a table and operated with both hands.

Solution of an Equation. Let us now solve the equation

$$10x^3 + 24x^2 + 9x - 7 = 0. \tag{1}$$

The process is as follows: First, close the instrument, then wind up the drums

[1] Professor of physics, University College, Dundee.

until each slide comes to the zero point of its scale, and all the cords are taut (Fig. 1). The arms will now move freely through an angle of 90°, with all the cords continuously taut. Now move the first slide 10 units—the coefficient of x^3 —downward, by moving the iron pin which always holds this slide in a fixed position; unwind 24 units—the coefficient of x^2—from the cord wound around the first drum; likewise, unwind 9 units—the coefficient of x—from the second drum; since the constant term is negative 7, wind up the last drum until the last cord

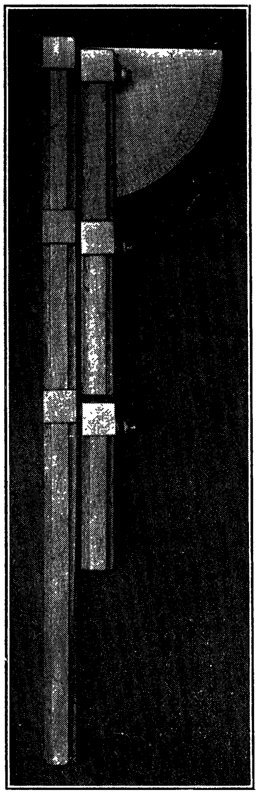

FIG. 1.

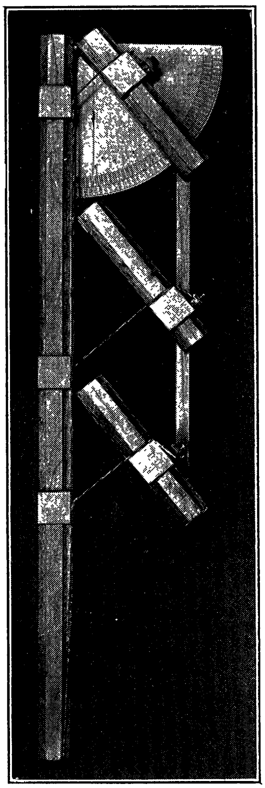

FIG. 2.

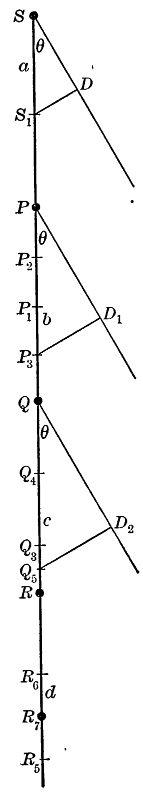

FIG. 3.

is shortened by 7 units. Now turn the arms through some angle until all the cords become taut, with the slides on the arms so adjusted that the cords attached to them shall be at right angles to the arms (Fig. 2). The reading on the scale under the first arm now shows one root of the equation to be .366. The exact root is $(\sqrt{3} - 1)/2$.

Proof. Let the instrument be closed, as shown in Fig. 1, with all the cords taut; and suppose the small bar connecting the outer ends of the arms (Fig. 2) to be removed for the time being, so that each arm can turn independently.

Let S, P, and Q (Fig. 3) be the zero positions of the slides on the main bar,

or more accurately, the "eyes" on these slides, when they are in line with the hinge pins. Then the slides which are on the arms, and carry the drums D, D_1, D_2, will be to the right of S, P, Q, respectively. Let R be a weight at some fixed point on the cord which is attached to the drum D_2 and whose lower end is fastened to the main bar.

Move the slide at S down to S_1, where it will remain fixed, making

$$SS_1 = a. \tag{2}$$

This will let all the slides move downward the same distance, the slide at P stopping at P_1, making $PP_1 = a$, also.

Now turn the first arm through an angle θ, the slide on it taking the position D, with the cord S_1D perpendicular to the arm SD. Since the cord attached to the drum D passes through the eye on the slide at S_1, and is made fast to the slide at P_1, this will pull the latter up to P_2, making

$$P_1P_2 = S_1D = a \sin \theta.$$

$$\therefore \quad PP_2 = PP_1 - P_1P_2 = a(1 - \sin \theta).$$

Next unwind a length b on the cord attached to the drum D. This will let slide P_2 move down to P_3, and since the slide Q has moved in precisely the same way as P, taking the same number of corresponding positions, and is now at Q_3, we have

$$PP_3 = QQ_3 = a(1 - \sin \theta) + b. \tag{3}$$

Then turn the second arm through the same angle θ, the slide on it taking the position D_1. Since the cord attached to D_1 passes through the eye on P, and is fastened to Q, this will pull Q up to Q_4, making

$$Q_3Q_4 = P_3D_1 = PP_3 \sin \theta.$$

$$\therefore \quad QQ_4 = QQ_3 - Q_3Q_4 = PP_3(1 - \sin \theta).$$

$$= a(1 - \sin \theta)^2 + b(1 - \sin \theta).$$

Now unwind a length c from drum D_1. This will move Q down to Q_5, and since the point R has moved in the same manner as Q, and is now at R_5, we have

$$QQ_5 = RR_5 = a(1 - \sin \theta)^2 + b(1 - \sin \theta) + c. \tag{4}$$

Then turn the third arm through the angle θ. Since the cord attached to D_2 passes through the eye on Q, this will draw the point R up to R_6, making

$$R_5R_6 = Q_5D_2 = QQ_5 \sin \theta.$$

$$\therefore \quad RR_6 = RR_5 - R_5R_6 = QQ_5(1 - \sin \theta),$$

$$= a(1 - \sin \theta)^3 + b(1 - \sin \theta)^2 + c(1 - \sin \theta).$$

value of the function when $x = 1$, that is, the sum of the coefficients of the function. As the arms are turned from 0° to 90°, keeping the slides properly adjusted and all the cords taut, the distance of this slide, R, from its initial position will be continuously the value of the function as x varies continuously from 1 to 0.

ON THE ORTHOCENTRIC QUADRILATERAL.[1]

By NATHAN ALTSHILLER-COURT, University of Oklahoma.

Introduction. (*a*) The altitudes AD, BE, CF of a triangle ABC meets in a point H, the orthocenter of ABC. The triangle DEF formed by the feet D, E, F, of the altitudes is frequently called the orthic triangle of ABC.

To Carnot[2] is due the credit for having called attention to the almost obvious fact that *each of the four points H, A, B, C, is the orthocenter of the triangle formed by the other three.*

The points A, B, C, H are referred to as an *orthocentric group of points*, or an *orthocentric quadrilateral*, and the four triangles determined by these four points as an *orthocentric group of triangles.*

(*b*) In 1821 Brianchon and Poncelet showed that the circumcircle (N) of the orthic triangle DEF of ABC passes through the mid-points A', B', C', of the sides BC, CA, AB, of ABC, and also through the mid-points P, Q, R, of the segments AH, BH, CH respectively.[3] That the circle through the first six points mentioned passes also through the last three becomes obvious if we observe that *DEF is the orthic triangle not only of ABC, but of each of the four triangles of the orthocentric group $ABCH$.*

(*c*) In 1822 Feuerbach proved[4] that the circle (N) is tangent to the four circles which touch the sides of the triangle ABC. It was not until 1861 that Sir William R. Hamilton pointed out that (N) is also tangent to the circles touching the sides of the triangles BCH, CHA, HAB.[5] Now since the orthic triangle DEF is common to the four triangles of the orthocentric group $ABCH$, the circumcircle (N) of DEF is the nine-point circle of each of these four triangles, and therefore Hamilton's extension of Feuerbach's theorem becomes self-evident.

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[1] Read before the American Mathematical Society, St. Louis, December 31, 1919. Readers of this article will be interested in comparing it with the first part of the author's earlier article "On the I-centre of a triangle" (*1918*, 241–246)—EDITOR.

[2] Carnot, *De la corrélation des figures de Géométrie*, 1801, p. 102.

[3] For the proof, see, for instance, J. Casey, *A Sequel to Euclid*, second edition, 1882, p. 58, or C. V. Durell, *Plane Geometry for Advanced Students*, vol. 1, pp. 30–31.

[4] For a proof see Casey, *l.c.*, pp. 58–61, or Durell, *l.c.*, pp. 46–47 and pp. 149–150.

[5] In making this statement Professor Altshiller-Court was possibly misled by Casey's reference to the result as "Sir William Hamilton's Theorem" (*Quarterly Journal of Mathematics*, 1861, p. 249) and by the fact that Sir William proposed the result as a problem in *Nouvelles annales de mathématiques*, 1861, vol. 20, p. 216.

The result was not, however, given originally by Sir William, but by T. T. Wilkinson, as prize-problem 1883 in *Lady's and Gentleman's Diary*, London, 1854, p. 72 (Solutions, *Diary*, 1855, pp. 67–69).—EDITOR.

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[1] Read before the American Mathematical Society, St. Louis, December 31, 1919. Readers of this article will be interested in comparing it with the first part of the author's earlier article "On the I-centre of a triangle" (*1918*, 241–246)—EDITOR.

[2] Carnot, *De la corrélation des figures de Géométrie*, 1801, p. 102.

[3] For the proof, see, for instance, J. Casey, *A Sequel to Euclid*, second edition, 1882, p. 58, or C. V. Durell, *Plane Geometry for Advanced Students*, vol. 1, pp. 30–31.

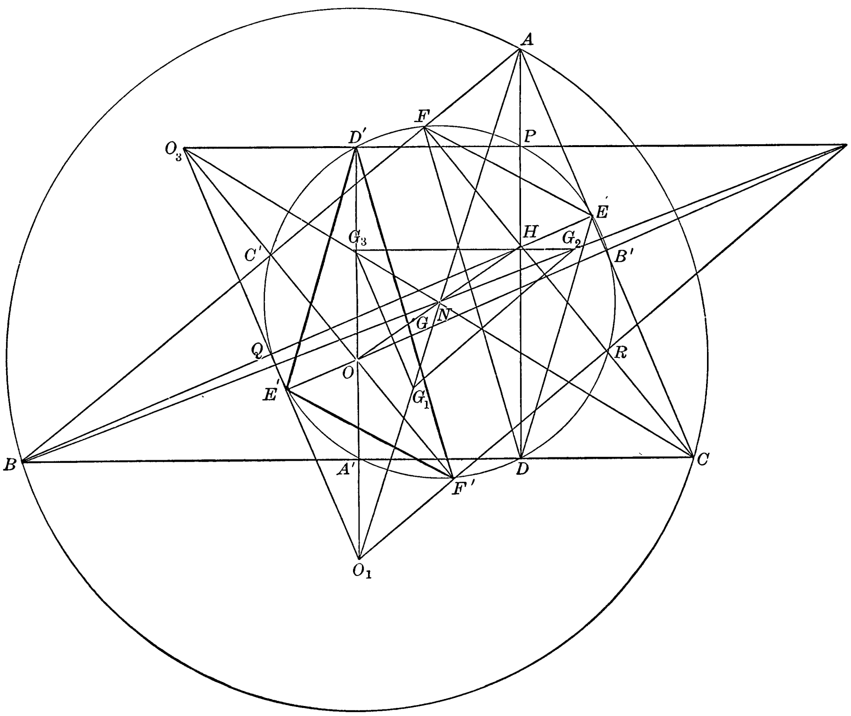
[4] For a proof see Casey, *l.c.*, pp. 58–61, or Durell, *l.c.*, pp. 46–47 and pp. 149–150.

[5] In making this statement Professor Altshiller-Court was possibly misled by Casey's reference to the result as "Sir William Hamilton's Theorem" (*Quarterly Journal of Mathematics*, 1861, p. 249) and by the fact that Sir William proposed the result as a problem in *Nouvelles annales de mathématiques*, 1861, vol. 20, p. 216.

The result was not, however, given originally by Sir William, but by T. T. Wilkinson, as prize-problem 1883 in *Lady's and Gentleman's Diary*, London, 1854, p. 72 (Solutions, *Diary*, 1855, pp. 67–69).—EDITOR.

These examples suggest that in certain connections it may be fruitful to consider the circle (N) as belonging not to the triangle ABC, but to the orthocentric group $ABCH$. The following considerations are based on this remark.

1. The center N of the nine-point circle (N) of the triangle ABC was shown by Benjamin Bevan, in 1804,[1] to lie midway between the orthocenter H and the circumcenter O of the triangle ABC. In other words, the circumcenter O of ABC is the symmetric of H with respect to N. But, as has been pointed out above, (N) is also the nine-point circle of the triangle BCH, whose orthocenter is the point A, hence the circumcenter O_1 of BCH is the symmetric of A with



respect to N. Similarly for the circumcenters O_2, O_3, of the triangles CHA, HAB. Consequently: *The four circumcenters of an orthocentric group of triangles form an orthocentric group which is the symmetric of the given group with respect to the nine-point center.*

2. From the symmetry of the two groups of points $ABCH$ and $O_1O_2O_3O$ follow immediately all the known properties of the circumcenters O_1, O_2, O_3, O. For instance:

(*a*) *The triangles $O_1O_2O_3$ and ABC are congruent*[2] and furthermore, *their sides are respectively parallel.* It may also be observed that these properties hold for the pairs of triangles O_2O_3O and BCH; O_3OO_1 and CHA; OO_1O_2 and HAB.

[1] For the proof compare Casey, or Durell, *l.c.*

[2] Durell, *l.c.*, p. 36, exercise 89.

corresponds to H in a similitude of ratio $-1/3$, the center of similitude being N.[1] But N is also the nine-point center of the triangle BCH, whose orthocenter is A, hence the centroid G_1 of BCH corresponds to A in a similitude of ratio $-1/3$ with N as center of similitude. Similarly for the centroids G_2, G_3, of the triangles CHA, HAB. Consequently: *The four centroids of an orthocentric group of triangles form an orthocentric group, the two groups being similar and similarly placed.*

8. Since the centroids G, G_1, G_2, G_3, form an orthocentric group, all the properties of such a group immediately follow, as, for instance, that G is the orthocenter of the triangle $G_1G_2G_3$, etc.

Again the similitude of the two groups $GG_1G_2G_3$ and $HABC$ puts into evidence a great many properties, as for instance, that G_1G_2 is parallel to AB and is equal to $1/3$ of its length; that the point of intersection of GG_1 and G_2G_3, which will be represented by (GG_1, G_2G_3), is collinear with N and $D \equiv (HA, BC)$; etc. The reader may find it interesting to formulate a number of these propositions.

9. In the similitude (7) by which the group $GG_1G_2G_3$ is derived from the group $HABC$, the center of similitude N is a double point. Hence: *An orthocentric group of triangles and the orthocentric group of their centroids have the same nine-point center.*

10. The orthocentric group $GG_1G_2G_3$ has been derived from the given orthocentric group $HABC$ by a similitude of center N and ratio $-1/3$. But the process may be reversed, and the orthocentric group $HABC$ may be derived from the orthocentric group $GG_1G_2G_3$, considered as given, by a similitude of ratio -3, the center remaining the same. Consequently: *The four points of an orthocentric group may be considered as the centroids of another orthocentric group of triangles, the two groups having the same nine-point center, this point being a center of similitude of the two groups, the ratio of similitude being* -3.

11. Since from (1) the two groups $HABC$ and $OO_1O_2O_3$ are symmetrical about the center N, therefore it follows from (10) that the two groups $GG_1G_2G_3$ and $OO_1O_2O_3$ admit N as a center of similitude, the ratio of similitude being $+3$. Hence: *The centroids and the circumcenters of an orthocentric group of triangles form two orthocentric groups of points having the same nine-point center, this point being a center of similitude of these two groups, the ratio of similitude being* $+3$.

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[1] Euler, *Novi comment. acad. sc. Petrop.*, vol. 11 (1765), 1767, p. 114.—Editor.

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the standard cubic is that:

$$(3) \qquad \begin{cases} v^3 + v^2 + xv = y \\ 3v^2 + 2v + x = 0. \end{cases}$$

This is also the necessary and sufficient condition that the point (x, y) lie on the envelope of the straight line. Every isoradical straight line is therefore tangent to the locus

$$(4) \qquad 4x^3 - x^2 + 18xy + 27y^2 - 4y = 0$$

obtained by eliminating v from (3).

All that is necessary, therefore, in order to solve a cubic, is to reduce it to standard form, obtaining the quantities x and y. Plot the corresponding point, and drop the three tangents to the locus (4). The slopes of the three tangents will be the three roots of the equation as written in standard form.

Note that the locus (4) can be plotted once for all, and in place of plotting a cubic equation, one need only plot a point for each problem to be solved.

QUESTIONS AND DISCUSSIONS.

EDITED BY W. A. HURWITZ, Cornell University, Ithaca, N. Y.

DISCUSSIONS.

Psychologists today are restoring to honor, albeit in much modified form, the Old Reliable Dream Book. Theologists and educators are seeking to read the future. It is not surprising that a claim should be made for the use of mathematics as a means of prediction. Such a claim is stated by Professor Weaver in the first discussion this month. His conception must not be dismissed as fantastic without an investigation into its meaning. He makes quite clear that no prediction of individual events or circumstances is intended; his hope is at most that the progress of history in the large or average sense may be forecasted. In support of his contention may be cited the service of mathematics in exactly this function of prediction in connection with the physical sciences. It may even be argued that an example of similar application to the social sciences is to be found in the generally accepted theory of the periodicity of economic cycles of prosperity and depression, with an approximate period of ten or eleven years. Against this argument may be placed the apparently hopeless complexity, as regards mathematical formulation, of the social problem, compared with the physical problem. It is doubtful whether the reduction to concise qualitative laws, or even the specification of the independent variables in terms of which such laws are to be stated lies within the mental potentialities of the human race.

Several details in Professor Weaver's account suggest comment. For example, it is not obvious that the analogy of the determination of all values of an everywhere analytic function by the values of the function and its derivatives

at a single point, is the most natural one. Would it not be more appropriate, in view of the suggestions of mathematical physics, to suppose that the history of the universe is contained in a vast system of differential equations, or in view of Volterra's hypothesis of "heredity," of integro-differential or other still more complicated equations, so that the knowledge of a *finite* set of initial conditions would suffice to determine a solution completely?

Also the notion of the "most probable" of all possible trends of natural phenomena may be much less simple than special cases would indicate. That determination of the history of a country or the world which will produce the "most probable" of all conceivable birth-rates may not coincide with the determination which will produce the "most probable" duration of life. This almost obvious comment is not meant as an objection to Professor Weaver's remarks, nor is it implied that a "most probable" configuration under the whole imaginable set of auxi iary conditions may not also exist; such questions would naturally have to be settled if the present vision should come within appreciable range of reality.

The second discussion brings us back to something more concrete. Dr. Sensenig indicates a form of derivation of the integral as the limit of a sum, which practically amounts to a proof, for the case of an analytic function, of the identity of the concepts *integral* and *anti-derivative*. It is at times useful to have such proofs at hand, even in elementary instruction, as aids in producing conviction in the minds of students. Of course, the use of the formula for Σn^m will prove an obstacle to the use of Dr. Sensenig's work for that purpose. It is doubtless universally familiar that the convergence of the Cauchy sum, in the case of a monotonically increasing or decreasing continuous function, can be cast in a geometric form so vivid as to be accepted with ease by the ordinary classes in calculus.

As the third discussion appears an alternative derivation of the expressions for the half-angles of a plane triangle in terms of the sides. Professor Baudin obtains the relations from the law of sines instead of from the law of cosines. The work is less simple than the ordinary methods; it is interesting, however, to see that it can be carried out.

I. Forecast.

By Warren Weaver, California Institute of Technology.

The foundations of science and scientific thought have been subjected in the last fifty years to examination of a most critical sort. This examination has resulted in development along two general lines; a filling of needful matter into the interstices of the even yet porous body of logical thought where the general trend of the previously accepted body has been found still tenable, and an opening up of new problems in those regions where the old theories have been found inadequate or merely approximate. As an example of the first type there comes to one's mind the rigor that has been brought to the fundamental concepts of the calculus by means of function theory. The outstanding illustration of the latter

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type of problem of the newer science is the recasting of the old concepts of time and space that is necessary under the modern theory of relativity, along with the fundamental and far-reaching alterations which this theory imposes upon mechanics and electrical theory.

In many such investigations a mode of attack is being used that reminds one of the older metaphysical and theological considerations of scientific problems, such as the proof of the Principle of Least Action given by Maupertius. As the type of problem being studied becomes more and more fundamental and ultimate in nature one is not surprised to find that philosophical considerations enter along with considerations of a more formal mathematical nature. So that it is a little less idle than hitherto for one to speculate how fundamental the truths really may be that he meets with in pure mathematics and in the mathematics of pure physics. Some of the theories that have hitherto seemed restricted to the formal considerations of the mathematician have indeed been emerging of late as truths the generality and broad applicability of which are not yet fully appreciated. Out of an almost indecent past shadily connected with the gaming table comes a theory of probability that, expanded by the researches of Gibbs and others, furnishes the one ultimate foundation for the kinetic theory of gases, kinetic theories of electrical conductivity, a broad class of thermodynamical problems, and the theory of statistics in all its branches. The fact that of all possible complexions of a system the trend of nature is always towards the most probable seems to be the statement of a principle whose breadth has been only of late and perhaps not even now fully appreciated. One may easily cite other modern questions to which one can give a broadly philosophical cast. As has been suggested in a recent note in *Science*[1] velocity is essentially of such nature that there is a *best* way to measure it, a conclusion which, if completely acceptable, may be taken as a fundamental argument for the relativity theory. The work of Dr. R. C. Tolman in developing what he has called the Principle of Similitude[2] has furnished upon some such general ground not only a startlingly immediate method of establishing many physical relationships known upon other grounds to be correct, but seems also to disclose these results as *inevitable* truths in a way that a more formal proof does not at all. The atomic structure of matter and the corpuscular nature of energy seem to be demanded by the fact that if nature is to tend towards some certain definite most probable configuration and a certain definite most probable distribution of energy it must be possible to recognize discrete elements of matter and energy.[3] Generalized relativity has been extended along certain lines to suggest that it may be the expression of a very fundamental truth that possibly and probably transcends the sciences of mathematics and physics, narrowly conceived, and is applicable to diversified considera-

[1] "The Nature of Velocity" by Tenney L. Davis, *Science*, Oct. 10, 1919.

[2] Articles on the Principle of Similitude and related subjects: *Physical Review*, April, 1914; Aug., 1914; *Phys. Rev.*, 6, 219 (1915); July, 1916; *Phys. Rev.*, 9, 237 (1917).

[3] *Eight Lectures on Theoretical Physics* by Max Planck, pp. 42–55. Columbia University Press.

tions regarding life.[1] Dr. H. Bateman has suggested, in the note in the *Philosophical Magazine* to which reference has just been made, the possible and probable existence of a deep underlying principle which, since it will be the ultimately true equation of motion, not only would furnish the results of ordinary relativity theory, but would also contain, among other things, an explanation of the fact that good designs are perpetuated in nature. We may conclude therefore that the trend of scientific thought at the present time renders it more hopeful and less inexcusable to contemplate analogies between the principles of mathematics and the principles of life, in the attempt to enrich and perhaps shed new light upon each.

It ought to seem queer to the scientist that history is so nearly exclusively a record of what man has done, and in so very small measure a study of what man will do. For the method of the scientist is to record past events only so far as is necessary to the accumulation of sufficient data to check theories which make possible the prediction of events in the future. It has probably been many times suggested that serious study be made of the possibility of a similar method being used in history, but perhaps the best-known comes from the versatile pen of H. G. Wells. In a discourse entitled "The Discovery of the Future"[2] delivered at the Royal Institution in London he gives his foundation for an optimistic hope that a bold and searching study may serve to give us eventually a good deal of general but exceedingly useful knowledge of the future. He says that "as one assimilates the broad conception of science, the persuasion comes into one's mind that the adequacy of causation is universal," so that "the man of science comes to believe at last that the events of the year A.D. 4000 are as fixed, settled, and unchangeable as the events of the year A.D. 1600." He cites the pushing back of our information to prehistoric times by means not of revelation, but by means of a keen and vigorous habit of inquiry, and continues; "and now if it has been possible for men by picking out a number of suggestive and significant looking things in the present, by comparing them, criticizing them, and discussing them, with a perpetual insistence upon 'Why?', without any guiding tradition and indeed in the teeth of established beliefs, to construct this amazing searchlight of inference into the remoter past, is it really, after all, such an extravagant thing to suggest that, by seeking operating causes instead of fossils and by criticizing them as consistently and thoroughly as the geologic record has been criticized it may be possible to throw a searchlight of inference forward instead of backward, and to attain a knowledge of coming things as clear, as universally convincing, and infinitely more important to mankind than the clear vision of the past geology has opened to us during the nineteenth century?" The lecture includes a discussion to show that the existence of the "exceptional man," such as Napoleon, Cæsar, or William the Conqueror need not hopelessly remove the possibility for success of such a scheme, and it is careful not to hold out any

[1] See, for example, the end of an article called "The Physical Aspect of Time," by H. Bateman, *Memoirs and Proc. of Manchester Literary and Philosophical Soc.*, Vol. 54, Part 3, 1909–1910: and, same author, *Phil. Mag.*, Vol. 37, Feb., 1919.

[2] Published by B. W. Huebsch, New York, in book form with the above title.

point.[1] In terms of our former analogy we may say that if we have a set of "cross-section" values of a function (which in our application would be a set of values for a constant t) and of its first derivative, and know that the function is characterized and specialized by the fact that it satisfies some very general condition (which condition while very general must get at the very root of the matter) we can completely determine the function at all other points. This general condition, in the case of ordinary potential and many allied problems is, of course, given by the ordinary Laplacian equation $\Delta^2 V = 0$. This suggests that if, by careful study, we could but obtain some universally applicable and fundamentally far-reaching principle—some truly dynamic theory of history, to use Henry Adams's phrase, we could probably thus cut down enormously upon the amount of contemporary data necessary.

This analogy may perhaps all be interpreted as merely strengthening Wells's point that in science effect seems inevitably to follow cause in an unending and inescapable chain of events. There is perhaps no more striking example of the adequacy of causation than that contained in the fact that the value and behavior of y a million million miles down the x axis is inevitably determined by the way in which the curve crosses the y axis; and it is a natural and, it would seem, not ridiculous hope to expect that some day there will be an historian bold enough and wise enough to write down a few of the equations of history which will hold not only for the minus sign, but also for the plus.

II. A Proof of the Definite Integral Formula.

By Wayne Sensenig, Conshohocken, Pa.

On pages 214–216 of Lamb's *Infinitesimal Calculus* several special cases of the definite integral are calculated "ab initio."[2] The following direct proof of the formula for the definite integral covers cases where y can be expressed as a *convergent* power series in x over the *closed* interval considered in the definite integral. The area under a curve is used as an illustration, but the form of proof is obviously independent of any special geometric interpretation. Use is made of the following well-known formula:[3]

$$\begin{aligned} S_m &= 1^m + 2^m + \cdots + n^m \\ &= \frac{n^{m+1}}{m+1} + \tfrac{1}{2}n^m + B_1 \frac{m}{2!} n^{m-1} - B_3 \frac{m(m-1)(m-2)}{4!} n^{m-3} \\ &\quad + B_5 \frac{m(m-1)(m-2)(m-3)(m-4)}{6!} n^{m-5} - \cdots, \end{aligned}$$

[1] If it is desired to calculate at points exterior to S the regularity of V at infinity is also necessary. The above mathematical fact does not correspond perfectly to the previous example, but the aspect of the logical content with which we are concerned is closely enough connected with the former to serve our purpose.

[2] Similar exercises are found in various text-books. Problems of the same character have from time to time appeared in the MONTHLY.—EDITOR.

[3] Hall and Knight, *Higher Algebra*, p. 337.

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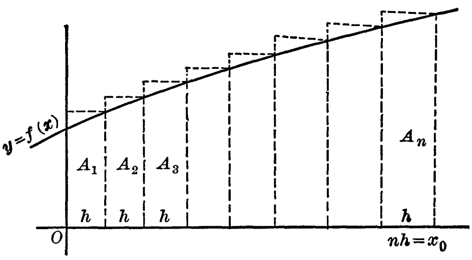
[3] Hall and Knight, *Higher Algebra*, p. 337.

ending with a term in n or n^2; the B's being the Bernoulli numbers:

$$B_1 = \tfrac{1}{6}, \quad B_3 = \tfrac{1}{30}, \quad B_5 = \tfrac{1}{42}, \quad B_7 = \tfrac{1}{30}, \quad B_9 = \tfrac{5}{66}, \quad \cdots.$$



Let the interval of integration be $0 \leqq x \leqq x_0$, and suppose

$$y = f(x) = a_0 + a_1x + a_2x^2 + \cdots,$$

the series converging in an interval extending beyond x_0. Then, since the convergence must be uniform in the closed interval $|x| \leqq x_0$, any given ϵ, however small, will determine a value of m, such that

$$y = a_0 + a_1x + a_2x^2 + \cdots + a_mx^m + R_m(x),$$

where

$$|R_m(x)| < \frac{\epsilon}{2x_0}, \qquad 0 \leqq x \leqq x_0.$$

Divide the interval into n equal parts each of length h, and form the Cauchy sum in the usual way. With reference to the accompanying figure, we have:

$$\begin{aligned} A_1 &= a_0h + a_1h^2 + a_2h^3 + \cdots + a_mh^{m+1} + hR_m(h), \\ A_2 &= a_0h + 2a_1h^2 + 2^2a_2h^3 + \cdots + 2^ma_mh^{m+1} + hR_m(2h), \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ A_n &= a_0h + na_1h^2 + n^2a_2h^3 + \cdots + n^ma_mh^{m+1} + hR_m(nh). \end{aligned}$$

Thus

$$A = A_1 + A_2 + \cdots + A_n = s_{m,\,n} + hr_{m,\,n},$$

where

$$\begin{aligned} s_{m,\,n} &= a_0nh + a_1h^2S_1 + a_2h^3S_2 + \cdots + a_mh^{m+1}S_m, \\ r_{m,\,n} &= h[R_m(h) + R_m(2h) + \cdots + R_m(nh)]. \end{aligned}$$

Evidently

$$|r_{m,\,n}| < hn\frac{\epsilon}{2x_0} = \frac{\epsilon}{2};$$

so that independently of n, $r_{m,\,n}$ can be made arbitrarily small.

Furthermore, we have

$$nh = x_0$$

$$\lim_{n\to\infty} h^2S_1 = \lim_{n\to\infty} h^2\left(\frac{n^2}{2} + \frac{n}{2}\right) = \frac{x_0^2}{2},$$

$$\lim_{n\to\infty} h^3S_2 = \lim_{n\to\infty} h^3\left(\frac{n^3}{3} + \frac{n^2}{2} + B_1\frac{2}{2!}n\right) = \frac{x_0^3}{3},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\lim_{n\to\infty} h^{m+1}S_m = \lim_{h\to\infty} h^{m+1}\left(\frac{n^{m+1}}{m+1} + \frac{n^m}{2} + \cdots\right) = \frac{x_0^{m+1}}{m+1};$$

and therefore

$$\lim_{n \to \infty} s_{m,\,n} = a_0 x_0 + a_1 \frac{x_0^2}{2} + \cdots + a_m \frac{x_0^{m+1}}{m+1}.$$

The last expression is the sum of $(m + 1)$ terms of the series

$$a_0 x_0 + a_1 \frac{x_0^2}{2} + a_2 \frac{x_0^3}{3} + \cdots.$$

If we choose n so great that $s_{m,\,n}$ differs from its limit by less than $\epsilon/2$, then since $|r_{m,\,n}| < \epsilon/2$ for all n, we see that A differs from the sum of $(m + 1)$ terms of the above series by less than ϵ. Since the series is readily shown to converge, it follows that the limit of A, as n becomes infinite, exists and is equal to the value of the series:

$$\int_0^{x_0} f(x)dx = a_0 x_0 + a_1 \frac{x_0^2}{2} + a_2 \frac{x_0^3}{3} + \cdots.$$

The result can easily be extended to the case in which the lower limit of integration is not zero.

III. On a Formula of Plane Trigonometry.

By M. C. Baudin, Miami University.

The formulas for finding the angles of a triangle when the three sides are given, namely:

$$\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}, \quad \tan\frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}, \quad \tan\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}},$$

where s is one half the perimeter of the triangle, may be derived trigonometrically from the law of sines in the following manner.

Consider the equations:

$$\text{(1)} \qquad \frac{\sin A + \sin B - \sin C}{\sin A + \sin B + \sin C} = \frac{s-c}{s},$$

$$\text{(2)} \qquad \frac{\sin A - \sin B + \sin C}{\sin A + \sin B + \sin C} = \frac{s-b}{s},$$

$$\text{(3)} \qquad \frac{-\sin A + \sin B + \sin C}{\sin A + \sin B + \sin C} = \frac{s-a}{s}.$$

Let us change the left-hand members of these equations into quotients of products. The numerator of (1) may be written

$$\text{(4)} \qquad \sin A + \sin B - \sin C + \sin (A + B + C).$$

Now

$$\sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2},$$

$$\sin (A + B + C) - \sin C = 2 \sin \frac{A+B}{2} \cos \left(C + \frac{A+B}{2} \right)$$

and (4) becomes

$$2 \sin \frac{A+B}{2} \left[\cos \frac{A-B}{2} + \cos \left(C + \frac{A+B}{2} \right) \right]$$

or

$$4 \sin \frac{A+B}{2} \cos \frac{B+C}{2} \cos \frac{A+C}{2}$$

or

$$4 \cos \frac{C}{2} \sin \frac{A}{2} \sin \frac{B}{2}.$$

The denominator of (1) may be written

$$\sin A + \sin B + \sin C - \sin (A + B + C).$$

Changing $\sin A + \sin B$ and then $\sin C - \sin (A + B + C)$ into products and adding the results, we get

$$2 \sin \frac{A+B}{2} \left[\cos \frac{A-B}{2} - \cos \left(C + \frac{A+B}{2} \right) \right]$$

or

$$4 \sin \frac{A+B}{2} \sin \frac{B+C}{2} \sin \frac{A+C}{2}$$

or

$$4 \cos \frac{C}{2} \cos \frac{A}{2} \cos \frac{B}{2}.$$

We therefore have

$$\frac{\sin A + \sin B - \sin C}{\sin A + \sin B + \sin C} = \tan \frac{A}{2} \tan \frac{B}{2}.$$

In the same way we obtain

$$\frac{\sin A - \sin B + \sin C}{\sin A + \sin B + \sin C} = \tan \frac{A}{2} \tan \frac{C}{2},$$

$$\frac{-\sin A + \sin B + \sin C}{\sin A + \sin B + \sin C} = \tan \frac{B}{2} \tan \frac{C}{2}.$$

Substituting these values for the left-hand members in (1), (2) and (3), and solving those equations, we get

$$\tan \frac{A}{2} = \pm \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}, \quad \tan \frac{B}{2} = \pm \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}, \quad \tan \frac{C}{2} = \pm \sqrt{\frac{(s-a)(s-b)}{s(s-c)}},$$

The positive sign must evidently be chosen before the radical, since $A/2$, $B/2$, $C/2$ cannot be greater than $\pi/2$.

RECENT PUBLICATIONS.

REVIEWS.

MATHEMATICAL PHILOSOPHY AND PHILOSOPHY OF MATHEMATICS.

Introduction to Mathematical Philosophy. By BERTRAND RUSSELL. New York, The Macmillan Company, 1919. 8 + 208 pages. Price $3.00.

The title of this book does not sufficiently indicate, perhaps no brief title could indicate clearly, the nature of the work. It is not so much an introduction to mathematical philosophy as to the philosophy of mathematics. The two 'philosophies' are very different things, the former being identified mainly by its method, the latter by its matter. An example of mathematical philosophy is the famous attempt of Spinoza to geometrize the philosophy of Descartes or his yet more famous attempt to construct the *Ethics* after the manner of Euclid. On the other hand, the philosophy of mathematics, as Mr. Russell himself here and elsewhere avows, is primarily concerned with such unanswered questions as arise in reflecting upon the nature of mathematics and the character of its foundations. This distinction being understood, we may say that there are two respects in which the present book may be properly said to be an introduction to the philosophy of mathematics. These respects may be indicated briefly as follows. When a question belonging to the philosophy of mathematics is at length acceptably answered, it ceases to be a question; the matter passes from the domain of philosophy to that of science, it becomes a genuine part of mathematics itself. Students of the logical foundations of mathematics are well aware, though the average professional philosopher is slow to learn, that this transition from speculation to knowledge has been a common and striking phenomenon in recent times. Many questions regarding the nature of such fundamentals as class, relation, number, continuity, infinity, and kindred matters, have been answered satisfactorily. The answers together with the ways of discovering them constitute the most fundamental branch of mathematical science. In the book before us Mr. Russell has endeavored to give the reader a stimulating acquaintance with some of these scientific results and methods. In this respect, then, the book is an introduction to the philosophy of mathematics, for the best approach to questions that have not been answered lies through acquaintance with the answers to those that have been answered. But the author has done more than that. Not only is the reader led in various directions *towards* the frontier of knowledge in this field but in several directions he is led *to* the frontier and is made to feel the challenge of unanswered questions that arise there. What is the characteristic—the necessary and sufficient criterion—of a mathematical proposition? What is to be the ultimate form of the theory of "logical types"? Is the multiplicative axiom, which is equivalent to that of Zermelo, true? In other words, is it true that, given any class of mutually exclusive classes, of which

none is empty, there is at least one class having exactly one term in common with each of the given classes? Is every infinite class reflexive? Such are some of the frontier questions which Mr. Russell's introduction makes the reader aware of.

The scope and content of the book are fairly indicated by the titles of its 18 chapters, which are as follows:

(1) The series of natural numbers; (2) Definition of number; (3) Finitude and mathematical induction; (4) The definition of order; (5) Kinds of relations; (6) Similarity of relations; (7) Rational, real and complex numbers; (8) Infinite cardinal numbers; (9) Infinite series and ordinals; (10) Limits and continuity; (11) Limits and continuity of functions; (12) Selections and the multiplicative axiom; (13) The axiom of infinity and logical types; (14) Incompatibility and the theory of deduction; (15) Propositional functions; (16) Descriptions; (17) Classes; (18) Mathematics and logic.

In the space allotted to this review it is impossible to present a digest of these chapters, for each of these is itself only a semi-popular digest, in some cases only a very compact and somewhat meager digest, of the more elaborate and more systematic and technical handling of the same topics in Whitehead and Russell's *Principia Mathematica* and in Russell's earlier work, *The Principles of Mathematics*. Accordingly I shall not attempt to offer a digest of these various digests but shall content myself with giving a sketchy account of some of them in the hope that the reader may be thereby drawn to the book itself and by it to the mentioned *Principia* and the great spiritual enterprise it represents.

The supreme thesis of the book is that logic and mathematics, rightly understood, are not two distinct sciences but that they together literally constitute one science without any inner division or breach from the most primitive things in what is traditionally called logic to the latest developments in what is traditionally called mathematics. Why, then, does Mr. Russell open his discussion with a consideration of the natural numbers, or ordinary integers? Are these assumed to be the most primitive of known things in logic (*i.e.* mathematics)? Far from it. In the *Principia* one must march through more than 350 large pages of solid developments before the concept of even the first of the integers is encountered. In an introduction designed to attract instead of repelling the uninitiated it was necessary to take most of those developments for granted; it was necessary to begin with something familiar; and nothing is more familiar than the so-called natural numbers, notwithstanding the fact that very few people, even among mathematicians, have taken the pains to ascertain the significance of these numbers in terms of logical primitives.

Mr. Russell's discussion of the natural numbers attaches itself to the great, albeit now superseded, work of Peano. "Having reduced all traditional pure mathematics to the theory of the natural numbers, the next great step in logical analysis was to reduce this theory itself to the smallest set of premises and undefined terms from which it could be derived. This work was accomplished by Peano. He showed that the entire theory of the natural numbers could be derived from three primitive ideas and five primitive propositions in addition to those of

pure logic." Having paid this deserved tribute to the pioneer work of the distinguished Italian, Mr. Russell proceeds to show in detail that and how Peano's three primitive (undefined) ideas—zero, number, successor—may be defined perfectly in purely logical terms and, moreover, that and how his five primitive (undemonstrated) propositions admit of rigorous demonstration by means of purely logical concepts and propositions. This beautiful achievement, accomplished in less than 30 pages, is very weighty as showing how the entire superstructure of traditional mathematics rests upon a purely logical basis.

The natural numbers are cardinal numbers. On the threshold of this subject three questions present themselves: (1) What is meant by saying that two given classes have the same cardinal number? (2) What is meant by the cardinal number of a given class? (3) What is a cardinal number? Two classes are *similar* if they can be paired in one-one fashion like the class whose members are the legs and the class whose members are the arms of a normal human body. The answer to question (1) is: two classes have the same cardinal number if they are similar. The answer to (2) is: the cardinal number of a given class c is the class C of all classes similar to c. *i.e.*; of all classes each of which has the same cardinal number as c. The answer to (3) is: the class C' of all classes C such that each of these has for its members all the classes c similar to one of them. It is interesting to note, in passing, how far the concept, cardinal number, is removed from the domain of perception.

Evidently cardinal numbers are signless, neither positive nor negative. In order to prepare the reader for definitions (in terms of logical constants) of positive and negative integers, rational numbers, real numbers, complex numbers and other sorts, it is necessary to acquaint him with a certain serviceable stock of relations and their chief properties—relations between individuals and classes, relations subsisting between classes, and relations, such as likeness of structure, holding between relations themselves. After such preparation, which the reader receives with pleasure and a growing sense of enlightenment, he readily discerns what manner of things the various kinds of number really are. He finds, for example, that a positive or a negative integer, instead of being, like a cardinal, a class of classes, is a specific kind of *relation* between such classes of classes. More precisely, if n be any cardinal (finite or infinite) and m be a finite cardinal, the positive integer, $+ m$, is the relation of the cardinal, $n + m$, to n, and the negative integer, $- m$, is the converse relation, namely that of n to $n + m$.

What is meant by a rational number? This, too, is a relation between cardinals but a different sort of relation from the foregoing one. The direct logical approach is through the notion of propositional function. A propositional function is any statement involving one or more variables. It has the form of a proposition but is not one, being neither true nor false since the variables in it are but blanks or empty symbols. If it contain but one variable, the function determines a class of terms, namely that composed of all the terms which, if substituted for the variable, yield true propositions. If it contains two variables, say x and y, it determines a dyadic relation, which for mathematics is a class

excellence, consists well with the warmer elements of a broader humanity. A citation or two may be in place. "Pure logic, and pure mathematics (which is the same thing), aims at being true, in Leibnizian phraseology, in all possible worlds, not only in this higgledy-piggledy job-lot of a world in which chance has imprisoned us." Again: "It is a disgrace to the human race that it has chosen to employ the same word 'is' in two such utterances as, Socrates is human, and Socrates is a man." Once more: "Men may be defined as featherless bipeds, or as rational animals, or (more correctly) by the traits by which Swift delineates the Yahoos."

CASSIUS J. KEYSER.

COLUMBIA UNIVERSITY.

Elements of Vector Algebra. By L. SILBERSTEIN. London, Longmans, 1919. 8vo. 4 + 42 pp. Price $1.60.

Extract from the Preface—"This little book was written at the instance of Messrs. Adam Hilger, and, in accordance with their desire, it contains just what is required for the purpose of reading and handling my *Simplified Method of Tracing Rays, etc.* (Longmans, Green and Co., London, 1918). With this practical aim in view, all critical subleties have been purposely avoided. In fact, it is scarcely more than a synoptical presentation of the elements of vector algebra covering the needs of those engaged in geometrical optics. At the same time, however, it is hoped that this booklet will serve a more general purpose, viz., to provide everybody unacquainted with the subject with an easy introduction to the use of vector algebra.

"It is scarcely necessary to explain that the deductions given in this book are based on Euclid's axioms, notably with the inclusion of his postulate of parallels—upon which the equality of vectors is most essentially based. Those readers who are desirous of seeing how the formal rules here given can be generalized so as to be valid independently of the axioms of congruence and of parallels, may consult the author's *Projective Vector Algebra* (Bell and Sons, 1919), and a sequel to it published in *Phil. Mag.* for July, 1919, pp. 115–143. It is, however, advisable for the student to become first thoroughly familiar with the Euclidean vector algebra as here presented."

Contents—Section 1. Vectors defined, 1–2; 2. Equality of vectors defined, 2–3; 3. Addition of vectors, 3–10; 4. Subtraction of vectors, 10–11; 5. Scalar product of two vectors, 11–17; 6. The vector product of vectors, 17–21; 7. Expansion of vector formulæ, 21–23; 8. Iteration of vectorial multiplication 23–25; 9. The linear vector operator, 25–38; 10. Hints on differentiation of vectors, 38–40; Index, 41–42.

NOTES.

The Elementary Differential Geometry of Plane Curves by R. H. FOWLER, fellow of Trinity College, Cambridge is the latest issue, number 20, of the Cambridge Mathematical Tracts (Cambridge University Press, price 6 shillings).

The beautiful large pages, large type, and wide margins which the *American Journal of Mathematics* maintained for forty-one years have made way for pages about one-half as large, smaller type, and narrower margins.

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Karpinski's *Robert of Chester's Latin Translation of the Algebra of al Khowarismi* (New York, 1915) 218–219.

Heft 12, in December, 1919, completed the jubilee volume of *Zeitschrift für mathematischen und naturwissenschaftlichen Unterricht aller Schulgattungen,* founded by J. C. V. Hoffmann (1825–1905) in 1869.

In 1911 Teubner published the first two volumes of the great edition of Leonard Euler's *Opera Omnia,* now being prepared under the auspices of the Swiss Mathematical Society, a section of the Society of Swiss Naturalists. It was planned that the works should contain about 70 volumes and be issued in three series: series I—opera mathematica, 28 volumes; series II—opera mechanica et astronomica, 27 volumes; series III—opera physica, miscellanea, epistolae, 15 volumes.

Thirteen of these volumes have been published: the two volumes of *Dioptrica* in series III; the two volumes of *Mechanica sive motus scientia analytice exposita* in series II; and the following nine volumes in series I—*Vollständige Anleitung zur Algebra;* the first of four volumes of *Commentationes arithmeticae;* the four volumes of *Institutiones calculi differentialis* and *Institutiones calculi integralis;* the first of the two volumes of *Commentationes analyticae* (integrals); and the two volumes of *Commentationes analyticae* (elliptic integrals).

In E. C. Moore's *What the War Teaches About Education and other Papers and Addresses* (New York, Macmillan, 1919) Chapter 6, pages 95–119, entitled "Does the study of mathematics train the mind specifically or universally?", was an address before the Association of Teachers of Mathematics in New England, April, 1917; chapter 7, pages 120–128, entitled "Mathematics and formal discipline again," was reprinted from *School and Society,* December 29, 1917, and is a reply to "The inadequacy of arguments against disciplinary values" by C. N. Moore; chapter 8, pages 129–151, entitled "Does the study of mathematics train the mind specifically or universally? A reply to a reply," is reprinted from *School and Society,* April 27, 1918, in reply to "Does the study of mathematics train the mind specifically or universally? A reply" by R. E. Moritz.

ARTICLES IN CURRENT PERIODICALS.

Annaes scientificas da Academia Polytechnica do Porto, volume 13, 1919, no. 2: "Pedro Nunes e os infinitamente pequenos" by R. Guimaraes, 65–71—No. 3: "Sur les surfaces réglées" by C. Servais, 129–151; "Sur l'octaëdre à faces triangulaires" by J. Neuberg, 161–171.

Bulletin de la Société Mathématique de France, volume 47, nos. 1–2, 1919: "Les fondements de la théorie des fonctions elliptiques" by H. Hancock, 37–42.

Bulletin des Sciences Mathématiques, volume 54, August, 1919: Review by H. Deltheil of G. Castelnuovo's *Calcolo delle Probabilità* (Milano-Roma-Napoli, 1919), 165–174; Review by E. Cartan of G. Bouligand's *Cours de géométrie analytique* (Paris, 1919), 175–178; "Sur les intégrales de Fresnel" by A. Bloch, 179–180.

Harvard Graduates Magazine, volume 28, March 1920: "An American liaison officer in Paris" by J. L. Coolidge, 394–408.

Mathematical Gazette, volume 9, no. 143, December, 1919: "Report of the Mathematical Association committee on the teaching of mathematics in public and secondary schools,"

UNDERGRADUATE MATHEMATICS CLUBS.

Edited by U. G. Mitchell, University of Kansas, Lawrence.

NOTES AND SUGGESTIONS.

A Junior Mathematics Club was organized at Cornell University in the early part of the present academic year. One feature of this club's work seems to the editor well worth incorporating into the program of other undergraduate clubs. In response to a request from the students, a member of the faculty devoted an hour to explaining to the club "the utility and value of each of the mathematical courses offered by the University beyond the freshman year; what courses would be of greatest value to the student of chemistry, of physics, of biology, of economics, of education, and for further mathematical development; and, in each case, some of the reasons why the course suggested was of particular value in the field indicated."

In response to a request published some time ago for suggestions as to mathematical games suitable for club uses in social meetings, Mrs. W. E. Beckwith of the College for Women, Western Reserve University, Cleveland, Ohio, sends the following: "Place a fairly good-sized mirror upright on a table and have each student draw a four-inch square with its diagonals by watching his movements in the mirror. I find it necessary to cover the hands to insure use of the mirror only. This undertaking is far more difficult than it would appear to be and is certain to 'break the ice' on any occasion."

In response to the same request Professor Elizabeth B. Cowley of Vassar College, Poughkeepsie, N. Y., writes that their club has used mathematical adaptations of some of the familiar guessing games. For example, a theorem in geometry was selected and each person of a group took one word of the theorem. The person who had been sent "out" returned and asked each a question to which she replied, using in the reply, her word in the theorem. This game was most successful when the theorem selected was quite familiar. Another game was that in which one member was sent out and all the others selected a mathematical term. When the person returned, she had the privilege of asking twenty questions about the term, selecting the persons to be questioned. Each person questioned must answer, but without mentioning the term. Of course, if the questions are skilful and the answers are less so, the term is quickly guessed. At one of its meetings the club played charades using mathematical terms as the words to be represented by acts.

CLUB ACTIVITIES.

The Mathematics Club of Brown University, Providence, R. I. [*1918*, 33–34; *1919*, 167; *1920*, 28.]

The Mathematics Club of Brown University issues annually a printed folder announcing the program for the year and containing the list of officers and

February 27: "Who's who in modern mathematics" by Professor Ernest B. Skinner.

March 13: "The trisection of an angle by means of conic sections" by Madge Ryan '20; "Curves" by Ethel Vasey '19.

April 16: Social evening at the home of Professor and Mrs. Dowling. Illustrated lecture on "Famous mathematicians" by Professor Dowling.

May 8: "Works of Archimedes" by Professor Charles S. Slichter.

May 22: Business meeting for election of officers. Officers elected for the year 1919–20: President, Margaret Lee '20; vice-president, Ruth-Marie Urban '20; secretary-treasurer, Gladys Baur '20.

October 21: Picnic.

October 30: "Classification of curves and surfaces" by Professor Van Vleck.

November 13: "Some surprises in the history of mathematics" by Professor Dowling.

December 4: "Mathematical recreations" by Estelle Stone '20 and Madge Ryan '20.

December 18: "Graphical calculation" by Professor Arnold Dresden.

January 8, 1920: "Numerals of antiquity" by Ruth-Marie Urban '20; "Life and works of Abel" by Alta Gudsos '20.

January 22: "Integers" by Professor Skinner.

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL AND OTTO DUNKEL.

Send all communications about problems and solutions to B. F. FINKEL, Springfield, Mo.

NOTE ON CAUSTICS.

By OTTO DUNKEL, Washington University.

The purpose of this note is to show that the formula (3), which was derived by Professor da Cunha in another way in his solution of problem 2768, below, has a somewhat wider application. If from a point F rays of light fall upon the concave side of a curve Γ in a plane containing the point, and the reflected rays intersect on the same side, the envelope γ of these reflected rays is called the caustic of Γ with respect to the point F. Let M be a point of Γ, F' the point in which the ray reflected from M has contact with γ, ω the angle of incidence, $R = MO$, the radius of curvature of Γ at M, $\delta = FM$ and $\delta' = MF'$. Then it may be proved that

$$\frac{1}{\delta} + \frac{1}{\delta'} = \frac{2}{R \cos \omega}. \tag{1}$$

A simple geometrical derivation of this result is given in Humbert's *Cours d'Analyse*, vol. 1, page 77. Let the point of intersection of the normal to Γ at M with the line FF' be denoted by T and the length MT by n. By dropping

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NOTE ON CAUSTICS.

By OTTO DUNKEL, Washington University.

The purpose of this note is to show that the formula (3), which was derived by Professor da Cunha in another way in his solution of problem 2768, below, has a somewhat wider application. If from a point F rays of light fall upon the concave side of a curve Γ in a plane containing the point, and the reflected rays intersect on the same side, the envelope γ of these reflected rays is called the caustic of Γ with respect to the point F. Let M be a point of Γ, F' the point in which the ray reflected from M has contact with γ, ω the angle of incidence, $R = MO$, the radius of curvature of Γ at M, $\delta = FM$ and $\delta' = MF'$. Then it may be proved that

$$\frac{1}{\delta}+\frac{1}{\delta'}=\frac{2}{R\cos\omega}. \tag{1}$$

A simple geometrical derivation of this result is given in Humbert's *Cours d'Analyse*, vol. 1, page 77. Let the point of intersection of the normal to Γ at M with the line FF' be denoted by T and the length MT by n. By dropping

perpendiculars from F and F' upon the normal MT and by considering two pairs of similar triangles, it is easily seen from a figure that

$$(2) \qquad \frac{n - \delta \cos \omega}{\delta' \cos \omega - n} = \frac{\delta}{\delta'} \quad \text{or} \quad \frac{1}{\delta} + \frac{1}{\delta'} = \frac{2 \cos \omega}{n}.$$

From (1) and (2) there results at once

$$(3) \qquad R = \frac{n}{\cos^2 \omega},$$

the formula mentioned above.

If the curvature of the curve Γ is such that the point F' lies on the side opposite to F, then by a very slight modification of the derivation given by Humbert it may be shown that (1) is again true provided that δ' be regarded as negative. In this case there is no caustic by reflection in the physical sense and it is not considered by Humbert. Also a proof similar to that indicated above shows that (2) also is true if δ' be regarded as negative. Thus (3) is true for both cases. It is thus somewhat more general and at the same time simpler than (1); it also has the advantage of being more convenient for geometrical constructions. Thus, if the point O has been found on the evolute of a curve Γ corresponding to the point M of Γ, the formula (3) gives an easy construction for the point F' on the caustic of Γ with respect to a given point F. If, for example, the curve Γ is a circle the caustic may be easily traced by this construction. If, on the other hand, Γ is a conic and F is a focus, then γ reduces to the other focus, or to a point at infinity in the case of a parabola, and the formula gives the construction given by Professor da Cunha.

Another construction for the center of curvature may be derived from a property of triangles. In any triangle FMF' let C be the middle point of the side FF', T the point in which the bisector of the angle at M meets FF', P the point in which the side FM is cut by the perpendicular to the bisector at T, O the point in which the perpendicular to FM at P meets the bisector MT (O is the center of curvature when F, F' and M have the meaning above) and finally Q the point in which TP meets CM. The property referred to is the fact that QO is perpendicular to FF'. A similar theorem is true for the external bisector. The second construction, which applies to the general case as well as to conics, is then as follows: Erect a perpendicular to the normal at the point where it cuts the axis and from the point in which this perpendicular meets the line joining the center with the given point on the curve drop a perpendicular to the axis and produce it to meet the normal. This point is the center of curvature.

PROBLEMS FOR SOLUTION.

2829. Proposed by E. S. PALMER, New Haven, Conn.

Given a set of arbitrary pairs of positive integers (a_p, b_p), $(p = 1, 2, \cdots, n)$: (*a*). Is it always possible to find a set of positive integers k_p, $(p = 1, 2, \cdots, n)$ such that

$$k_p a_p + k_p b_p > \sum_{r=1}^{r=n} k_r a_r, \qquad (p = 1, 2, 3, \cdots, n).$$

(*b*) If or when possible, show how to find k_p?

resulting expression is $(-1)^n\lambda^n[(n+1)a-\lambda]$, and is of the form demanded by our theorem by the case of $n+1$. The left-hand member of the resulting expression may be written so that its determinant is of order $n+1$,

$$\frac{na+a-\lambda}{na-\lambda}\cdot\begin{vmatrix} -\lambda, & 0, & 0, & \cdots, & 0 \\ a, & a-\lambda, & a, & \cdots, & a \\ a, & a, & a-\lambda, & \cdots, & a \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a, & a, & a, & \cdots, & a-\lambda \end{vmatrix}. \tag{1}$$

Here the $-\lambda$ of our multiplier appears as the element in the first row and first column of the determinant of order $n+1$. Add the second, third, and each of the following rows in turn to the first row obtaining an equal expression, the new determinant being exactly the same as that in (1) above except that the elements of the first row are now all $na-\lambda$. Multiply the factor preceding this determinant into the elements of the first row obtaining a third determinant which is again exactly like (1) except that the elements of the first row are now all $na+a-\lambda$. Subtract the second, third, and each of the following rows from the first, thus obtaining a determinant of order $n+1$ and of precisely the form demanded by our theorem if it is to be true in the case of $n+1$. Thus, if the relation is true for n, it is also true for $n+1$, and the induction is complete.

Also solved by A. L. Candy, P. J. da Cunha, L. D. Hand, A. M. Harding, R. A. Johnson, L. C. Mathewson, H. L. Olson, A. Pelletier, J. L. Riley, G. Y. Sosnow, Elijah Swift, and C. C. Yen.

2744 [1919, 37]. Proposed by E. B. ESCOTT, Chicago, Ill.

An insurance company computes its quarterly premiums by adding 6 per cent to the annual premium and dividing by 4. If a policyholder pays quarterly, what rate of interest is he paying?

I. Solution by Elijah Swift, University of Vermont.

If we assume that this means that the policyholder sets aside the annual premium at the beginning of the year, pays the first of the quarterly premiums out of it, lets the remainder lie at interest for three months, then deducts the second premium, and so on, the interest will be compounded quarterly, and the present worth of the four premiums at the beginning of the year must equal the annual premium. If we call the annual premium, $4P$, and the (unknown) annual interest rate, $4i$, each quarterly premium will be $1.06P$ and we have the equation

$$1.06P+\frac{1.06P}{1+i}+\frac{1.06P}{(1+i)^2}+\frac{1.06P}{(1+i)^3}=4P.$$

This cubic may be solved by Horner's method, whence $4i = 16.11$ per cent.

If interest be compounded semi-annually, we have a quadratic,

$$1.06P+\frac{1.06P}{1+i}+\frac{1.06P}{1+2i}+\frac{1.06P}{(1+i)(1+2i)}=4P$$

whence $4i = 16.33$ per cent.

If interest be reckoned as simple (compounded annually) we have to solve the cubic

$$1.06P+\frac{1.06P}{1+i}+\frac{1.06P}{1+2i}+\frac{1.06P}{1+3i}=4P.$$

whence $4i = 16.54$ per cent.

In any case, then, the policyholder must pay over 16 per cent for the accommodation.

II. Solution by the Proposer.

A (by algebra).

Let P = annual premium, p = quarterly premium $=\frac{1.06}{4}P$, r = annual rate of interest.

$$P=\frac{400}{106}p.$$

We have the equation

$$(P - p)\left[1 + \frac{r}{4}\right]^3 - p\left[1 + \frac{r}{4}\right]^2 - p\left[1 + \frac{r}{4}\right] - p = 0.$$

Putting

$$1 + \frac{r}{4} = x$$

and substituting value of P, we have the equation

$$294x^3 - 106x^2 - 106x - 106 = 0.$$

Solving by Horner's Method, we have

$$x = 1 + \frac{r}{4} = 1.04028,$$

$$r = .16112 = 16.112 \text{ per cent compounded quarterly.}$$

B (by arithmetic). A more elementary and more "practical" method is the method by trial and error. A few trials will show that the rate is something over 16 per cent.

First Trial. Taking the rate as 16 per cent and the annual premium as 100, we have the scheme,—

Annual premium due	100.00
First quarterly premium paid	26.50
	73.50
Interest for three months	2.94
	76.44
Second quarterly premium paid	26.50
	49.94
Interest for 3 months	2.00
	51.94
Third quarterly premium paid	26.50
	25.44
Interest for 3 months	1.02
	26.46
Fourth quarterly premium	26.50
First error	0.04

We see that 16 per cent is slightly too small.

Second trial. Taking the rate as 16.2 per cent., we have, in the same way as before, an error of $+ .03$.

Forming a table

Rate	Error
16	$-.04$
16.2	$+.03$

By interpolation, the rate that will give zero error is

$$16 + \frac{4}{7} \times .2 = 16.114 \text{ per cent.}$$

If greater accuracy were required, repeat the computation with the last rate and interpolate again.

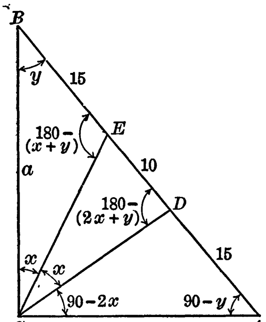
Also solved by G. N. Armstrong, H. N. Carleton, and H. L. Olson.

2747 [1919, 72]. Proposed by DANIEL KRETH, Wellman, Iowa.

In the right triangle ABC, right angle C, we have given on the hypotenuse the segments $AD = 15$, $DE = 10$, $EB = 15$, and the angle DCE equal to the angle ECB. Find the angle DCE, and the sides AC and BC.

SOLUTION BY MARCIA LATHAM, New York City.

Let $BC = a$, $AC = b$, $DC = c$, $\angle DCE = \angle ECB = x$, $\angle ABC = y$. Then $\angle DCA = 90 - 2x$, $\angle EDC = 180 - (2x + y)$, and $\angle BEC = 180 - (x + y)$.



$$\sin y = b/40 \qquad \text{and} \qquad \cos y = a/40. \tag{1}$$

Since EC bisects $\angle BCD$,

$$a/c = 15/10 = 3/2. \tag{2}$$

In the triangle, DCA, by the law of sines,

$$c/15 = \cos y/\cos 2x. \tag{3}$$

Combining (1), (2), and (3), $2a/45 = a/40 \cos 2x$, or $\cos 2x = 9/16$, $x = 1/2 \cos^{-1} 9/16 = 27° \, 53' \, 8''$. In the triangle, BCD, by law of sines, $\sin (2x + y)/\sin y = a/c$. Whence, by (1) and (2)

$$\sin (2x + y) = 3b/80. \tag{4}$$

Also, $a/25 = \sin (2x + y)/\sin 2x$; whence, by (4), $\sin 2x = 15b/16a$. But $\sin^2 2x + \cos^2 2x = 1$. Then $(15b/16a)^2 + (9/16)^2 = 1$; whence

$$b^2 = 7a^2/9. \tag{5}$$

Now, in the triangle, ABC, $a^2 + b^2 = (40)^2$; whence, by (5), $a^2 + 7a^2/9 = 1{,}600$; whence, $a = 30$ and from (5), $b = 10\sqrt{7}$.

Also solved by A. M. HARDING, POLYCARP HANSEN, C. E. HORNE, R. A. JOHNSON, ELMER LATSHAW, E. W. MARTIN, LOUIS ORDANKSY, A. PELLETIER, J. L. RILEY, H. M. ROESER, L. SMITH, D. L. STAMY, H. TSAI, and L. G. WELD.

2760 [1919, 124]. Proposed by CHARLES N. SCHMALL, New York City.

In an arithmetical progression, if s_n be the sum of the first n terms, s_{2n} the sum of the first $2n$ terms, and s_{3n} the sum of the first $3n$ terms of the same series, prove that $s_{2n} - s_n = \frac{1}{3}s_{3n}$.

SOLUTION BY EMMA M. GIBSON, Springfield (Mo.) High School.

The sum of n terms of an arithmetical progression is expressed by the formula

$$s_n = \frac{n(a_1 + a_n)}{2},$$

where a_1 and a_n are the first and nth terms, respectively.

Hence, $s_n = n(a_1 + a_n)/2$, $s_{2n} = 2n(a_1 + a_{2n})/2$, and $s_{3n} = 3n(a_1 + a_{3n})/2$ are the sums of the first n terms, the first $2n$ terms, and the first $3n$ terms, respectively. Now $a_{2n} = a_n + nd$, $a_{3n} = a_{2n} + nd = a_n + 2nd$, d being the common difference.

Then $s_{2n} = 2n(a_1 + a_n + nd)/2$ and $s_{3n} = 3n(a_1 + a_n + 2nd)/2$ and

$$s_{2n} - s_n = 2n(a_1 + a_n + nd)/2 - n(a_1 + a_n)/2 = n(a_1 + a_n + 2nd)/2 = s_{3n}/3.$$

Also solved by R. D. BOHANNAN, H. L. BRIDGES, JR., H. N. CARLETON, W. F. CHENEY, JR., P. J. DA CUNHA, H. C. GOSSARD, WILLIAM HERBERG, C. N. MILLS, LOUIS O'SHAUGHNESSEY, H. L. OLSON, A. PELLETIER, J. B. REYNOLDS, I. S. SUN, and ELIJAH SWIFT.

2768 [1919, 171]. Proposed by PAUL CAPRON, U. S. Naval Academy.

Given the center, a focus, and a point of a conic, construct geometrically the circle of curvature at the point.

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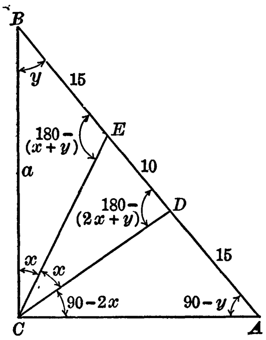
Let $BC = a$, $AC = b$, $DC = c$, $\angle DCE = \angle ECB = x$, $\angle ABC = y$. Then $\angle DCA = 90 - 2x$, $\angle EDC = 180 - (2x + y)$, and $\angle BEC = 180 - (x + y)$.



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are the tangents at these points. The conic is, therefore, determined; for we have three points M, A, A' and the tangent at each of them, known.

From this point on the construction may be completed in any one of three ways:

(1) By the use of Pascal's Theorem, find the point H in which the normal at M cuts the conic again. Draw the circle on HM as diameter. Find the fourth point of intersection G of this circle with the conic. Draw HG. Through M draw MK parallel to HG and determine the point K in which this line cuts the conic again. The circle tangent to the conic at M and passing through K is the required circle of curvature at M. (Cf. L. Cremona, *Elements of Projective Geometry*, Oxford, 1885, p. 190).

(2) By the method previously described both the ellipse and the hyperbola that have center C, focus F, and point M are determined. The construction may, therefore, be completed by making use of the fact that at the intersection of two confocal conics the center of curvature of either is the pole with respect to the other of the tangent to the former at the intersection.

(3) Through F, draw a line parallel to the tangent at M. By Desargues' Theorem, find the points R and R' in which this line meets the conic. On MF take Q so that $MQ = RR'$. Then the circle through Q and tangent to the conic is the required circle of curvature; for, the focal chord of curvature is equal to the focal chord of the conic drawn parallel to the tangent at the point. For results in (2) and (3) the reader may consult Salmon's *Conic Sections*, 6th ed., 1879, pp. 374–376.

It may be noted that if C is at infinity, the conic becomes a parabola with the axis CF, the focus F, and the point M given. Two parabolas may be drawn satisfying these conditions according to which of the bisectors of the angle made by MF and the diameter through M is taken as tangent. The above methods of construction then apply to this case also.

Also solved by H. HALPERIN, A. PELLETIER, J. B. REYNOLDS, and the Proposer.

2771 [1919, 191]. Proposed by GEORGE PAASWELL, New York City.

A circle is revolved through an angle of 90° about a vertical chord which does not pass through the center of the circle. Taking the origin at the lower extremity of the chord, the z-axis along the chord and the x- and y-axes in the boundary planes, pass a plane through the x-axis making a given angle with the xy plane. Determine the portion of the area of the surface above the plane and between the xz and yz planes.

I. SOLUTION BY J. B. REYNOLDS, Lehigh University.

A solution by vector analysis. Let $r = a\cos\theta + b\sin\theta$ be the polar equation of the circle revolved. From this we see that for a vector equation of the surface generated, we may write

$$r = (a\cos t + b\sin t)\cos t\cos u\cdot i + (a\cos t + b\sin t)\cos t\sin u\cdot j + (a\cos t + b\sin t)\sin t\cdot k;$$

or

$$r = \tfrac{1}{2}\{a + a\cos 2t + b\sin 2t\}\cos u\cdot i + \tfrac{1}{2}\{a + a\cos 2t + b\sin 2t\}\sin u\cdot j + \tfrac{1}{2}\{b + a\sin 2t - b\cos 2t\}\cdot k$$

from which

$$\frac{\partial r}{\partial t} = \{-a\sin 2t + b\cos 2t\}\cos u\cdot i + \{-a\sin 2t + b\cos 2t\}\sin u\cdot j + \{a\cos 2t + b\sin 2t\}\cdot k,$$

$$\frac{\partial r}{\partial u} = -\tfrac{1}{2}\{a + a\cos 2t + b\sin 2t\}\sin u\cdot i + \tfrac{1}{2}\{a + a\cos 2t + b\sin 2t\}\cos u\cdot j,$$

giving the vector product

$$\begin{aligned}\frac{\partial r}{\partial t}\times\frac{\partial r}{\partial u} = &-\tfrac{1}{2}\{a + a\cos 2t + b\sin 2t\}\{a\cos 2t + b\sin 2t\}\cos u\cdot i\\ &-\tfrac{1}{2}\{a + a\cos 2t + b\sin 2t\}\{a\cos 2t + b\sin 2t\}\sin u\cdot j\\ &+\tfrac{1}{2}\{-a\sin 2t + b\cos 2t\}\{a + a\cos 2t + b\sin^2 2t\}\cdot k.\end{aligned}$$

[1] Of course it is unnecessary to describe the circle through F', F, M, for the tangent and normal bisect angles given at M; nor is it necessary to invoke an involution in order to get A and A' since we have given $F'M + FM = 2a$.—EDITOR.

are the tangents at these points. The conic is, therefore, determined; for we have three points M, A, A' and the tangent at each of them, known.

From this point on the construction may be completed in any one of three ways:

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or

$$r = \tfrac{1}{2}\{a + a \cos 2t + b \sin 2t\} \cos u \cdot i + \tfrac{1}{2}\{a + a \cos 2t + b \sin 2t\} \sin u \cdot j + \tfrac{1}{2}\{b + a \sin 2t - b \cos 2t\} \cdot k$$

from which

$$\frac{\partial r}{\partial t} = \{- a \sin 2t + b \cos 2t\} \cos u \cdot i + \{- a \sin 2t + b \cos 2t\} \sin u \cdot j + \{a \cos 2t + b \sin 2t\} \cdot k,$$

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[1] Of course it is unnecessary to describe the circle through F', F, M, for the tangent and normal bisect angles given at M; nor is it necessary to invoke an involution in order to get A and A' since we have given $F'M + FM = 2a$.—EDITOR.

Now since a differential element of the surface generated is the absolute value of $\frac{\partial r}{\partial t} \times \frac{\partial r}{\partial u}$ multiplied by $dr\,du$, we have

$$S = \int\int \left[\frac{\partial r}{dt} \times \frac{\partial r}{\partial u}\right]_0 dudt$$

$$= \frac{\sqrt{a^2+b^2}}{2} \int\int \{a + a\cos 2t + b \sin 2t\} dtdu.$$

In order to get the limits of this integral, we must find the value of t for which the plane $r = xi + yj + y \tan\alpha \cdot k$, the plane through the x-axis making an angle α with the xy plane, cuts the surface of revolution. Equating coefficients of i, j, and k in the two equations, we find

$$x = \tfrac{1}{2}\{a + a\cos 2t + b\sin 2t\}\cos u, \qquad y = \tfrac{1}{2}\{a + a\cos 2t + b \sin 2t\}\sin u$$

$$y \tan\alpha = \tfrac{1}{2}\{b + a \sin 2t - b\cos 2t\},$$

the latter two of which give us the equation

$$b\cot\alpha - a\sin u + (a\cot\alpha - b\sin u)\sin 2t - (b\cot\alpha + a\sin u)\cos 2t = 0;$$

whence, $t = \tan^{-1}(-a/b)$ or $\tan^{-1}(\sin u/\cot\alpha)$ and since the first of these two values of t is for the special point where $r = 0$, we have

$$S = \frac{\sqrt{a^2+b^2}}{4}\int_0^{\pi/2} du\{2at + a\sin 2t - b\cos 2t\}_{\tan^{-1}\frac{\sin u}{\cot\alpha}}^{\pi/2}$$

$$= \frac{\sqrt{a^2+b^2}}{2}\int_0^{\pi/2}\left\{\frac{\pi a}{2} - a\frac{\cot\alpha\sin u}{\csc^2\alpha - \cos^2 u} + b\frac{\cot^2\alpha\csc^2 u}{\csc^2\alpha + \cot^2\alpha\cot^2 u} - a\tan^{-1}\frac{\sin u}{\cot\alpha}\right\}du$$

$$= \frac{\sqrt{a^2+b^2}}{2}\left\{\frac{\pi^2 a}{4} + \frac{\pi b\cos\alpha}{2} + a\frac{\cos\alpha}{2}\log\left[\frac{1-\sin\alpha}{1+\sin\alpha}\right]\right\} - a\frac{\sqrt{a^2+b^2}}{2}\int_0^{\pi/2}\tan^{-1}\left(\frac{\sin u}{\cot\alpha}\right)du$$

for the required surface. If $\alpha = \pi/2$, $S = 0$. If $a = 0$, $S = \pi b^2/4 \cos\alpha$, one fourth the surface of a sphere of diameter b less half a zone of height $b/2(1 - \cos\alpha)$. If $\alpha = 0$ and $b = 0$, $S = \pi^2 a^2/8$ one eighth the surface of a torus. If $\alpha = 0$,

$$S = \frac{\pi^2 a\sqrt{a^2+b^2}}{8} + \frac{\pi b\sqrt{a^2+b^2}}{4},$$

as it should, since a semicircle of length $\frac{\pi}{2}\sqrt{a^2+b^2}$ is revolved through 90° about an axis at a distance $a/2 + b/\pi$ from its center of gravity. Since for $\alpha < 45°$, $\tan^{-1}(\sin u/\cot\alpha)$ will expand into an absolutely convergent series we may use for such values of α to any desired degree of accuracy

$$\int_0^{\pi/2}\tan^{-1}\left(\frac{\sin u}{\cot\alpha}\right)du = \sum_{n=1}^{n=\infty}(-1)^{n+1}(\tan\alpha)^{2n-1}\frac{2\cdot4\cdot6\cdots 2n-2}{3\cdot5\cdot7\cdots 2n-1}\frac{1}{2n-1}.$$

II. Solution by B. F. Finkel, Drury College.

Those of our readers who are unfamiliar with quaternions or in whose hands this subject is not an easily operated instrument of investigation can obtain the integral S by the following process:

Using the same notation; that is, taking r, t, and u as polar coördinates, we have for the equation of the surface of revolution $r = a\cos t + b\sin t$, and for the element of arc on circle revolved $ds = \sqrt{a^2+b^2}\cdot dt$.

The required area is expressed by the double integral

$$S = \int\int r\cos t\, du\, ds$$

$$= \sqrt{a^2+b^2}\int\int (a\cos^2 t + b\sin t\cos t)\, du\, dt.$$

Also solved by C. A. BARNHART, H. HALPERIN, H. L. OLSON, and A. PELLETIER.

2773 [1919, 212]. Proposed by JOSEPH ROSENBAUM, Milford, Conn.

Point out the fallacy in the proof of the following problem:

In the triangle $A_1B_1C_1$ let m be a point such that the sum of the distances from it to the sides is a maximum; also let $A_2B_2C_2$ be a triangle formed by drawing lines through the vertices A_1, B_1, and C_1 parallel to their opposite sides. Then the sum of the distances from m to the sides of the triangle $A_2B_2C_2$ is a minimum.

Proof.—Because the sides of the two triangles are parallel in pairs, the sum of the distances from a variable point P in triangle $A_1B_1C_1$ to the six sides of the two triangles is constant. Now by hypothesis M is a point for which one part of this constant sum is a maximum, and hence it follows that the other part is a minimum.

SOLUTION BY H. L. OLSON, University of Wisconsin.

This proof is correct, with the understanding that if a point P is on the opposite side of BC, for example, to the vertex A, the distance to the side BC is to be regarded as negative. It is easy to see, however, that the point M does not exist, and that the proposition is therefore vacuous. Represent the perpendicular distances from P to the sides BC, AC, and AB by α, β, and γ respectively. If we denote by Δ the area of the triangle ABC, we are to minimize the function $\alpha + \beta + \gamma$, subject to the condition $a\alpha + b\beta + c\gamma = 2\Delta$. ($a$, b, and c represent, as is customary, the sides BC, AC, and AB, respectively.) Eliminating γ, we have, as the function to be minimized,

$$\left(1 - \frac{a}{c}\right)\alpha + \left(1 - \frac{b}{c}\right)\beta + \frac{2\Delta}{c}.$$

Hence, the derivatives, $\left(1 - \frac{a}{c}\right)$, and $\left(1 - \frac{b}{c}\right)$, of this function with respect to α and β must vanish; but for the general triangle they do not vanish and hence M does not exist. If, however, $a = b = c$, the sum of the distances is the constant $2\Delta/c$; likewise the sum of the distances for the corresponding triangle $A_2B_2C_2$ is constant.

Also solved by A. PELLETIER and A. L. WECHSLER.

2774 [1919, 212]. Proposed by FRANK IRWIN, University of California.

Evaluate the circulants

$$\begin{vmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ n & 1 & 2 & \cdots & n-2 & n-1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 2 & 3 & 4 & \cdots & n & 1 \end{vmatrix}, \quad \text{and} \quad \begin{vmatrix} a_1 & a_2 & a_3 & \cdots & a_{n-1} & a_n \\ a_n & a_1 & a_2 & \cdots & a_{n-2} & a_{n-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_2 & a_3 & a_4 & \cdots & a_n & a_1 \end{vmatrix},$$

where, in the latter, $a_1, a_2, \cdots, a_n$ form an arithmetical progression.

I. SOLUTION BY P. J. DA CUNHA, University of Lisbon, Portugal.

Denote the first of these circulants by Δ and the second by Δ^4. Let

$$s_n = \frac{1+n}{2} n$$

be the sum of the first n positive integers. Add to the elements of the last line of Δ the sum of the corresponding elements of all the preceding lines. We obtain a determinant which we can write as the product

$$\Delta = s_n \begin{vmatrix} 1 & 2 & 3 & \cdots & n-2 & n-1 & n \\ n & 1 & 2 & \cdots & n-3 & n-2 & n-1 \\ n-1 & n & 1 & \cdots & n-4 & n-3 & n-2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 4 & 5 & 6 & \cdots & 1 & 2 & 3 \\ 3 & 4 & 5 & \cdots & n & 1 & 2 \\ 1 & 1 & 1 & \cdots & 1 & 1 & 1 \end{vmatrix}$$

If we subtract from the elements in the first $n-1$ columns of this determinant the corresponding elements of the last column, we shall obtain a determinant which we can easily reduce to the order $n-1$. Now subtract from the elements of the first $n-2$ lines of this new determinant the corresponding elements of the last line. We obtain after a little reduction

$$\Delta = -s_n \begin{vmatrix} -n & -n & -n & \cdots & -n \\ 0 & -n & -n & \cdots & -n \\ 0 & 0 & -n & \cdots & -n \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & -n \end{vmatrix} = (-n)^{n-1}\frac{1+n}{2},$$

the determinant being of order $n-2$.

Pass now to the determinant Δ'. Putting

$$a_k = a_1 + (k-1)r \quad (k = 1, 2, 3, \cdots, n),$$

we shall have

$$\Delta' = \begin{vmatrix} a_1 & a_1+r & a_1+2r & \cdots & a_1+(n-2)r & a_1+(n-1)r \\ a_1+(n-1)r & a_1 & a_1+r & \cdots & a_1+(n-3)r & a_1+(n-2)r \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ a_1+r & a_1+2r & a_1+3r & \cdots & a_1+(n-1)r & a_1 \end{vmatrix}.$$

The same reasoning will lead us to

$$\Delta' = (-n)^{n-1}r^{n-1}\left(a_1 + \frac{n-1}{2}r\right),$$

as it is easy to see.

II. Solution by Otto Dunkel, Washington University.

The reduction of these special circulants can be made to depend upon the known fact that the general circulant reduces to the product $f(\omega_1)f(\omega_2)\cdots f(\omega_n)$, where the ω's are the roots of $x^n - 1 = 0$ and $f(x) = a_1 + a_2x + a_3x^2 + \cdots + a_nx^{n-1}$ (see Cesàro, *Elementares Lehrbuch der algebraischen Analysis* etc., page 25). If we set $a_k = a_1 + (k-1)r$, then

$$\begin{aligned} f(\omega) &= a_1[1 + \omega + \omega^2 + \cdots + \omega^{n-1}] + r[\omega + 2\omega^2 + 3\omega^3 + \cdots + (n-1)\omega^{n-1}], \\ &= 0 + \frac{rn}{\omega - 1}, \text{ if } \omega \neq 1, \\ &= a_1n + \frac{r(n-1)n}{2} = n\left[a_1 + \frac{n-1}{2}r\right], \text{ if } \omega = \omega_1 = 1, \end{aligned}$$

whence

$$\Delta' = \frac{r^{n-1}n^n\left[a_1 + \frac{n-1}{2}r\right]}{(\omega_2 - 1)(\omega_3 - 1)\cdots(\omega_n - 1)} = (-1)^{n-1}r^{n-1}n^{n-1}\left[a_1 + \frac{n-1}{2}r\right].$$

Also solved by H. L. Olson and A. Pelletier.

2777 [1919, 268]. Proposed by W. D. Cairns, Oberlin College.

Prove that the two series

$$1 + \frac{\pi^4}{2^4\cdot 4!} + \frac{\pi^8}{2^8\cdot 8!} + \cdots$$

and

$$\frac{\pi^2}{2^2\cdot 2!} + \frac{\pi^6}{2^6\cdot 6!} + \frac{\pi^{10}}{2^{10}\cdot 10!} + \cdots$$

are equal.

Solution by H. S. Uhler, Yale University.

By substituting x, $-x$, ix, and $-ix$ for y in the absolutely convergent series

$$e^y = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \cdots + \frac{y^n}{n!} + \cdots$$

it will be found at once that

$$\tfrac{1}{4}[(e^x + e^{-x}) + (e^{ix} + e^{-ix})] = 1 + \frac{x^4}{4!} + \frac{x^8}{8!} + \cdots + \frac{x^{4n}}{(4n)!} + \cdots, \tag{1}$$

$$\tfrac{1}{4}[(e^x + e^{-x}) - (e^{ix} + e^{-ix})] = \frac{x^2}{2!} + \frac{x^6}{6!} + \frac{x^{10}}{10!} + \cdots + \frac{x^{4n-2}}{(4n-2)!} + \cdots, \tag{2}$$

where $n = 1, 2, 3, \cdots$.

Again, since $e^{i\theta} = \cos\theta + i\sin\theta$ it follows that $e^{i(\pi/2)} + e^{-i(\pi/2)} = i - i = 0$.

Consequently by substituting $\pi/2$ for x in formulas (1) and (2) we see at a glance that each of the given series has the same limit $\frac{1}{4}(e^{\pi/2} + e^{-\pi/2})$, that is, the series are "equal."

Also solved by W. W. BEMAN, P. J. DA CUNHA, A. M. HARDING, H. L. OLSON, A. PELLETIER, S. W. REAVES, ELIJAH SWIFT, E. H. WORTHINGTON, and the Proposer.

2785 [1919, 366]. Proposed by W. H. ECHOLS, University of Virginia.

If on the sides, as bases, of any closed plane polygon, there be constructed similar triangles similarly placed, all outward or all inward, then the centroid of the vertices of these triangles coincides with the centroid of the corners of the polygon.

SOLUTION BY THE PROPOSER.

Let $Z_1, \cdots, Z_n \equiv Z_1$ be the n corners of the polygon, the Z's being complex numbers. The sides of the polygon are respectively

$$\Delta Z_r \equiv Z_{r+1} - Z_r, \qquad (r = 1, \cdots, n-1)$$

and $\Sigma\Delta Z_r = 0$, since the polygon is closed.

The n vertices of the similar triangles constructed similarly on the sides are

$$w_r = Z_r + k\Delta Z_r \cdot e^{i\alpha}, \qquad (r = 1, \cdots, n-1)$$

k being a real constant factor and α a real constant angle.

Hence,

$$\Sigma w_r = \Sigma Z_r + ke^{i\alpha}\Sigma\Delta Z_r$$

and therefore,

$$\frac{1}{n}\Sigma w_r = \frac{1}{n}\Sigma Z_r.$$

Also solved by S. W. REAVES and ELIJAH SWIFT.

NOTES AND NEWS.

EDITED BY E. J. MOULTON, Northwestern University, Evanston, Ill.

At Ohio State University, Messrs. VAN B. TEACH, V. B. CARIS and D. L. HOLL have been assistants in mathematics for the present year.

H. R. BRAHANA, of Princeton University, has been appointed instructor in mathematics at the University of Illinois for 1920–1921.

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At Brown University Associate Professor H. P. MANNING has resigned after twenty nine consecutive years of service as a teacher in the department of mathematics. Dr. R. F. BORDEN, of the University of Illinois, has been appointed instructor, and Messr sC. D. WENTWORTH and R. L. WILDER graduate assistants, for 1920–21.

At Cornell University, Dr. H. C. M. MORSE, of Harvard University, and Mr. P. A. FRALEIGH, of Dartmouth College, have been appointed instructors in mathematics for 1920–21.

Dr. C. C. CAMP, of the University of Illinois, has been appointed assistant professor of mathematics at Iowa State College.

Dr. CHESTER SNOW has resigned as professor of mathematics at the Uniersity of Idaho to accept a position as physicist in the Bureau of Standards, Washington, D. C.

According to *Science*, Professor A. W. BUTTERFIELD, of the department of mathematics at the Worcester Polytechnic Institute, has resigned to become educational director for the Norton Company.

Dr. C. N. REYNOLDS, of Wesleyan University, has been appointed instructor at Dartmouth College for 1920–21.

Dr. O. D. KELLOGG, who has been lecturer in mathematics at Harvard University during the past year, has been promoted to be professor of mathematics.

Miss MARIAN M. TORREY, of Brown University, who has been teaching for two years in the Phebe Anna Thorne Model School of Bryn Mawr College, has been appointed instructor in mathematics at the University of West Virginia.

At the University of Nancy Dr. LÉOPOLD LEAU (*1920*, 89) has been appointed professor of differential and integral calculus to replace Dr. A. S. E. HUSSON.

At the University of Lyons, Dr. L. SIRE, maître de conférence at the University of Rennes has been appointed professor of applied mathematics in place of the late Professor D. J. B. FLAMME.

At the University of Lille, Dr. A. CHATELET has been appointed professor of general mathematics.

Mr. W. E. H. BERWICK has been appointed lecturer in mathematics in the University of Leeds. *Nature* states:

"Mr. Berwick was assistant lecturer in the University of Bristol for two years, and afterwards became lecturer in mathematics in University College, Bangor. For two years he was engaged on the staff of the anti-aircraft experimental section of the Munitions Inventions Department at Portsmouth, where he made important contributions to the experimental and computative theory of gunnery. He has published a long series of papers in the *Proceedings of the London Mathematical Society* and elsewhere."

Professor G. E. FISHER, of the department of mathematics in the University of Pennsylvania, died March 28, 1920 within about three weeks of his fifty-seventh birthday. He was instructor in mathematics at Cornell University 1887–89, assistant professor at the University of Pennsylvania 1889–1908, and professor since 1908. He was a charter member of the ASSOCIATION.

With Professor I. J. SCHWATT he collaborated in issuing the authorized translation of the fourth German edition of H. Durège's *Elements of the Theory of Functions* (Philadelphia, 1896); *Text-book of Algebra*, part 1 (Philadelphia, 1898); *Rudiments of Algebra*, 1899; *School Algebra*, 1899; and other algebras.

Professor G. D. BIRKHOFF, of Harvard University, has been elected a member of the Royal Danish Academy of Sciences.

Professor L. E. DICKSON, of the University of Chicago, has been elected a member of the American Philosophical Society.

Mr. J. H. JEANS, of Dorking, England, formerly professor of applied mathematics at Princeton University, has been nominated as secretary of the Royal Society of London.

Professor D. R. CURTISS, of Northwestern University, gave an address before the Schoolmasters Club at Ann Arbor, April 3, on "Mathematics."

Professor E. W. BROWN delivered a lecture on the history of mathematics before the Gamma Alpha fraternity of Yale University on February 26. The Astronomical Society of the Pacific has awarded to Professor Brown its Bruce Medal for 1920.

The Silliman lectures at Yale University for 1920 were delivered by Professor JACQUES HADAMARD, of the Collège de France, on April 30, May 3 and May 5. The general subject of the lectures was "Some topics in linear partial differential equations."

At the meeting of the National Research Council on April 28, the Division of Physical Sciences voted to increase its membership by adding a representative of the MATHEMATICAL ASSOCIATION OF AMERICA, and a tenth member-at-large, who should represent mathematics. There were previously in the division fourteen physicists, three representatives of the American Mathematical Society (L. E. DICKSON, O. VEBLEN, and H. S. WHITE), three representatives of the American Astronomical Society, one meteorologist and one geodesist.

The fourteenth regular meeting of the American Mathematical Society was held at the University of Chicago, April 9–10. Nineteen papers were presented. At the symposium (*1920*, 144) Professor MAX MASON discoursed on "The electromagnetic field equations" and Professor A. C. LUNN on "The theory of relativity." There was an attendance of over a hundred at the meeting. At the

dinner on the evening of April 9 Professor L. E. DICKSON reported recommendations of the Committee on Bibliography of which Professor R. C. ARCHIBALD is chairman. On the presentation of a report, including these recommendations, to the Council in New York City on April 23, it was resolved that the Council is of the opinion that a journal of mathematical abstracts is very desirable, and the Committee was authorized to take steps toward securing the financial aid necessary to found such a journal and was requested to present to the Council plans for its organization. The plan for the establishment of a journal of mathematical abstracts was also heartily approved on April 28 by the Division of Physical Sciences of the National Research Council.

At the annual meeting of the National Academy of Sciences in Washington April 26–28 the following papers were read by members of the Association: By A. G. WEBSTER, (a) "The Springfield rifle and the heduc formulæ"; (b) "Some new methods in internal ballistics of the Springfield rifle"; (c) "Preliminary measurements on the pressures in the 'onde de choe'"; (d) "On the connection of the specific heats with the equation of state of a gas"; by L. E. DICKSON, "Recent notable progress in the theory of numbers"; by E. KASNER, "Geodesics and relativity."

Professor M. E. C. JORDAN of the Collège de France was elected a foreign associate of the Academy, and Professor H. F. BLICHFELDT, of Stanford University, the thirteenth member in the Section of Mathematics. The other twelve members are: G. D. BIRKHOFF, G. A. BLISS, O. BOLZA, L. E. DICKSON, E. KASNER, E. H. MOORE, W. F. OSGOOD, W. E. STORY, E. B. VAN VLECK, O. VEBLEN, H. S. WHITE, and E. T. WILCZYNSKI.

In the American Association for the Advancement of Science what has been prior to 1920, Section A, Mathematics and Astronomy, has been divided into two Sections: Section A, Mathematics, and Section B, Astronomy. The officers of Section A are as follows—Vice-president, D. R. CURTISS; Secretary, W. H. ROEVER; Members of Sectional Committee: 5 years, D. JACKSON; 4 years, A. D. PITCHER; 3 years, G. A. BLISS; 2 years, J. PAGE; 1 year, H. L. RIETZ; Member of the Council, G. A. MILLER; Member of the General Committee, E. V. HUNTINGTON. In Section B the Vice-president is J. STEBBINS, and the Secretary F. R. MOULTON.

At the meeting of the Association of Teachers of Mathematics in New England at Springfield, Mass., March 13, 1920, the following papers by college teachers were presented: by Professor J. W. YOUNG, Dartmouth College, "The work of the National Committee on Mathematical Requirements and the National Council of Teachers of Mathematics"; by Professor ELEANOR C. DOAK, Mount Holyoke College, "Inscriptible polygons"; by Mr. J. S. MIKESH, Yale University, "Courses for teachers of mathematics in secondary schools." At the spring meeting of the Association, at Boston University, May 1, the following papers

were read by members of the Association: By L. R. PERKINS, "College courses for teachers of mathematics"; by C. L. E. MOORE, "Einstein's theory."

At the meeting of Mathematics Teachers of New Jersey in Elizabeth, N. J., April 17, 1920, addresses were delivered by the following members of the ASSOCIATION: By F. DURELL, "The organization of graphic methods" (presidential address); by O. VEBLEN, "Displacements and symmetries in three dimensions"; by H. E. WEBB, (*a*) "Note on the pure quadratic equation," (*b*) "Note on the proof of Euler's theorem $e^{i\theta} = \cos\theta + i\sin\theta$," and (*c*) "Certain questions arising from the report of the National Committee."

A new organization, to be known as the National Council of Teachers of Mathematics, was launched at Cleveland, Ohio, on February 24. About 150 persons were present, representing 20 different states and many different organizations of primary and secondary school teachers in various parts of the country. The following officers were elected—President, C. M. Austin, Oak Park, Ill.; vice-president, H. O. Rugg, New York, N. Y.; secretary-treasurer, J. A. Foberg, Chicago, Ill.; executive committee: Marie Gugle, Columbus, O. (3 years); J. Rorer, Philadelphia, Pa. (3 years); H. Wheeler, Worcester, Mass. (2 years); W. D. Reeve, Minneapolis, Minn. (1 year). Two members are to be appointed by the committee itself.

Mathematics Teacher, in reorganized form, will probably be the official organ of the Council, administered by an editorial board of from three to five members and an editor in chief. This board is to consist of teachers of elementary and secondary mathematics, and to include a member representing the college group in an advisory capacity.

Sir THOMAS MUIR has recently deeded 2500 books from his mathematical library to the South African Public Library at Cape Town. This collection is mainly made up of serial publications and writings bearing on the theory of determinants, and allied matters. So far as is known the library is in this latter respect the most complete in existence.

Additional announcements (cf. *1920*, 192–194) of mathematics courses in Summer Sessions are as follows:

Leland Stanford University, June 22–September 3. By Professor H. F. BLICHFELDT, Columbus, 4 hrs.; Coördinate geometry, 4 hrs.; Advanced course. By H. W. BRINKMAN, Trigonometry, 5 hrs.; Algebra, 4 hrs.

University of Missouri, June 21–August 14. By Professor L. INGOLD, Second calculus and Seminar. By Mr. E. ALLEN, Trigonometry and Analytic geometry.

Ohio State University, June 21–August 13. By Professor H. W. KUHN, Fundamental concepts of algebra and geometry, 4 hrs.; Modern higher algebra, 3 hrs. By Professor S. E. RASOR, Geometrical representation of functions of real and complex variables, 3 hrs.; Integral calculus, 5 hrs. By Assistant Professor HORTENSE RICKARD, Trigonometry, 5 hrs.; Analytic geometry, 5 hrs.

University of Wisconsin, June 28–August 6. By Professor A. DRESDEN, W. W. HART, E. P. LANE, H. W. MARCH, and E. B. SKINNER, and Messrs R. W. BABCOCK, J. E. DAVIS, and H. L. SMITH courses in Algebra; Analytic geometry; Differential calculus; Commerce algebra; Elementary solid geometry; Elementary mathematical analysis; Integral calculus; Teaching of secondary teachers; Differential equations; Theoretical mechanics; Modern analytic geometry; Differential geometry; Special topics in algebra; Elliptic integrals; Axioms of geometry; Point sets; and Differential equations of mathematical physics.

N.B. *University of Chicago:* The first term opens June *21* and not June *1* as stated *1920,* 192.

In view of the wide-spread and increasing interest in applied mathematics and the probability that a larger proportion of graduate students may in future be disposed to consider including technical, or semi-technical, subjects in their programs, the departments of Mathematics and Physics at the *Massachusetts Institute of Technology* announce the following list of courses available for graduate students. It will be understood that a particular course may be withdrawn in case a very small number of students should apply for it.

The departments will endeavor to carry out a periodic plan under which, while the more fundamental subjects may be given each year, others will be offered once in two or three years, in rotation. The possible range of subject matter may thus be capable of extension beyond the present list.

In addition to the titles given, there will also be opportunity—as heretofore—for the study of the usual courses in Advanced calculus, Theory of functions, Modern geometry, Modern analysis, etc., also experimental physics.

Analytical mechanics (MOORE). A problem course; three hours per week.
Mathematical laboratory (LIPKA). A treatment of alignment charts and other methods of graphical computation; two hours per week during the second and third terms.
Fourier's series (BAILEY). Two hours per week.
Application of mathematics to chemistry (HITCHCOCK). An application of thermodynamics to chemical problems; three times per week.
Mathematical theory of investment (TAYLOR). Three hours per week for one term.
Advanced mechanics (PHILLIPS). Two hours per week; including Analytical dynamics, Statistical mechanics and Theory of radiation.
Relativity and Einstein's theory of gravitation (PHILLIPS and MOORE). Two hours per week.
Heat conduction (PHILLIPS). Two hours per week for one term.
Aeronautics, 1st course (MOORE). Three hours per week.
Aeronautics, 2d course (WILSON). Two hours per week.
Electrochemistry (H. M. GOODWIN). Four hours per week.
Theoretical physics (WILSON). Three hours per week.
Electromagnetism (WILSON). Including (*a*) Electrodynamics, (*b*) Electromagnetic theory, (*c*) Applied electromagnetism, of which only one would be offered in a particular year.
Kinetic theory (W. S. FRANKLIN). Two hours per week for two terms.
Constitution of matter (WILSON). Two hours per week.
Research in mathematical physics (WILSON).

Each course will continue through the year (three terms) except where the contrary is stated.

The 1920 Summer Meeting of the Association

The fifth summer meeting of the Mathematical Association of America will be held at the University of Chicago on Monday, September 6, 1920, followed by the summer meeting of the American Mathematical Society on Tuesday, and the ninth Colloquium of the Society Wednesday to Saturday. This arrangement of the meetings was decided upon by the combined committees of the Association and Society after careful consideration of various possible plans and consultation with some thirty persons who attended the summer meetings and Colloquium at Cambridge in 1916. The principal reasons for thus condensing all the meetings within one week, rather than, for example, putting the Association meetings on the Friday and Saturday preceding, are (a) the saving of expense and the avoidance of too prolonged tension for those attending the meetings of both organizations; (b) the inability to secure the room accommodations at the University earlier than September fifth or later than September eleventh.

The Association coöperates heartily with the Society in magnifying the Colloquium by limiting its own sessions to one day and making it convenient for its members to attend the sessions of the Society and the Colloquium on the following days, which, it is believed, very many will desire to do. The Colloquium speakers will be Professor G. D. Birkhoff of Harvard University and Professor F. R. Moulton of the University of Chicago.

At no little inconvenience the University of Chicago has consented to open two of its dormitories, one for men and one for women, during the week of these meetings, at a nominal price for rooms (compared with hotel rates). The rooms may be occupied from Sunday, September 5, to Saturday, September 11. The most advantageous arrangements possible for meals will also be arranged.

Full details regarding rooms and board will accompany the program, which will be mailed under first-class postage early in August. If there is any doubt as to prompt forwarding of your mail to your summer address, please inform the Secretary and he will send your notice directly to you wherever you may be.

This will be the only preliminary announcement of the summer meeting. Reservation cards for rooms will accompany programs, and these should be returned immediately on receipt, in order to insure accommodations.

The American Mathematical Monthly

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1620

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ELIMINATION OF SKIDDING DUE TO STEERING MECHANISM ON MOTOR CARS

By A. L. CANDY, University of Nebraska.

When a wheel rolls along on the ground, if there is motion in the direction perpendicular to the plane of the wheel, then there is some sliding on the ground as well as a rolling forward in the plane of the wheel. This sliding is what I shall call "skidding." When a wheel rolls along a curved path, if the plane of the wheel is always tangent to its path, there is no skidding, but only a slight rotation about the point of contact with the ground, for the center of curvature of the path will then be on the projection of the axis of the wheel upon the ground.

In the ordinary wagon, or traction engine, the front axle turns on a single pivot at the center, so that the axes of the front wheels are always in the same straight line, viz., that of the front axle. If such a vehicle is set in motion, and the angle θ (Fig. 1) between the axles is kept constant, the four wheels, A, B, C, D, will describe concentric circles with center at O, the intersection of the lines of the axles, produced. Under these conditions there will be no skidding.

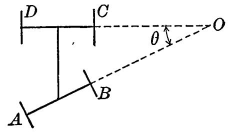

Fig. 1.

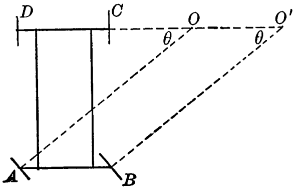

Fig. 2.

But in the automobile each front wheel turns on a separate pivot at the ends of the front axle. Hence the axes of the two front wheels are never in the same line, except when the car is moving straight forward. Let the two front wheels be turned through the same angle, and held rigidly. Then their axes, being parallel, will meet the line of the rear axle, produced, in two distinct points, O, and O' (Fig. 2), and make equal angles, θ, with it.

Now, if the car be made to turn about the point O as a center, the wheel B will skid. Likewise, if the car be made to turn about O' the wheel A will skid. If the car be made to turn about any other point on the line DC, both front wheels will skid. Under these conditions, if the car runs along any curved path whatever, there will be some skidding.

If, however, the inner wheel B is turned through a greater angle than the outer wheel A, so that the points O' and O coincide, then the car will run on a circle with O as center without any skidding whatever. Now in the steering gear of an automobile the cranks are not made parallel to the planes of the front wheels, so that the rod connecting the cranks is shorter than the distance between the two hinge joints. Hence, in the process of steering, the inner wheel is actually turned through a greater angle than the outer wheel.

The object of the following investigation is to determine whether by means of this simple scheme, all skidding can be practically eliminated. If so, what shall be the angle between the crank and the plane of the wheel?

Let $C(a, 0)$ and C' $(-a, 0)$ be the positions of the two pivot joints at the ends of the front axle (Fig. 3).

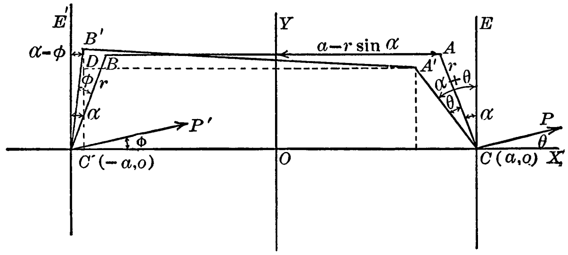

Fig. 3.

Let CE and $C'E'$ be perpendicular to $C'C$, and in the same plane with the cranks CA and $C'B$, of length r. Then CE and $C'E'$ will be horizontal diameters of the front wheels when the car is moving straight forward.

Let $\alpha = \angle ECA = \angle E'C'B$, the constant angles the cranks make with the planes of the wheels.

Let the inner crank CA be turned through an angle θ, to the position CA', then the outer crank $C'B$ will be turned through some angle φ, to the position $C'B'$, making

$$\angle\, ECA' = \alpha + \theta, \qquad \text{and} \qquad \angle\, E'C'B' = \alpha - \varphi.$$

It is required to find the locus of the intersection of the two lines CP and $C'P'$, which are perpendicular to the planes of the wheels, that is, the lines representing the axles of the wheels, produced.

Draw $A'D$ parallel to the x-axis ($C'OC$), meeting the perpendicular from B' on the x-axis, in D. Then

$$A'D^2 + B'D^2 = A'B'^2, \qquad A'D = 2a - r\,[\sin(\alpha + \theta) + \sin(\alpha - \varphi)],$$

$$D\,B' = r\,[\cos(\alpha - \varphi) - \cos(\alpha + \theta)],$$

$$A'B' \equiv AB = 2(a - r\sin\alpha), \text{ a constant;}$$

$$\therefore [2a - r[\sin(\alpha + \theta) + \sin(\alpha - \varphi)]]^2 + r^2[\cos(\alpha - \varphi) - \cos(\alpha + \theta)]^2 = 4(a - r\sin\alpha)^2. \tag{1}$$

This equation reduces to the form

$$2a[\sin(\alpha + \theta) + \sin(\alpha - \varphi)] + r\cos(2\alpha + \theta - \varphi) = r\cos 2\alpha + 4a\sin\alpha, \tag{2}$$

from which the variables θ and φ must be eliminated.

The equations of the two lines CP and $C'P'$ are, respectively,

$$y = (x - a)\tan\theta, \qquad \text{and} \qquad y = (x + a)\tan\varphi,$$

whence

$$\tan\theta = \frac{y}{x - a}, \quad \text{and} \quad \tan\varphi = \frac{y}{x + a}.$$

Let $\lambda_1 \equiv (x-a)^2 + y^2$, and $\lambda_2 \equiv (x+a)^2 + y^2$. Then

$$\sin\theta = \frac{y}{\sqrt{\lambda_1}}, \qquad \cos\theta = \frac{x-a}{\sqrt{\lambda_1}}, \qquad \sin\varphi = \frac{y}{\sqrt{\lambda_2}}, \qquad \cos\varphi = \frac{x+a}{\sqrt{\lambda_2}},$$

$$\sin(\alpha+\theta) = \frac{(x-a)\sin\alpha + y\cos\alpha}{\sqrt{\lambda_1}}, \qquad \sin(\alpha-\varphi) = \frac{(x+a)\sin\alpha - y\cos\alpha}{\sqrt{\lambda_2}},$$

$$\sin(\theta-\varphi) = \frac{2ay}{\sqrt{\lambda_1\lambda_2}}, \qquad \cos(\theta-\varphi) = \frac{x^2+y^2-a^2}{\sqrt{\lambda_1\lambda_2}},$$

and

$$\cos(2\alpha+\theta-\varphi) = \cos 2\alpha\cos(\theta-\phi) - \sin 2\alpha\sin(\theta-\phi)$$

$$= \frac{(x^2+y^2-a^2)\cos 2\alpha - 2ay\sin 2\alpha}{\sqrt{\lambda_1\lambda_2}}.$$

Substituting these values in (2), and letting $s = \sin\alpha$, $c = \cos\alpha$, $s_2 = \sin 2\alpha$, $c_2 = \cos 2\alpha$, and $k = rc_2 + 4as$, we get

$$2a\left[\frac{s(x-a)+cy}{\sqrt{\lambda_1}} + \frac{s(x+a)-cy}{\sqrt{\lambda_2}}\right] + r\left[\frac{c_2(x^2+y^2-a^2)-2as_2y}{\sqrt{\lambda_1\lambda_2}}\right] = k, \tag{3}$$

which is the equation of the locus.

Multiplying this equation by $\sqrt{\lambda_1\lambda_2}$ and rationalizing, we get

$$\begin{aligned}\{&-64a^4(s^2x^2-c^2y^2+as_2y-a^2s^2)^2 - 32a^2kr(s^2x^2-c^2y^2+as_2y\\ &-a^2s^2)[c_2(x^2+y^2-a^2)-2as_2y] - 2k^2r^2[c_2(x^2+y^2-a^2)-2as_2y]^2\\ &-8a^2k^2[\lambda_2(sx+cy-as)^2+\lambda_1(sx-cy+as)^2]\}\lambda_1\lambda_2 + k^4\lambda_1^2\lambda_2^2\\ &+16a^4[\lambda_2(sx+cy-as)^2+\lambda_1(sx-cy+as)^2]^2 + r^4[c_2(x^2+y^2-a^2)\\ &-2as_2y]^4 - 8a^2r^2[\lambda_2(sx+cy-as)^2+\lambda_1(sx-cy\\ &+as)^2][c_2(x^2+y^2-a^2)-2as_2y]^2 = 0.\end{aligned} \tag{4}$$

On expanding this equation, restoring the values of λ_1 and λ_2, which are both functions of x and y, and also restoring the value of $k \equiv rc_2 + 4as$, we obtain an equation of the 8th degree in which the coefficients of x^8, x^6, x^4, x^2, and the constant term all vanish. The equation can then be divided by $32a^2y$. Hence the x-axis may be regarded as a part of the locus.

Then, after an enormous amount of work in collecting and simplifying the twenty-five coefficients involved, we get the following equation (5) as the final equation of the locus:

$$
\begin{array}{llllll}
2s^2 & [4a^2s^2 & +4arsc_2 & +r^2c_2^2 & &]x^6y \\
+2s^2 & [4a^2(4s^2-1) & +12arsc_2 & +3r^2c_2^2 & &]x^4y^3 \\
+2s^2 & [4a^2(5s^2-2) & +12arsc_2 & +3r^2c_2^2 & &]x^2y^5 \\
+2s^2 & [4a^2(2s^2-1) & +4arsc_2 & +r^2c_2^2 & &]y^7 \\
-2s_2 & [4a^3s^2 & +4a^2rs(1-3s^2) & +ar^2c_2(1-6s^2) & -r^3sc_2^2 &]x^6 \\
+2s_2 & [4a^3(1-3s^2) & -4a^2rs(3-7s^2) & -2ar^2c_2(1-7s^2) & +3r^3sc_2^2 &]x^4y^2 \\
+2s_2 & [4a^3(1-s^2) & -4a^2rs(3-5s^2) & -ar^2c_2(1-10s^2) & +3r^3sc_2^2 &]x^2y^4 \\
+2s_2 & [4a^3s^2 & -4a^2rs(1-s^2) & +2ar^2c_2s^2 & +r^3sc_2^2 &]y^6 \\
-2 & [4a^4s^2 & +4a^3rs(1-7s^2+6s^4) & +a^2r^2(1-16s^2+56s^4-44s^6) & +ar^3sc_2(1-2c_2^2) \quad -r^4s^2c^2c_2^2 &]x^4y \\
+2 & [4a^4(1-8s^2+10s^4) & -8a^3rs(1-9s^2+10s^4) & -a^2r^2(1-18s^2+80s^4-72s^6) & -2ar^3sc_2(1-2c_2^2) +2r^4s^2c^2c_2^2 &]x^2y^3 \\
-2 & [12a^4s^2(1-2s^2) & +4a^3rs(1-11s^2+14s^4) & -2a^2r^2s^2(1-12s^2+14s^4) & +ar^3sc_2(1-2c_2^2) \quad -r^4s^2c^2c_2^2 &]y^5 \\
+6a^2s_2 & [4a^3s^2 & +4a^2rs(1-3s^2) & +ar^2c_2(1-6s^2) & -r^3sc_2^2 &]x^4 \\
-as_2 & [8a^4 & -16a^3rs(1-s^2) & -2a^2r^2(3-20s^2+24s^4) & +4ar^3sc_2^2 \quad +r^4c_2(1-2c_2^2) &]x^2y^2 \\
+as_2 & [24a^4s^2 & -8a^3rs(1+s^2) & -4a^2r^2(1-6s^2+6s^4) & +2ar^3sc_2^2 \quad -r^4c_2(1-2c_2^2) &]y^4 \\
+2a^2 & [4a^4s^2(2-3s^2) & +4a^3rs(2-17s^2+18s^4) & +a^2r^2(2-35s^2+124s^4-100s^6) & +2ar^3sc_2(1-2c_2^2) -2r^4s^2c^2c_2^2 &]x^2y \\
-\tfrac{1}{2}a^2 & [48a^4s^2(1-2s^2) & +16a^3rs(2-19s^2+22s^4) & +4a^2r^2(1-19s^2+84s^4-76s^6) & +8ar^3sc_2(1-2c_2^2) - r^4(1-6s_2^2c_2^2) &]y^3 \\
-6a^4s_2 & [4a^3s^2 & +4a^2rs(1-3s^2) & +ar^2c_2(1-6s^2) & -r^3sc_2^2 &]x^2 \\
+a^3s_2 & [24a^4s^2 & +8a^3rs(1-5s^2) & -2a^2r^2(1-2s^2-4s^4) & -2ar^3sc_2^2 \quad +r^4c_2(1-2c_2^2) &]y^2 \\
-2a^4 & [4a^4s^2(1-2s^2) & +4a^3rs(1-9s^2+10s^4) & +a^2r^2(1-18s^2+64s^4-52s^6) & +ar^3sc_2(1-2c_2^2) \quad -r^4s^2c^2c_2^2 &]y \\
+2a^6s_2 & [4a^3s^2 & +4a^2rs(1-3s^2) & +ar^2c_2(1-6s^2) & -r^3sc_2^2 &]=0.
\end{array}
\tag{5}
$$

Since y is a factor of the seventh degree terms in equation (5), the locus has an asymptote parallel to the x-axis. The equation of this asymptote is found by putting the sum of the x^6y and x^6 terms equal to zero.

This gives

$$y = 2\frac{c}{s}(a - rs) \equiv 2 \cot \alpha(a - r \sin \alpha). \tag{6}$$

Throughout the limited range of motion (variation of θ) that is possible in the automobile, the curve lies very close to this asymptote, and on the same side of it as the x-axis, which is in the position of the front axle.

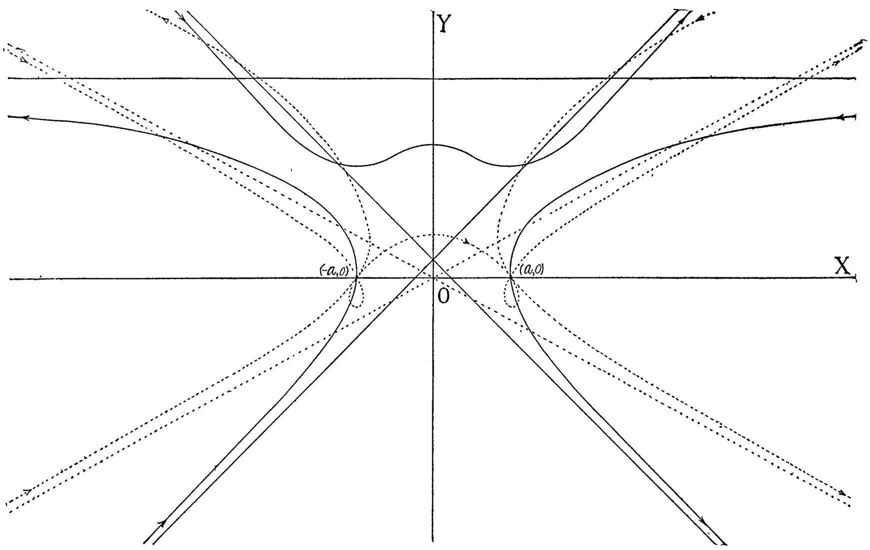

Fig. 4.

For continuous curve, $\alpha = 30°$; for dotted curve, $\alpha = 15°$.

Therefore, if, with given values of a and r, we find a value of α which makes the ordinate of this asymptote a little greater than the wheelbase of the car, there will be very little skidding due to the steering mechanism when the car runs on a curved path.

If $a = 2.1$ ft., $r = \frac{1}{2}$ ft., and $\alpha = 21°$, we get $y = 10.05$ ft.
These would be good values for a car with a wheel-base of 9.5 ft.

It is evident from equation (6) that y increases as α diminishes. Therefore, since a and r may be considered as practically the same in all cars, the greater the wheel-base the smaller the angle α must be.

So far it has been assumed that the cranks are behind the front axle. If the cranks project forward, as in some cars, they must then diverge from the planes of the wheels. In order to get the corresponding equations for this case, it will be sufficient to change the sign of r.

With $a = 2$ and $r = 1$ we get for this asymptote [Eq. (6)] the following equations:

When $\alpha = 30°$, $y = 3\sqrt{3}$. [See Fig. 4].
When $\alpha = 15°$, $y = 13-$. Not shown in Fig. 4.
When $\alpha = 45°$, $y = 2.6$. See dot and dash line in Fig. 5.
When $\alpha = 60°$, $y = 1.3$. See dotted line in Fig. 5.
When $\alpha = 90°$, $y = 0$.
When $\alpha \doteq 0$. $y \doteq \infty$.

If we let

$$l^2 = 4a^2s^2 + 4arsc_2 + r^2c_2^2 = (2as + rc_2)^2,$$

and

$$m^2 = c_2(4a^2 - 4ars - r^2c_2),$$

the 7th degree terms may then be written in the form

$$2s^2y(l^2x^2 - m^2y^2)(x^2 + y^2)^2.$$

Hence there is also a pair of asymptotes parallel to the two lines

$$l^2x^2 - m^2y^2 = 0.$$

After another long process of reduction we find the equations of these asymptotes to be

$$y = \pm\frac{l}{m}x + \frac{as(4a^2s^2 - 4ars^3 - r^2c_2)}{cm^2} \tag{7}$$

Let $a = 2$, and $r = 1$.[1] Then for this pair of asymptotes we get the following equations:

When $\alpha = 30°$, $y = \pm\frac{5}{\sqrt{23}}x + \frac{20\sqrt{3}}{69} = \pm x \tan 46°11.'5 + .5$. [See Fig. 4]

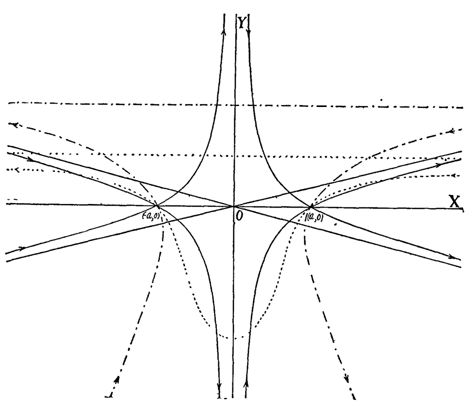

Fig. 5.
For continuous curve $\alpha = 0°$; for dot-dash curve, $\alpha = 45°$; for dotted curve, $\alpha = 60°$.

When $\alpha = 15°$, $y = \pm x \tan 29°29' + .03$. [See Fig. 4]

When $\alpha = 0$, $x^2 - 15y^2 = 0$. [See Fig. 5]

When $\alpha = 44°59'$,

$$y = \pm x \tan 88°53'.5 + 2948.$$

x-intercepts $= \pm 55.75$.

When $\alpha > 45°$, these asymptotes are imaginary, since m^2 contains the factor c_2, which is then negative.

As $\alpha \doteq 45°$, these asymptotes move off into the infinite region. Hence that part of the curve below the x-axis [Fig. 5] approaches in the limit the form of a parabola with infinite branches extending downward.

[1] All the curves shown in Figs. 4 and 5, were constructed with $a = 2$ in. and $r = 1$ in. Hence the unit of the scale in these figures is half the distance from O to $(a, 0)$.

Thus these points are related to this branch of the locus in the same way as the origin is related to the conchoid $(x^2 + y^2)(x - a)^2 = b^2x^2$.

The coördinates (0, 0) always satisfy this equation, and when $y = 0$, we get $x = 0, 0, a \pm b$, but the curve does not cut the x-axis four times unless b $> a$. That is, the origin is a double point, or a conjugate point according as $b > a$, or $b < a$. The same thing is true in regard to the limaçon

$$(x^2 + y^2 - ax)^2 = b^2(x^2 + y^2).$$

NOTE ON THE INCENTERS OF A QUADRILATERAL.

By F. V. Morley, Johns Hopkins University.

The rectangular configuration of the incenters of an inscribed quadrilateral was known to Professor Neuberg of Liège before 1906.[1] It was rediscovered, as an isolated fact, by the writer in 1914, and several proofs were supplied by his father, Professor Morley. When the Monthly enlarged its scope in 1916 the proposition was sent in as a problem, appearing in March, 1917; the obvious extension and the writer's solution were published soon after.[2] In June, 1918, Professor Altshiller-Court published an article in this Monthly, rediscovering the configuration.[3]

The proposition as an isolated fact with reference to the inscribed quadrilateral was therefore worth discovery; but it will now be shown that the proposition appears in a well-known theory.

I.

There are several beautiful chains of theorems concerned with the elementary geometry of n-lines in a plane. One of these is Clifford's chain, and others of similar type are well known to students of metric or reflexive geometry. With slight modifications, they apply to directed as well as to undirected lines.[4]

One of these chains is concerned with the incenters of n-lines. In the present note it will be simpler to consider the lines as directed. For example, a triangle composed of three undirected lines has four incenters, using the term in its general meaning; but if the three lines are thought of as directed, there is only one circle tangent to all three in the proper sense, and hence only one incenter.

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[2] November, 1917, volume 24, pp. 429–430.

[3] Volume 25, 1918, pp. 241–246; for comment, see volume 26, 1919, pp. 65–66. Still more recently, the subject has been discussed in the comprehensive article by J. W. Clawson, in the *Annals of Math.*, vol. 20, p. 254 (1919). [Mr. Morley's paper was in the hands of the Editor some time before Professor Clawson's paper was published.—Editor.]

[4] Cf. F. Morley, *Transactions of the American Mathematical Society*, Vol. 1, pp. 97–115; and F. H. Loud, *Transactions of the American Mathematical Society*, Vol. 1, pp. 323–338.

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Four directed lines may be taken three at a time in four ways; each triangle will have its incenter, and according to the theorem of Steiner which is the basis of the chain, these four incenters lie on a circle.[1] This circle has been called the center-circle[2] of the quadrilateral; its center is a unique point of the quadrilateral, and may be called by analogy its incenter. The notation commonly used is C_4 for the center-circle of four directed lines, and c_4 for the incenter.

The statement of the chain proceeds as follows.[3]

1. *Five directed lines have five points* c_4, *the incenters of the lines taken four at a time; these five points are on a circle,* C_5, *whose center,* c_5, *is the incenter of the 5-line. Six directed lines have six points* c_5, *on a circle* C_6; *and in general, n directed lines have n points of the type* c_{n-1}, *which are on a circle* C_n, *the center of which is the incenter,* c_n, *of the n-line.*

2. *Five directed lines have five circles* C_4; *these circles meet in a point,* N_5, *called the node of the 5-line; and in general, n directed lines have n circles* C_{n-1} *on a node* N_n.

These general theorems apply to the circles C_{n-1} and the centers c_{n-1} of n lines. The configuration has associated with it a variety of other circles, C_{n-2}, C_{n-3}, . . . , with corresponding centers, on which there are definite conditions; but the number and complexity of these in general makes their investigation of doubtful value. Special cases occur, however, in which the subcircles and centers simplify, and one of these has recently aroused some interest.

II.

In dealing with four directed lines, the reversal of any one direction will produce an entirely different arrangement of the circles and centers. The same directed quadrilateral will have all the features of five, six, seven, or eight lines, according to the number of its sides which are counted both ways and as the general configuration will be considerably simplified, new properties of the quadrilateral may be emphasized.

Eight directed lines will have 56 points of the type c_3, four at a time on 70 circles C_4. There will be 70 centers c_4, five at a time on 56 circles C_5. The 56 centers c_5 are six at a time on 28 circles C_6; the 28 centers c_6 are seven at a time on 8 circles, C_7 whose centers c_7 form the center-circle of the 8-line. So much the general theorem prescribes; and when the quadrilateral, by reversing its sides, is considered as eight lines, these facts will hold. The centers will double up, however, and there will appear only 28 c_3, 38 c_4, 28 c_5, 16 c_6, 4 c_7; moreover, not all of

[1] Stated without proof in *Annales de mathématiques* (Gergonne), Tome 18, 1828, p. 302. A proof was furnished by J. Mention in *Nouvelles annales de mathématiques*, vol. 21, 1862, p. 16f. The theorem did not, however, originate with Steiner but with L. Puissant; see *Correspondance sur l'Ecole Polytechnique*, Tome 1, 1806, p. 193.—EDITOR.

[2] The term "*center-circle*" was used by F. Morley, *l.c.*, p. 99, for the circle on the circumcenters of four lines taken three at a time. But later, *Trans. Am. Math. Soc.*, vol. 8 (1907), p. 20, he alters the name for the circle on the circumcenters to "*centric-circle.*" Now four directed lines have two distinct circles, one concerned with circumcenters (the centric-circle) and one on the four incenters. This last is the one with which we are concerned, and it is now called the "*center-circle.*" Cf. Hodgson, *Trans. Am. Math. Soc.*, (1912), p. 203.

[3] Loud (*l.c.*), p. 325.

these are distinct, for the centers c_7 coalesce and unite with the incenter of the configuration. The points c_6 will then lie on a circle, and the points c_5 will be affected somewhat.

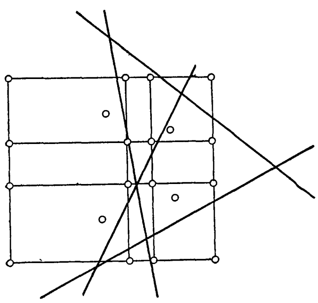

Fig. 1.

It is worth investigation to see how these comparatively few points c_5 behave in the case where the quadrilateral is considered as a special case of eight directed lines; and it is interesting to note in Figure 1 their resemblance to the configuration discussed above; namely, the rectangular net formed by the incenters of an inscribed quadrilateral.

The figure of the circles, C_5, is also of interest in this case, (Figure 2). As their centers are constrained to the rectangular net it is to be expected that they are in families of four; in addition they all pass through a point.

III.

The configuration suggested by the figures is readily proved. It is convenient to consider first a quadrilateral with two sides counted both ways; by repeating this figure the whole configuration may be obtained, but all that is essential is contained in Figure 3.

Here there are six directed lines. By the general theorem there is a center-circle for all six, expressed in the manner of Loud (*l.c.*, p. 325), as

$$x = a_0 - a_1 t$$

where the constants are complex and t an orthogonal number. The six circles C_5 are included in the double infinity of circles

$$x = a_0 - a_1(t + t') + a_2 tt'$$

which are known to pass through a point. Figure 2 is thus confirmed; all the circles C_5 pass through a point, which turns out to be the Clifford point of the quadrilateral.

On examination the six points c_5 in Figure 3 are seen to fall into two sets; four on a rectangle, and two separate. Consider now a triangle with two sides counted both ways; by drawing the centers c_4 it is found that the incenter c_5 coincides with the circumcenter of the triangle. Here there are two such triangles, formed by omitting the single lines 3 and 4 in turn. It follows that the two separate points noticed above are the circumcenters of those triangles, and in passing to the case of Figure 1, the four points c_5 which do not lie on the rectangular net are found to be the circumcenters of the four triangles of the original quadrilateral. They each count for 3 of the c_5, making up with the other 16 the total of 28. It is well known that the circumcenters of a quadrilateral lie on a circle;[1] in this particular case the node or Clifford point is also on this circle.

[1] Steiner, *Gesammelte Werke*, Band 1, p. 323.

QUESTIONS AND DISCUSSIONS.

EDITED BY W. A. HURWITZ, Cornell University, Ithaca, N. Y.

NEW QUESTIONS.

The following interesting questions were formulated by Professor S. Kakeya at the request of the editors, and transmitted through Professor Birkhoff. Professor Kakeya has treated some problems of similar nature in various papers in the *Science Reports* of the Tôhoku Imperial University. They involve considerations of decided difficulty in the analytic formulation of geometric relations.

39. There are certain problems in geometry which are simple in statement but can be reduced only to very complicated problems in transcendental analysis. Following are several examples of the type of problem in question.

1. What is the smallest plane area within which a given figure can be turned through a complete revolution? It is not implied that the figure should revolve about a fixed point, but merely that in the course of its motion the figure should have every possible orientation in the plane. The problem may be modified by considering only convex areas.

An interesting special case is that in which the given figure is a segment of a straight line. In this case it has been conjectured by Professors Osgood and Kubota that the smallest area may be bounded by a three-cusped hypocycloid; if we consider only convex areas, perhaps the result will be an equilateral triangle. I have no indication of a proof.

2. For every closed convex curve of area P there is an n-sided circumscribed polygon of least area Q and an n-sided inscribed polygon of greatest area R. For a fixed value of the integer n and for all closed convex curves, what is the upper limit of Q/P and what is the lower limit of R/P? I have succeeded only in proving that for the case $n = 3$, the upper limit of Q/P is 2.

3. Let the area of a given simple closed curve A be a. Remove from A the greatest possible area a_1 similar to another given simple closed curve B. From the remaining figure remove the greatest possible area a_2 similar to B. Continue this process indefinitely. Is it or is it not true that

$$a_1 + a_2 + a_3 + \dots = a?$$

I have proved the statement to be true in the special case that A is convex and B is a circle.

4. Let a given closed convex curve K have the property that a given triangle whose angles are incommensurable with π can be revolved completely within K (see part 1 of this question), always remaining inscribed in K. What may the curve K be? Can any other curve except a circle satisfy the conditions?

DISCUSSIONS.

For a special type of quintic equation whose coefficients involve two parameters, Mr. C. B. Haldeman, in the first discussion following, obtains an algebraic solution by means of elementary operations and the extraction of roots; he also derives expressions for the roots in trigonometric form, and gives a graphical construction by the use of a certain cubic curve and a circle.

Professor Trevor, who has previously given instances of the use of mathematics in thermodynamics (*1919*, 444–447; *1920*, 55–57), presents an interesting set of results in connection with the study of a state of thermodynamic equilibrium in a homogeneous fluid. Several transformations of independent variables in a fundamental relation yield corresponding interpretations in the physical problem.

Professor Ransom gives a simple formula which may be used as a check in connection with the ambiguous case in the solution of plane triangles. Of course it applies only to problems which yield two solutions. Apparently there is a similar formula for the analogous case of spherical triangles:

$$\tan \tfrac{1}{2}(c' + c'') = \tan b \cos A,$$

and another for the dual case of ambiguity.

Professor Dick discusses a relationship between the dimensions of the "King's Chamber" in the Great Pyramid, the regular pentagon, and the regular icosahedron and dodecahedron. The implication of a connection between this relationship, the occurrence of the numbers of Fibonacci in plant life, and certain mystic statements of the ancient mathematicians will perhaps seem to most readers to be somewhat far-fetched. It should also be stated that the relation between a regular pentagon and the inscription of a square in a semicircle is clearly implied by the usual construction.

I. Resolution of a Certain Quintic Equation and a Geometrical Construction for its Roots.

By C. B. Haldeman, Ross, Butler County, Ohio.

1. The transformation

$$y = x - \frac{a}{x}$$

will reduce the equation $y^5 + 5ay^3 + 5a^2y + 2b = 0$ to $x^{10} + 2bx^5 - a^5 = 0$; from which we find x, and consequently

$$y = \sqrt[5]{-b + \sqrt{b^2 + a^5}} + \sqrt[5]{-b - \sqrt{b^2 + a^5}}.$$

If $b^2 + a^5$ be negative it appears the roots of the given equation are trigonometrical functions of the coefficients; for the transformation

$$y = -2S\sqrt{-a}$$

will reduce the given equation to

$$16S^5 - 20S^3 + 5S = \frac{b}{a^2\sqrt{-a}};$$

and since

$$16 \sin^5 A - 20 \sin^3 A + 5 \sin A = \sin 5A,$$

we may take

$$S = \sin A, \qquad \frac{b}{a^2\sqrt{-a}} = \sin 5A$$

and get

$$y = -2\sqrt{-a} \sin \tfrac{1}{5} \sin^{-1} \frac{b}{a^2\sqrt{-a}},$$

$$y = -2\sqrt{-a}\sin\frac{1}{5}\left(2\pi + \sin^{-1}\frac{b}{a^2\sqrt{-a}}\right),$$

$$y = -2\sqrt{-a}\sin\frac{1}{5}\left(\pi - \sin^{-1}\frac{b}{a^2\sqrt{-a}}\right),$$

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2. The equation

$$xy\sqrt{-(b^2 + a^5)} = a^2y^3 + by^2 + 3a^3y + 2ab \tag{1}$$

represents a real trident of Newton[1] when $b^2 + a^5$ is negative, and

$$x^2 + y^2 = -4a \tag{2}$$

is the equation of a circle, which will be real when a is negative. Eliminating x from (1) by means of (2), the result may be placed under the form

$$(y^5 + 5ay^3 + 5a^2y + 2b)(a^2y + 2b) = 0.$$

The first of these factors is identical with the quintic expression given above and has five real roots when $b^2 + a^5$ is negative. From this it appears the five real roots may be represented by the ordinates of the intersections of a trident and a circle. The five intersections, whose ordinates are the five real roots of this equation, are the vertices of a regular pentagon inscribed in a circle whose radius is $2\sqrt{-a}$, as may be seen by reference to the above values of y.

II. Certain Mathematical Features of Thermodynamics.

By J. E. Trevor, Cornell University.

Let e, v, s denote the energy, volume, and entropy of unit mass of a body of homogeneous fluid in a state of thermodynamic equilibrium under the pressure p at the temperature θ. The energy e is then a continuous function of v, s, and equilibrium subsists when and only when

(1) $$de = -pdv + \theta ds.$$

Hence the conditions of equilibrium are the equations

(2) $$-p = \partial e/\partial v, \qquad \theta = \partial e/\partial s.$$

The state of equilibrium is stable, with respect to small displacements, when

[1] One of the four canonical forms (no. 108) to which Newton has reduced the general equation of cubic curves in his *Enumeratio linearum tertii ordinis*, first printed in Newton's *Optics*, London, 1704.—Editor.

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and only when all possible sets of variations δe, δv, δs of the energy, volume, and entropy of the body satisfy the inequality

$$\delta e + p\delta v - \theta\delta s > 0.$$

Hence the necessary and sufficient conditions of stability are

$$\frac{\partial^2 e}{\partial v^2}\frac{\partial^2 e}{\partial s^2} - \left(\frac{\partial^2 e}{\partial v \partial s}\right)^2 > 0, \qquad \frac{\partial^2 e}{\partial v^2} > 0.$$

A consequence is that $\partial^2 e/\partial s^2 > 0$.

For various purposes it is desirable to employ v, θ or p, s or p, θ as independent variables, the remaining three of the variables e, p, v, θ, s then being functions of those chosen. The determinants of these transformations are

$$\frac{\partial(v, \theta)}{\partial(v, s)} = \frac{\partial^2 e}{\partial s^2}, \qquad \frac{\partial(p, s)}{\partial(v, s)} = -\frac{\partial^2 e}{\partial v^2}, \qquad \frac{\partial(p, \theta)}{\partial(v, s)} = -\frac{\partial^2 e}{\partial v^2}\frac{\partial^2 e}{\partial s^2} + \left(\frac{\partial^2 e}{\partial v \partial s}\right)^2,$$

and by the conditions of stability none of these determinants vanish. When functions of the new variables are defined by the equations

$$f(v, \theta) = e - \theta s, \qquad g(p, s) = e + pv, \qquad h(p, \theta) = e + pv - \theta s,$$

differentiation and comparison with (1) yields the criterion of equilibrium in the forms

$$(3) \qquad df = -pdv - sd\theta, \qquad dg = vdp + \theta ds, \qquad dh = vdp - sd\theta.$$

The conditions of integrability of the second members of (1) and (3) are Maxwell's "thermodynamic relations,"

$$(4) \qquad -\frac{\partial p}{\partial s} = \frac{\partial \theta}{\partial v}, \qquad \frac{\partial p}{\partial \theta} = \frac{\partial s}{\partial v}, \qquad \frac{\partial v}{\partial s} = \frac{\partial \theta}{\partial p}, \qquad \frac{\partial v}{\partial \theta} = -\frac{\partial s}{\partial p}.$$

It may be asked whether p, v and θ, s can serve as sets of independent variables. We have that

$$-\frac{\partial(v, p)}{\partial(v, s)} = \frac{\partial(\theta, s)}{\partial(v, s)} = \frac{\partial^2 e}{\partial v \partial s};$$

and, as discussed below, it may be taken to be a fact of observation that these determinants vanish, if at all, only at points on a line $\partial^2 e/\partial v^2 = 0$. The circumstance that any one of the variables p, v, θ, s is a function of any two of the others gives rise to the consideration of six pairs of variables that can be taken successively independent, of twelve functions of pairs of variables, and of a set of twenty-four first derivatives of these functions. The derivatives of the set may conveniently be assigned to two classes, Class I containing the derivatives of the "work variables" p, v, and of the "heat variables" θ, s, with regard to each other, and Class II containing the derivatives of a work variable with regard to a heat variable, and the reverse. Making use of the thermodynamic relations (4)

connecting the cross derivatives, the proposed classification is exhibited in the following table:

Class I	Class II	
$-\left(\dfrac{\partial p}{\partial v}\right)_\theta = \dfrac{\Delta}{e_{22}}$	$-\left(\dfrac{\partial p}{\partial s}\right)_v = +\left(\dfrac{\partial \theta}{\partial v}\right)_s = \dfrac{e_{12}}{1} =$	e_{12}
$-\left(\dfrac{\partial p}{\partial v}\right)_s = \dfrac{e_{11}}{1}$	$-\left(\dfrac{\partial p}{\partial \theta}\right)_v = -\left(\dfrac{\partial s}{\partial v}\right)_\theta = \dfrac{e_{12}}{e_{22}} =$	f_{12}
$+\left(\dfrac{\partial \theta}{\partial s}\right)_p = \dfrac{\Delta}{e_{11}}$	$-\left(\dfrac{\partial v}{\partial s}\right)_p = -\left(\dfrac{\partial \theta}{\partial p}\right)_s = \dfrac{e_{12}}{e_{11}} =$	$-g_{12}$
$+\left(\dfrac{\partial \theta}{\partial s}\right)_v = \dfrac{e_{22}}{1}$	$-\left(\dfrac{\partial v}{\partial \theta}\right)_p = +\left(\dfrac{\partial s}{\partial p}\right)_\theta = \dfrac{e_{12}}{\Delta} =$	$-h_{12}$

Using the notation $e_{11} = \partial^2 e/\partial v^2$, $e_{12} = \partial^2 e/\partial v \partial s$, $e_{22} = \partial^2 e/\partial s^2$, $\Delta = e_{11}e_{22} - e_{12}{}^2$, the tabulated values of the derivatives in terms of second derivatives of $e(v, s)$ are found from the equations

$$- dp = e_{11}dv + e_{12}ds, \quad d\theta = e_{12}dv + e_{22}ds,$$

obtained by differentiating the conditions of equilibrium (2). The derivatives of Class II have the values e_{12}, f_{12}, $-g_{12}$, $-h_{12}$; *i.e.*, they form the terms of the conditions of integrability of the second members of the equations (1) and (3). The tabulated derivatives and their reciprocals constitute the set of twenty-four derivatives. By the "reciprocal" of $(\partial p/\partial v)_\theta = -\Delta/e_{22}$, for example, is meant the derivative $(\partial v/\partial p)_\theta = -e_{22}/\Delta$.

The signs of the twenty-four rates of change of the quantities p, v, θ, s with regard to one another afford extensive information concerning the thermodynamic properties of fluids. By the conditions of stability we have that, for all realizable states of equilibrium, e_{11}, e_{22}, Δ are positive. It appears then that, for realizable states of equilibrium, the signs of the derivatives of Class I are determined by the conditions of stability, while the signs of the derivatives of Class II are determined by the sign of e_{12}, which must be found by observation. From $(\partial v/\partial \theta)_p = -e_{12}/\Delta$ we observe that e_{12} is negative when the specific volume rises on heating at constant pressure, and that e_{12} changes sign if the representative point crosses a line of maximum density at constant pressure. It is clear that the fluid states ordinarily observed lie in a field A (in which $e_{12}<0$), but that a field B (in which $e_{12}>0$) appears if, as in the case with liquid water, a locus of maximum density occurs. The derivatives whose values (having regard to their signs) are $e_{12}, f_{12}, -g_{12}, -h_{12}$ are negative in the field A, positive in the field B, and zero on the "zero line" $e_{12} = 0$. A moment's reflection serves to refer a given derivative, with the proper sign, to e_{12} or f_{12}, etc. When "heating" is understood to mean increase of entropy or rise of temperature, and "expansion" means increase of volume or fall of pressure, the circumstance that e_{12} is positive

When the path is any other curve whose projection on the v, θ-plane has a horizontal tangent at the zero line, its projection on the s, θ-plane is a curve of two distinct branches that meet the zero line in a cusp. The common slope $ds/d\theta$ of these branches at the point of meeting determines the value, at this point, of the specific heat c with regard to the path.

When the path is any curve whose projection on the v, θ-plane has not a horizontal tangent at the point of meeting the zero line, its projection on the s, θ-plane is a curve tangent to the zero line at the point of meeting. Hence the slope $ds/d\theta$ of the zero line at this point determines the value there of the specific heat c with regard to the path. The isobar and the isometric (the curve of constant volume) through the point are such paths.

III. A Check Formula for the Ambiguous Case in Plane Triangles.

By W. R. Ransom, Tufts College.

In the solution of a triangle for which sides a and b and angle A are given, two values B' and B'' are first obtained for the angle opposite b; then two angles C' and C'' are found, and finally two sides c' and c''. The obvious relation $\frac{1}{2}(c' + c'') = b \cos A$ may be used to discover the presence of an error in either of the two triangles that have been computed. This formula does not appear in any text book with which I am acquainted: has it not been employed by some one?

IV. The "King's Chamber" and the Geometry of the Sphere.

By F. J. Dick, Râja-Yoga College.

That the designers of the Great Pyramid possessed a thorough knowledge of the geometry of the sphere has been recognized by some, although the usual view[1] is confined to the recognition of their knowledge of the value of π. The length of the "King's Chamber" is exactly double the breadth, while its height is exactly half the diagonal of the floor.[2] Thus if the width be called 2, and the length 4, the "cubic diagonal" of the chamber is 5.

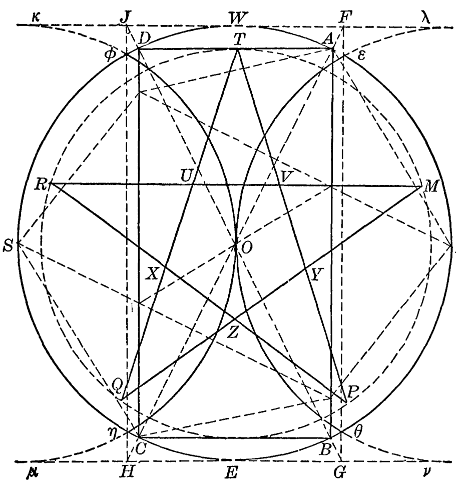
Attention is drawn to the significance of this in connexion with the geometry of the sphere. Let the rectangle $DABC$, 4 units by 2, represent the floor plan (a shape, by the way, found in many ancient temples). Let the circumscribed circle represent the diametral section of a sphere and let two other spheres touch at the center as shown forming the double *vesicá piscis* $\phi O\eta S$ and $\epsilon N\theta O$, determining the planes JH and FG cutting the cylindrical envelope $\kappa\lambda\mu\nu$. The diagonals JG and FH coincide with the diagonals of $DABC$, and are each 5 in length, *i.e.*, they are of the length of the cubic diagonal of the "King's Chamber." These are the traces of cones cutting the sphere $WNES$ in the small circles whose diameters are AB, CD. Join AN, and draw the circle $TMPQR$. Then AN measures a side of the pentagon $TMPQR$ whose diagonals only are shown. Pro-

[1] W. M. F. Petrie, *The Pyramids and Temples of Gizeh*, London, 1883.

[2] *Op. cit.*, p. 195.

jecting this and its reflex on AB, CD we have of course the projection of the icosahedron on the plane containing two opposite edges AN, SC, while $UVYZX$ is one face of the internal dodecahedron formed by joining the 12 angular points of the icosahedron.

This diagram appears to show a relation between the *vesica piscis* and the "length-double-the-width" features of some archaic architecture. Incidentally it shows that the inscription of a square in a semicircle places the icosahedron in the sphere at one stroke, so to say.



Taking AN or TM, an edge of the icosahedron, as unity, the edge of the internal dodecahedron is represented by the limit of the sequence

$$\tfrac{1}{2},\ \tfrac{1}{3},\ \tfrac{2}{5},\ \tfrac{3}{8},\ \tfrac{5}{13},\ \tfrac{8}{21},\ \tfrac{13}{34},\ \cdots,$$

the elements of which are formed in obvious fashion from the elements of the series of Fibonacci.[1] This series is found in nature. In certain species of plants, the denominators give the number of shoots or twigs corresponding to a number of spiral circuits given by the numerator.

It seems just possible that the geometers of ancient Egypt who, like the later Pythagorean and Platonic schools, derived their knowledge from ancient Âryâvarta, knew well what they meant when suggesting that the world-universe was built on number and the geometry of the dodecahedron. And it may be that we possess the merest fragments of what was actually taught in the temples of old. But we do have some of their mighty works in stone. Have they been read and fully understood?

RECENT PUBLICATIONS.

REVIEWS.

Euclid in Greek. Book I, with Introduction and Notes. By Sir Thomas L. Heath. Cambridge, 1920. Pp. x + 240. Price 10s.

Nearly ten years ago Sir George Greenhill, sitting at his baize-covered work table in Staple Inn, Holborn, in an old-world library well known to many scholars from many lands, made the remark to a visitor from over seas that he felt that the only way to teach plane geometry was by a study of Euclid in the original Greek. The remark led to an interesting discussion upon the present state of

[1] See the article by R. C. Archibald in this Monthly for May, 1918, vol. 25, p. 235.

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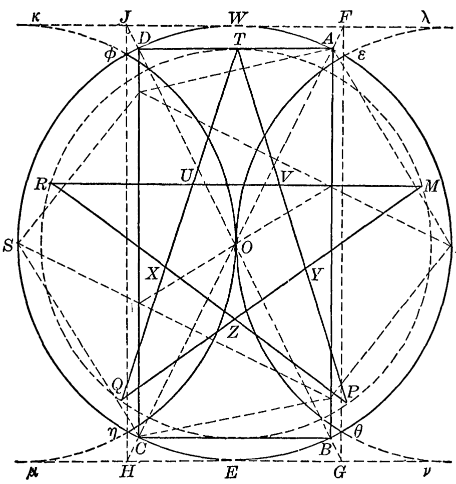

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education, with the result that the distinguished scholar took the ground that his opinion was by no means chimerical; while his visitor, although lamenting the present debasement of scholarly ideals on the part of certain educational leaders, maintained that such a plan was hopeless of accomplishment in the present generation.

And now, of all times, when the world seems bent solely upon selfish class activity, when every group seems determined to profit by the misfortunes of every other group, and when idealism seems to have died the death of the martyr, there comes like a voice from another sphere one of the most interesting little volumes that has appeared from the educational press in many a year,—scholarly in composition, delightful in style, and dignified and artistic in its typographical features. When one thinks of the state of mind in which Europe finds herself at present, the book is a surprise; when he looks at the other books which have recently come from her various presses, with their poor paper and poor press work, the feeling of mere surprise becomes one of pleased astonishment.

Anticipating the protest of *οἱ πολλοί* in the group of modern educators Sir Thomas faces the issue very frankly in his preface:

"In these days when Greek is supposed to be on its trial and Euclid happily defunct, it may well seem a wildly reactionary proceeding to suggest to teachers a combination of the two, a piling (so it might be thought) of one inutility on another. But, first, we must bear in mind that it is only compulsory Greek that is threatened: when that is gone, the study of Greek will be no whit less necessary to a complete education. Generation after generation of men and women will still have to go to school to the Greeks for the things in which they are our masters; and for this purpose they must continue to learn Greek. Again, Euclid can never at any time be more than apparently in abeyance; he is immortal. Elementary geometry will also continue to form part of a complete education; and elementary geometry *is* Euclid, however much the editors of text-books may try to obscure the fact."

Such words may properly be looked upon as food for thought, not merely in England and in Europe generally, but in America as well. Surely the reign of the educational destructionists must be nearly over in this country,—that is, of the young and vigorous group of those who destroy without rebuilding, setting individual tastes above all world experience, and unceasingly talking about social uplift while encouraging in every possible way all intellectual debasement. If this reign is in reality drawing to a close, as indications seem happily to show, and if we are about to rebuild our educational structure, then the question may very properly arise as to whether elective Greek may not find a welcome in our newer type of progressive senior high schools.

If such should be the case, there might well replace some of the literary classics studied in the past that scientific classic, read more widely than any of the other great pieces of literature that Greece produced, published in more editions than all the other scientific works of the ancient world combined, namely, the Elements of Euclid, of which Sir Thomas Heath has already given us the best complete

sequence for his propositions or some new foundation on which to build,—to such a teacher these notes will seem like the words of one having authority and not as those of the educational scribes and Pharisees.

Not least among the valuable features of the work is the index of Greek terms and the index of proper names, aids which readers so often miss in books of this general nature.

In America the book will serve an immediate purpose, in that it is one of the few books on geometry that no teacher can afford to be without, that is indispensable in the library of any well-equipped high school, and that the general reader with scholarly taste will welcome as a pleasant relief from most of our current educational literature. But it is also to be hoped that it will serve still another purpose, the one already referred to as supplying a new classic for those elective courses which may very likely come with the development of a better and more modern type of senior high school in this country.

David Eugene Smith.

Theory of Maxima and Minima. By Harris Hancock. Boston, Ginn and Company, 1917. xiv + 194 pages. Octavo. Price $2.50.

The little treatise deals with maxima and minima, the first half being concerned principally with the case of functions of two variables. The first chapter disposes effectually of the one dimensional problem, and some sections treat in particular examples of three independent variables. The second half, chapters V to VIII, treat the general case. Ordinary and extraordinary maxima and minima, the latter occurring only at irregular points of the function are considered, and considerable space is given relative extremes in addition to the usual discussion of absolute extremes. The discussion treats the problem from many points of view and is particularly rich in elements common to other mathematical disciplines.

The principal topics emphasized are homogeneous forms, definite, semidefinite, and indefinite, and their relation to the determination of maxima and minima; the fallacy of the Lagrangian criterion by which the point in question is approached on straight lines only; various attempts to improve the theory particularly those of Stolz, Scheefer, and v. Dantscher; homogeneous quadratic forms in many variables; the treatment of conditions by obvious and also by more symmetric methods; applications to geometrical and physical problems; subsidiary conditions as inequalities rather than equations of condition; Gauss's principle in problems of mechanics; reversion of series; certain fundamental conceptions in the theory of analytic functions, such as analytic and algebraic dependence, and algebraic singularities.

The readers of the Monthly are doubtless most interested in the question as to the place in a college curriculum this book may be expected to fill. The author mentions in his preface, "As introductory to a courese of lecture on the calculus of variations, I have for a number of years given a brief outline of the

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which can come as near to g as we wish . . .," p. 136, while the character of the region of definition is treated as wholly arbitrary.

ALBERT A. BENNETT.

Jahrbuch über die Fortschritte der Mathematik . . . herausgegeben. Von E. LAMPE† und A. KORN. Band 45. Jahrgang 1914-1915. (In 3 Heften) Heft 1. Berlin und Leipzig, Vereinigung Wissenschalflicher Verleger, 1919, 12+368 pp.

The Heft opens with a fine portrait and a seven page appreciation of Emil Lampe's life and work. He was an editor of the "*Fortschritte*" since Jahrgang 1883. The Heft covers History and philosophy, Algebra, Arithmetic, and about twenty five pages of the fourth section on Combinatory analysis and the calculus of probabilities. The number of pages for the first three sections is about eighty more than for the corresponding sections of Jahrgang 1913, and about sixty more than for a similar portion of Jahrgang 1912.

The Theory of the Imaginary in Geometry together with the Trigonometry of the Imaginary. By J. L. S. HATTON, Cambridge, at the University Press, 1920. Royal 8vo. 8 + 216 pp. Price 18 shillings.

Preface: "The position of any real point in space may be determined by means of three real coördinates, and any three real quantities may be regarded as determining the position of such a point. In geometry as in other branches of pure mathematics the question naturally arises, whether the quantities concerned need necessarily be real. What, it may be asked, is the nature of the geometry in which the coördinates of any point may be complex quantities of the form $x + ix'$, $y + iy'$, $z + iz'$? Such a geometry contains as a particular case the Geometry of real points. From it the geometry of real points may be deduced (a) by regarding x', y', z' as zero, (b) by regarding x, y, z as zero, or (c) by considering only those points, the coördinates of which are real multiples of the same complex quantity $a + ib$. The relationship of the more generalized conception of geometry and of space to the particular case of real geometry is of importance, as points, whose determining elements are complex quantities, arise both in coördinate and in projective geometry.

"In this book an attempt has been made to work out and determine this relationship. Either of two methods might have been adopted. It would have been possible to lay down certain axioms and premises and to have developed a general theory therefrom. This has been done by other authors. The alternative method, which has been employed here, is to add to the axioms of real geometry certain additional assumptions. From these, by means of the methods and principles of real Geometry, an extension of the existing ideas and conceptions of geometry can be obtained. In this way the reader is able to approach the simpler and more concrete theorems in the first instance, and step by step the well-known theorems are extended and generalized. A conception of the imaginary is thus gradually built up and the relationship between the imaginary and the real is exemplified and developed. The theory as here set forth may be regarded from the analytical point of view as an exposition of the oft quoted but seldom explained 'Principle of Continuity.'

"The fundamental definition of Imaginary points is that given by Dr. Karl v. Staudt in his *Beiträge zur Geometrie der Lage;* Nuremberg, 1856 and 1860. The idea of (α, β) figures, independently evolved by the author, is due to J. V. Poncelet, who published it in his *Traité des Propriétés Projectives des Figures* in 1822. The matter contained in four or five pages of Chapter II is taken from the lectures delivered by the late Professor Esson, F.R.S., Savilian professor of geometry in the University of Oxford, and may be partly traced to the writings of v. Staudt. For the remainder of the book the author must take the responsibility. Inaccuracies and inconsistencies may have crept in, but long experience has taught him that these will be found to be due to his own deficiencies and not to fundamental defects in the theory. Those who approach

the subject with an open mind will, it is believed, find in these pages a consistent and natural theory of the imaginary. Many problems however still require to be worked out and the subject offers a wide field for further investigations."

Contents—I: Imaginary points and lengths on real straight lines, Imaginary straight lines, Properties of semi-real figures, 1–40; II: The circle with a real branch, The conic with a real branch, 41–69; III: Angles between imaginary straight lines, Measurement of imaginary angles and of lengths on imaginary straight lines, Theorems connected with projection, 70–124; IV: The general conic, 125–140; V: The imaginary conic, 141–163; VI: Tracing of conics and straight lines, 164–195; VII: The imaginary in space, 196–212; Index of Theorems, 213–215; Index of terms and definitions, 216.

Aeronautics, A Class Text. By EDWIN BIDWELL WILSON. New York, Wiley, 1920. 8vo. 8 + 265 pages. Price $4.00.

Preface: "For several years I have been giving, at the Massachusetts Institute of Technology, courses of lectures on those portions of dynamics, both rigid and fluid, which are fundamental in aeronautical engineering. The more elementary parts of these courses, covering about ninety out of one hundred fifty lectures, are found in this book. Although it has been customary to teach the two subjects of rigid and of fluid dynamics in parallel or in rapid alternation, so that they are both developed as needed for each other and for the accompanying courses on airplane and airship design, it has seemed better in making a presentation in book form to separate them. The student should have completed Chaps. IX–XII of the fluid mechanics before undertaking the latter part of Chap. VI.

"A number of topics which might well be included in a work on aeronautics have been omitted from the book, as they are from my lectures, because they can be taken up so much better in the parallel courses on design. In the preparation of the selected material I have had constantly in mind my own experience and needs relative to effective classroom instruction, particularly in the matter of lists of exercises. Although my students are supposed to have completed thorough courses in calculus, including the elements of differential equations, and in theoretical and applied mechanics, it has seemed better to assume too little, rather than too much, as retained in usable form. I hope, therefore, that with the present interest in aeronautics in particular, and in applied mathematics in general, this work may prove stimulating to other than technical students of aeronautical engineering.

"Nobody can issue a book on aeronautics at this time without lamenting the fact that much, if not most, of the progress in theory which has been made during the war, particularly in England, has not yet been released for publication. To wait, however, until its release and subsequent digestion would mean a long delay. Indeed from one viewpoint no time is more appropriate for the printing of these elementary, introductory, and orienting lectures than just now when there impends a deluge of material for advanced study . . ."

Contents: *Introduction*—I. Mathematical preliminaries; 3–10; II. The pressure on a plane, 11–21; III. The skeleton airplane, 22–36. *Rigid Mechanics*—IV. Motion in a resisting medium, 37–56; V. Harmonic motion, 57–80; VI. Motion in two dimensions, 81–106; VII. Motion in three dimensions, 107–121; VIII. Stability of the airplane, 122–151. *Fluid Mechanics*—IX. Motion along a tube, 152–164; X. Planar motion, 165–181; XI. Theory of dimensions, 182–198; XII. Forces on an airplane, 199–217; XIII. Stream function, velocity potential, 218–235; XIV. Motion of a body in a liquid, 236–250; XV. Motion in three dimensions, 251–262. *Index*—263-265.

How to make and use Graphic Charts. By A. C. HASKELL, with an introduction by R. T. Dana. New York, Codex Book Co., 1919. 8vo. 7 + 539 pp.

Contents: Introduction, 1–7; Rectilinear charts, 8–11; Logarithmic charts, 12–16; Semi-logarithmic charts, 17–24; Polar charts, 25–26; Geometric charts, 27–29; Trilinear charts, 30–36; Nomographic or alignment charts, 37–53; General principles pertaining to the use of charts, 54–77; Organization and management charts, 78–167; Cast and cast analysis charts, 168–228; Scheduling and progress charts, 229–261; Operating characteristics, 262–308; Charts showing the results of tests and experiments, 309–325; Trends, tendencies and statistical prediction shown by charts, 326–341; Computation, arithmetical and geometrical, by charts, 342–444; Charts as an aid to designing and estimating, 445–508; Miscellaneous uses of charts, 509–534; Index, 535–539.

There are numerous exact references to the literature of the subject on pages 165–167, 227–228, 260–261, 307–308, 324–325, 340–341, 442–444, 506–508, 533–534.

NOTES.

An author and subject index (4 + 152 pages) for the first fifty years of the *Sitzungsberichte der mathematisch-physikalischen Klasse der kgl. Bayerischen Akademie der Wissenschaften zu München* was prepared by A. Hilsenbeck and published in 1913.

D. M. Y. Sommerville's *The Elements of Non-Euclidean Geometry* has appeared with "Chicago . . . Open Court . . . , 1919" on the title page. The work does not differ, except in title page, from that published by Bell of London, in 1914.

Rand McNally of Chicago published in 1918 *Plane Geometry* (Price $1.25) by MABEL SYKES and C. E. COMSTOCK. Mr. Comstock is a member of the Mathematical Association of America and a professor of mathematics in the Bradley Polytechnic Institute.

Two other members of the Association, MATILDA AUERBACH, supervisor of mathematics in the Ethical Culture High School, New York City, and C. B. WALSH, principal of the Friends Central School, Philadelphia, are the authors of *Plane Geometry* (Philadelphia, Lippincott, 1920, 18 + 383 pp. price $1.32), in the Lippincott's School Text Series edited by W. F. Russell, dean of the College of Education, State University of Iowa. The three volumes of *Applied Arithmetic, the Three Essentials* in this same series are by Professor N. J. LENNES, of the University of Montana, and Professor FRANCES JENKINS, of the University of Cincinnati. 1919–1920. 12mo. 11 + 283 pp.; 9 + 294 pp.; 9 + 340 pp. Price 72 + 80 + 88 cents).

Quotation from *The Degradation of the Democratic Dogma.* By Henry Adams. New York, Macmillan, 1919; "The Rule of Phase applied to history" (1909) pages 272–274:

". . . Static electricity already lay beyond the legitimate domain of sensual science, while beyond static electricity lay a vast supersensual ocean roughly called the ether, which the physicists and chemists, on their old principles, were debarred from entering at all, and had to be dragged into, by Faraday and his school. Beyond the ether, again, lay a vast region, known to them as the only substance which they knew or could know—their own thought,—which they positively refused to touch.

"Yet the physicists here, too, were helpless to escape the step, for where they refused to go as experimenters, they had to go as mathematicians. Without the higher mathematics they could no longer move, but with the higher mathematics, metaphysics began. There the restraints of physics did not exist. In the mathematical order, infinity became the invariable field of action, and not only did the mathematician deal habitually and directly with all sorts of infinities, but he also built up hyper-infinities, if he liked, or hyper-spaces, or infinite hierarchies of hyper-space. The true mathematician drew breath only in the hyper-spaces of Thought; he could exist only by assuming that all phases of material motion merged in the last conceivable share of immaterial motion—pure mathematical thought.

The physicist, in self-defence, though he may not deny, prefers to ignore this rigorous con-

sequence of his own principles, as he refused for many years to admit the consequences of Faraday's experiments; but at least he can surely rely upon this admission being the last he will ever be called upon to make. No phase of hyper-substance more subtle than thought can ever be conceived, since it could exist only as his own thought returning into itself. Possibly, in the inconceivable domains of abstraction, the ultimate substance may show other sides or extensions, but to man it can be known only as hyperthought,—the region of pure mathematics and metaphysics,—the last and universal solvent.

"There even mathematics must stop. Motion itself, ended; even thought became merely potential in this final solution. The hierarchy of phases was complete."

ARTICLES IN CURRENT PERIODICALS.

Annals of Mathematics, second series, volume 21, no. 2, December, 1919 (published February, 1920): "A property of cyclotomic integers and its relation to Fermat's last theorem" by H. S. Vandiver, 73–80; "Surfaces of rotation in a space of four dimensions" by C. L. E. Moore, 81–93; "The circle nearest to n given points, and the point nearest to n given circles" by J. L. Coolidge, 94–97; "Singular solutions of differential equations of the second order" by E. M. Coon, 98–103; "Note on a class of integral equations of the second kind" by C. E. Love, 104–117; "Concerning sense on closed curves in non-metrical plane analysis situs" by J. R. Kline, 118-119; "On the theory of summability" by G. James, 120–127; "On the consistency and equivalence of certain generalized definitions of the limit of a function of a continuous variable" by L. L. Silverman, 128–140.

Bulletin of the American Mathematical Society, volume 26, no. 4, January, 1920: "The October meeting of the American Mathematical Society" by F. N. Cole, 145–151; "The October meeting of the San Francisco Section" by B. A. Bernstein, 152–155; "On the proof of Cauchy's integral formula by means of Green's formula" by J. L. Walsh, 155–157; "A set of completely independent postulates for the linear order η" by M. G. Gaba, 158–159; "Certain properties of binomial coefficients" by W. D. Cairns, 160–164; "The work of Poincaré on automorphic functions" [review of *Oeuvres de Henri Poincaré*, tome II (Paris, 1916)] by G. D. Birkhoff, 164–172; "A brief account of the life and work of the late Professor Ulisse Dini" by W. B. Ford, 173–177; "Shorter Notices," 177–183; "Notes," 184–188; "New publications," 189–192—No. 5, February: "Integro-differential equations with constant limits of integration" by I. A. Barnett, 193–203; "On a pencil of nodal cubics" by N. Altshiller-Court, 203–211; "Definition and illustrations of new arithmetical group invariants" by E. T. Bell, 211–223; "Matrices and determinoids" [review of C. E. Cullis's *Matrices and Determinoids* (Cambridge, 1913–1918)] by J. B. Shaw, 224–233; "Notes," 233–237; "New publications," 237–240.

L'Enseignement Mathématique, volume 20, no. 6, January, 1920: "Sur l'élimination algébrique" by C. Riquier, 405–421; "Nouveaux théorèmes sur le viriel de forces et leurs applications géométriques et mécaniques" by F. Boulad, 421–432; "Sur les foyers rationnels d'une courbe algébrique" by E. Turrière, 433–436; Chronique, Bibliographie, Bulletin Bibliographique, 436–462; Table de Matières, 463–468; Supplément, "Spaco" by R. de Saussure, 12 pages [Specimen of the international language "Espérantide (conciliation de l'Esperanto et de l'Ido)." The first sentences are: "En spaco existas sep, kay nur sep, figuron, kiun estas nur pozician, t.e., kiun entenas nenia grando. Lu estas (fig. 1): 1. La *punkto* (P). Irg ni punkto estas nura pozicio kay havas neni amplexo; punkto povas rotaci omnimanere sur si self, ne cestante esti la sama punkto.

2. La *reglo* (R), or rekta senfina linio, konsiderata kom spacelemento, kom nedivizibla tuto (kay ne kom serio de punkton). Irg ni reglo estas nura pozicio kay enhavas nenia grando; reglo povas gliti or rotaci sur si self, ne cestante esti la sama reglo."].

The Mathematics Teacher, volume 12, no. 2, December, 1919: "Certain undefined elements and tacit assumptions in the first book of Euclid's Elements" by H. E. Webb, 41–60; "Association of mathematics teachers of New Jersey. Report of the committee of first-year high-school mathematics," 61–74; "New books," 75–76; "Notes and News," 77–88.

Monist, volume 30, no. 1, January, 1920: "The analytical treatment of Newton's problems" by P. E. B. Jourdain, 19–36 [First paragraph: "It is a great help to the proper understanding of the meaning of Newton's propositions to consider the formulation of these propositions in analytical language. In this article, after a mention is made of Leibniz's analytical work of 1689 (§ I), Varignon's of 1690 (§ II), Maclaurin's of 1742 (§ III), and of some work of Johann Bernoulli, Keill, Hermann, and others near the beginning of the eighteenth century (§ IV), the contributions

sequence of his own principles, as he refused for many years to admit the consequences of Faraday's experiments; but at least he can surely rely upon this admission being the last he will ever be called upon to make. No phase of hyper-substance more subtle than thought can ever be conceived, since it could exist only as his own thought returning into itself. Possibly, in the inconceivable domains of abstraction, the ultimate substance may show other sides or extensions, but to man it can be known only as hyperthought,—the region of pure mathematics and metaphysics,—the last and universal solvent.

"There even mathematics must stop. Motion itself, ended; even thought became merely potential in this final solution. The hierarchy of phases was complete."

ARTICLES IN CURRENT PERIODICALS.

Annals of Mathematics, second series, volume 21, no. 2, December, 1919 (published February, 1920): "A property of cyclotomic integers and its relation to Fermat's last theorem" by H. S. Vandiver, 73–80; "Surfaces of rotation in a space of four dimensions" by C. L. E. Moore, 81–93; "The circle nearest to n given points, and the point nearest to n given circles" by J. L. Coolidge, 94–97; "Singular solutions of differential equations of the second order" by E. M. Coon, 98–103; "Note on a class of integral equations of the second kind" by C. E. Love, 104–117; "Concerning sense on closed curves in non-metrical plane analysis situs" by J. R. Kline, 118-119; "On the theory of summability" by G. James, 120–127; "On the consistency and equivalence of certain generalized definitions of the limit of a function of a continuous variable" by L. L. Silverman, 128–140.

Bulletin of the American Mathematical Society, volume 26, no. 4, January, 1920: "The October meeting of the American Mathematical Society" by F. N. Cole, 145–151; "The October meeting of the San Francisco Section" by B. A. Bernstein, 152–155; "On the proof of Cauchy's integral formula by means of Green's formula" by J. L. Walsh, 155–157; "A set of completely independent postulates for the linear order η" by M. G. Gaba, 158–159; "Certain properties of binomial coefficients" by W. D. Cairns, 160–164; "The work of Poincaré on automorphic functions" [review of *Oeuvres de Henri Poincaré*, tome II (Paris, 1916)] by G. D. Birkhoff, 164–172; "A brief account of the life and work of the late Professor Ulisse Dini" by W. B. Ford, 173–177; "Shorter Notices," 177–183; "Notes," 184–188; "New publications," 189–192—No. 5, February: "Integro-differential equations with constant limits of integration" by I. A. Barnett, 193–203; "On a pencil of nodal cubics" by N. Altshiller-Court, 203–211; "Definition and illustrations of new arithmetical group invariants" by E. T. Bell, 211–223; "Matrices and determinoids" [review of C. E. Cullis's *Matrices and Determinoids* (Cambridge, 1913–1918)] by J. B. Shaw, 224–233; "Notes," 233–237; "New publications," 237–240.

L'Enseignement Mathématique, volume 20, no. 6, January, 1920: "Sur l'élimination algébrique" by C. Riquier, 405–421; "Nouveaux théorèmes sur le viriel de forces et leurs applications géométriques et mécaniques" by F. Boulad, 421–432; "Sur les foyers rationnels d'une courbe algébrique" by E. Turrière, 433–436; Chronique, Bibliographie, Bulletin Bibliographique, 436–462; Table de Matières, 463–468; Supplément, "Spaco" by R. de Saussure, 12 pages [Specimen of the international language "Espérantide (conciliation de l'Esperanto et de l'Ido)." The first sentences are: "En spaco existas sep, kay nur sep, figuron, kiun estas nur pozician, t.e., kiun entenas nenia grando. Lu estas (fig. 1): 1. La *punkto* (P). Irg ni punkto estas nura pozicio kay havas neni amplexo; punkto povas rotaci omnimanere sur si self, ne cestante esti la sama punkto.

2. La *reglo* (R), or rekta senfina linio, konsiderata kom spacelemento, kom nedivizibla tuto (kay ne kom serio de punkton). Irg ni reglo estas nura pozicio kay enhavas nenia grando; reglo povas gliti or rotaci sur si self, ne cestante esti la sama reglo."].

The Mathematics Teacher, volume 12, no. 2, December, 1919: "Certain undefined elements and tacit assumptions in the first book of Euclid's Elements" by H. E. Webb, 41–60; "Association of mathematics teachers of New Jersey. Report of the committee of first-year high-school mathematics," 61–74; "New books," 75–76; "Notes and News," 77–88.

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AMERICAN DOCTORIAL DISSERTATIONS.

F. R. MORRIS, "Classification of involutory cubic space transformations," *University of California Publications in Mathematics*, vol. 1, no. 11, February, 1920, pp. 223–240. (University of California, 1918).

A. R. WILLIAMS, "On a birational transformation connected with a pencil of cubics," *University of California Publications in Mathematics*, vol. 1, no. 10, February, 1920, pp. 211–222. (University of California, 1916).

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL AND OTTO DUNKEL.

[Send all communications about problems and solutions to B. F. Finkel, Springfield, Mo.]

PROBLEMS FOR SOLUTION.

[N.B. The editorial work of this department would be greatly facilitated if, on sending in problems, the proposers would also enclose their solutions—*when they have them.* If a problem proposed is not original the proposer is requested *invariably* to state the fact and to give an exact reference to the source.]

2834. Proposed by OTTO DUNKEL, Washington University.

In any triangle ABC let M and N be, respectively, the points in which the median and the bisector of the angle at A meet the side BC, Q and P the points in which the perpendicular at N to NA meets MA and BA, respectively, and O the point in which the perpendicular at P to BA meets AN produced. Prove that the straight line QO is perpendicular to BC and the similar theorem for the external bisector of the angle at A.

This proposition shows the relation between two constructions for the center of curvature O of a conic for which B and C are foci and A is a point of the conic. (The figure also gives an easy proof of the Law of Tangents for triangles compare *1920*, 53–54. See also *1920*, 226.)

2835. Proposed by J. L. RILEY, Stephenville, Texas.

If x, y, z, u are finite, and not all zero, and satisfy the equations

$$x = by + cz + du, \qquad y = ax + cz + du, \qquad z = ax + by + du, \qquad u = ax + by + cz,$$

and if none of the quantities a, b, c, d, have the value -1, then will

$$\frac{a}{1+a} + \frac{b}{1+b} + \frac{c}{1+c} + \frac{d}{1+d} = 1.$$

2836. Proposed by W. V. N. GARRETSON, Rutgers College.

A ladder 40 feet long rests with one end on the ground against the foot of a building and the other end against the side of a second building directly across the street from the first. A second ladder 25 feet long inclines in a similar manner from the foot of the second building against the side of the first building, the two ladders crossing at a point 15 feet above the ground. How wide is the street?

2837. Proposed by B. F. FINKEL, Drury College.

Assuming v to be the velocity of sound; $\pm u_s$ the velocity of S, the source of sound, and n its frequency; $\pm u_r$ the velocity of R, the receiver; and $\pm w$ the velocity of M, the medium, discuss fully Döppler's Principle for the apparent frequency n'. The double signs are used to indicate that the discussion is to include the cases when source and receiver are approaching and when separating and the same consideration with reference to the medium. Limiting cases are especially desirable; e.g., when $w = 0$, $u_s = 0$, and $-u_r = 2v$; $w = 0$, $+u_s = v$, and $u_r = 0$; etc.

2838. "A rope is supposed to be hung over a wheel fixed to the roof of a building; at one end of the rope a weight is fixed, which exactly counterbalances a monkey which is hanging on to the other end. Suppose that the monkey begins to climb the rope, what will be the result?"

This problem was invented by Lewis Carroll in December, 1893 (S. D. Collingwood, *The Life and Letters of Lewis Carroll* (*Rev. C. L. Dodgson*), New York, 1899, pp. 317–318) and in his diary he remarked: "Got Professor Clifton's answer [R. B. Clifton, professor of physics at Oxford] to the 'Monkey and Weight Problem.' It is very curious, the different views taken by good mathematicians. Price [Bartholomew Price, professor of physics at Oxford] says that the weight goes *up* with increasing velocity; Clifton (and Harcourt [A. G. Vernon-Harcourt, professor of chemistry at Oxford]) that it goes *up*, at the same rate as the monkey; while Sampson [probably E. F. Sampson, lecturer, tutor and censor of Christ Church, Oxford] says that it goes *down*." Yet another solution by Rev. A. Brook is given on page 268 of *The Lewis Carroll Picture Book* . . . edited by S. D. Collingwood (London, 1899), namely that "the weight remains stationary."

The problem has been recently discussed in *School Science and Mathematics*, volume 17, December, 1917, p. 821; volume 19, December, 1919, p. 815; and volume 20, February, 1920, pp. 172–173. The editors of the MONTHLY invite mathematical solutions of the problem.

2839. By translating the steps of the construction of a regular pentagon from plane geometry into algebra show that one of the fifth roots of unity is equal to

$$\tfrac{1}{4}(\sqrt{5} - 1) + \frac{i}{4}\sqrt{10 + 2\sqrt{5}}.$$

(This problem is proposed for solution in Wilczynski and Slaught, *College Algebra with Applications*, Boston, 1916, p. 193.)

2840. Proposed by NORMAN ANNING, University of Maine.

It is observed in a table of values of

$$\log_{10}(\operatorname{colog}_{10} x)$$

that second differences are zero for values of x in the neighborhood of 0.37. Prove that this must be the case. (Cf. Chappell's *Five-Figure Mathematical Tables*, Edinburgh, 1915, p. 180.)

2841. Proposed by WILLIAM HOOVER, Columbus, Ohio.

The mixed number,

$$9\frac{49}{64} = 9 + \frac{49}{64} = 3^2 + \frac{7^2}{8^2},$$

is of the type form

$$k^2 + \frac{(2k+1)^2}{(2k+2)^2};$$

how may the *forms* of the terms of the fractional part be determined *deductively?*

Generally required that

$$k^2 + \frac{\{\varphi_1(k)\}^2}{\{\varphi_2(k)\}^2}$$

be a perfect square; show how $\varphi_1(k)$ and $\varphi_2(k)$ may be found.

2842. Proposed by H. S. UHLER, Yale University.

Express explicitly the following sextic in x as the product of a quadratic and a biquadratic:

$$3x^6 - 6k_1x^5 + (7k_1^2 - 9k_2^2)x^4 - 2(2k_1^3 - 4k_1k_2^2 - 3k_3^3)x^3 + [(k_1^2 - k_2^2)^2 - 9k_1k_3^3]x^2$$
$$- (k_1^2 - 2k_2^2)(k_1k_2^2 - 9k_3^3)x + (k_1^2 - 3k_2^2)(k_2^4 - 3k_1k_3^3).$$

SOLUTIONS OF PROBLEMS.

2746 [1919, 72]. Proposed by S. A. COREY, Des Moines, Iowa.

Establish the following algebraic identity without actually performing the indicated operations:

$$\begin{aligned}2(t_1t_2 + c_1t_3t_4 + c_2t_5t_6 + c_1c_2t_7t_8)(r_1r_2 + c_1r_3r_4 + c_2r_5r_6 + c_1c_2r_7r_8)\\ = (r_1t_1 - c_1r_3t_3 - c_2r_5t_5 + c_1c_2r_7t_7)(r_2t_2 - c_1r_4t_4 - c_2r_6t_6 + c_1c_2r_8t_8)\\ + (r_1t_2 - c_1r_3t_4 - c_2r_5t_6 + c_1c_2r_7t_8)(r_2t_1 - c_1r_4t_3 - c_2r_6t_5 + c_1c_2r_8t_7)\\ + c_1(r_1t_3 + r_3t_1 - c_2r_5t_7 - c_2r_7t_5)(r_2t_4 + r_4t_2 - c_2r_6t_8 - c_2r_8t_6)\end{aligned}$$

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$$-(k_1^2-2k_2^2)(k_1k_2^2-9k_3^3)x+(k_1^2-3k_2^2)(k_2^4-3k_1k_3^3).$$

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Proof of the given identity could have been obtained by the use of the ordinary complex quantities of algebra, but the above proof is given because of its greater generality and novelty.

It will be observed that the given identity as well as (1), (2), and the above 16-square theorem, all being homogeneous algebraic quadratic identities, may be given geometric interpretations by letting certain of the letters represent vectors and then taking the scalars of the resulting expressions.

2748 [1919, 72]. Proposed by J. B. REYNOLDS, Lehigh University.

The vertices of a triangle are (0, 0), $(2a, 0)$, and $(2x, 2y)$. Where are the vertices of the triangle of least area having its vertices on the perpendicular bisectors of the sides of the given triangle and the same center of gravity as the given triangle?

SOLUTION BY A. M. HARDING, University of Arkansas.

Let $Q_1(x_1, y_1)$, $Q_2(x_2, y_2)$, $Q_3(x_3, y_3)$ be the vertices of the required triangle. Since Q_1, Q_2, Q_3, are on the perpendicular bisectors of the sides of the triangle $P_1P_2P_3$, we have

$$y_1y + x_1(x - a) - x^2 + a^2 - y^2 = 0, \tag{1}$$

$$y_2y + x_2x - x^2 - y^2 = 0, \tag{2}$$

$$x_3 = a. \tag{3}$$

If the triangles $P_1P_2P_3$ and $Q_1Q_2Q_3$ have the same center of gravity

$$\frac{x_1 + x_2 + x_3}{3} = \frac{2x + 2a}{3}, \qquad \frac{y_1 + y_2 + y_3}{3} = \frac{2y}{3},$$

or

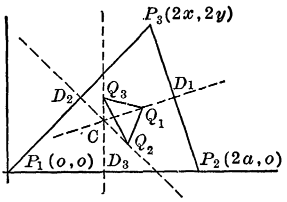
$$x_1 + x_2 = 2x + a, \tag{4}$$

$$y_1 + y_2 + y_3 = 2y. \tag{5}$$

From equations (1), (2), (4), (5), we find

$$\left.\begin{aligned} ax_1 &= -yy_3 + ax + a^2, \\ ax_2 &= yy_3 + ax, \\ ay_1 &= (x - a)y_3 + ay, \\ ay_2 &= -xy_3 + ay. \end{aligned}\right\} \tag{6}$$

The area of triangle $Q_1Q_2Q_3$ is given by



$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2a^2}\begin{vmatrix} ax_1 & ay_1 & 1 \\ ax_2 & ay_2 & 1 \\ ax_3 & ay_3 & 1 \end{vmatrix}$$

$$= \frac{1}{2a^2}\begin{vmatrix} -yy_3 + ax + a^2, & (x - a)y_3 + ay, & 1 \\ yy_3 + ax, & -xy_3 + ay, & 1 \\ a^2, & ay_3, & 1 \end{vmatrix}$$

whence $2a\Delta = 3yy_3^2 - 2(x^2 + y^2 - ax + a^2)y_3 + a^2y$.

The area will be a minimum when $(d/dy_3)(2a\Delta) = 0$; that is, when

$$3yy_3 = x^2 + y^2 - ax + a^2.$$

It may be easily shown that the center of the circumcircle of $\Delta P_1P_2P_3$ is $C(a, y_0)$, where $yy_0 = x^2 + y^2 - ax$. Hence $3yy_3 = yy_0 + a^2$, or $y_3 = y_0/3 + a^2/3y$. The coördinates of the other vertices may now be found from equations (6).

Note. It may be shown that if the triangles $P_1P_2P_3$ and $Q_1Q_2Q_3$ have the same center of gravity,

$$\frac{Q_1D_1}{P_2P_3} = \frac{Q_2D_2}{P_3P_1} = \frac{Q_3D_3}{P_1P_2},$$

where D_1, D_2, D_3 are the mid-points of the sides of $\Delta\ P_1P_2P_3$. This property of the triangles might have been used in this problem.

Also solved by E. H. CLARKE, C. E. HORNE, and A. PELLETIER.

2749 [1919, 72]. Proposed by C. N. SCHMALL, New York City.

In the parabola, $y^2 = 4ax$, two normals to the curve are drawn at the ends of a focal chord. Show that the area between these normals and the curve is $20a^2/(3 \sin^3 2\phi)$ where ϕ is the angle between one of the normals and the x-axis.

SOLUTION BY H. M. ROESER, Bureau of Standards, Washington, D. C.

The tangents to a parabola at the extremities of a focal chord intersect on the directrix at right angles. (Tanner and Allen, *Analytic Geometry*, page 227.) The tangents and normals will, therefore, form a rectangle of which the focal chord is a diagonal and whose area is equal to the product of the lengths of the tangents from their intersection on the directrix to the points of tangency. The area sought is the area of one of the triangular halves of the rectangle plus two-thirds of the area of the other triangle or five-sixths of the area of the rectangle.

Let m = slope of one of the normals. Then $y = mx - 2am - am^3$ is the equation of one normal and $y = -x/m + 2a/m + a/m_0$ is the equation of the other normal. $y = -x/m - am$ is the equation of one tangent, and $y = mx + a/m$ is the equation of the other tangent. The tangents intersect at the point $(x, y) \equiv [-a, a(1 - m^2)/m]$ and touch the curve at $(x, y) \equiv [am^2, -2am]$ and $(x, y) \equiv [a/m^2, 2a/m]$, respectively.

The lengths of the tangents are $l_1 = a(1 + m^2)\sqrt{1 + m^2}/m$ and $l_2 = a(1 + m^2)\sqrt{1 + m^2}/m^2$. The area sought is therefore $5l_1l_2/6 = 5a^2(1 + m^2)^3/6m^3 = 5a^2/(6 \sin^3 \phi \cos^3 \phi) = 20a^2/(3 \sin^3 2\phi)$.

Also solved by E. H. CLARKE, H. H. DOWNING, POLYCARP HANSEN, C. E. HORNE, MARCIA L. LATHAM, A. PELLETIER, and the PROPOSER.

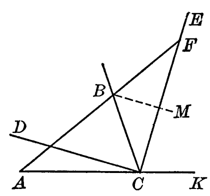
2750 [1919, 72]. Proposed by A. CAMPBELL, St. Johnsburg, Vermont.

Given the base, the sum of the sides of the triangle and the difference of the base angles, to construct the triangle.

SOLUTION BY THE PROPOSER.

Let b, be the given base; $a + c$, the sum of the other two sides, and $\alpha = C - A$ the difference of the base angles.

On the line AK lay off $AC = b$ and at C construct an angle $ACD = \frac{1}{2}(C - A) = \frac{1}{2}\alpha$. Draw CE perpendicular to DC. With A as a center and a radius equal to $a + c$ describe an arc intersecting CE in F. Draw AF. Construct the angle BCF equal to the angle BFC. Then the triangle ABC is the required triangle.



For, triangle BCF is an isosceles triangle having its base angles equal by construction. Hence, $BC = BF$, and, therefore, $AB + BC = AF$.

Also, angle CBF = angle A + angle C, or angle MBC (BM being the bisector of angle CBF) = $\frac{1}{2}$ angle CBF = $\frac{1}{2}$(angle A + angle C) = angle BCD = angle BDC = angle A + angle DCA; whence angle DCA = $\frac{1}{2}$(angle C − angle A) = $\frac{1}{2}\alpha$.

Also solved by C. L. ARNOLD, GEORGE AQUIS, MARY BEJSORIC, P. J. DA CUNHA, CHANG CHIH-CHEN, H. H. DOWNING, A. M. HARDING, C. E. HORNE, MARCIA L. LATHAM, A. PELLETIER, MARIAN M. TORREY, and LOUIS WEISNER.

2751 [1919, 72]. Proposed by ENOS E. WITMER, Senior in Franklin and Marshall College.

Investigate the problem of solving the equation

$$x^4 + ay^4 = w^2 + av^2. \tag{1}$$

Carmichael's *Diophantine Analysis*, Problem 18, p. 54.

Solution by the Proposer.

It appears that x, y, v, w, and a are to be rational numbers and $a \neq 0$.
If $v^2 = y^4$, $w^2 = x^4$, and a solution is given by $v = \pm y^2$ $w = \pm x^2$.
If $v^2 \neq y^4$ we may proceed as follows: From the given equation, we have

$$x^4 - w^2 = a(v^2 - y^4); \tag{2}$$

whence,

$$\frac{x^4 - w^2}{v^2 - y^4} = a = \frac{(x^4 - w^2)(v^2 - y^4)}{(v^2 - y^4)^2}, \tag{3}$$

or

$$\left(\frac{x^2 v \pm w y^2}{v^2 - y^4}\right)^2 - \left(\frac{x^2 y^2 \pm wv}{v^2 - y^4}\right)^2 = a. \tag{4}$$

Letting

$$\frac{x^2 v \pm w y^2}{v^2 - y^4} = b, \tag{5}$$

$$\frac{x^2 y^2 \pm wv}{v^2 - y^4} = c \tag{6}$$

and solving (5) and (6) for x^2 and w,

$$x^2 = bv - cy^2 \tag{7}$$

and

$$w = cv - by^2. \tag{8}$$

From (7),

$$v = \frac{x^2 + cy^2}{b}, \tag{9}$$

Substituting the value of v from (9) in (8), we have

$$w = \frac{cx^2 + c^2y^2 - b^2y^2}{b}.$$

But from (4), (5), and (6), $b^2 - c^2 = a$. Hence if we put $b - c = m$, $b + c = a/m$; then

$$b = \frac{1}{2}\left(\frac{a}{m} + m\right), \qquad c = \frac{1}{2}\left(\frac{a}{m} - m\right).$$

Hence

$$x = r, \qquad y = s,$$

$$v = \frac{2r^2 + \left(\frac{a}{m} - m\right)s^2}{\frac{a}{m} + m} = \frac{2mr^2 + (a - m^2)s^2}{a + m^2},$$

$$w = \frac{\left(\frac{a}{m} - m\right)r^2 - 2as^2}{\frac{a}{m} + m} = \frac{(a - m^2)r^2 - 2ams^2}{a + m^2}.$$

2765 [1919, 171]. Proposed by A. M. HARDING, University of Arkansas.

ABC is an equilateral triangle. A point D is taken in BC such that BD is $\frac{1}{3}$ of BC and E is taken in CA such that CE is $\frac{1}{3}$ of CA. If the lines AD and BE intersect at O, show that OC is perpendicular to AD.

Solution by the late L. G. Weld.

Since CD is twice CE and $\angle DCE = 60°$ the auxiliary line DE is perpendicular to CE; whence the point E is in the circumference of a circle described upon CD as a diameter. Since the triangles BCE and BOD are similar

$$BD \cdot BC = BO \cdot BE.$$

Hence, O, as well as E, lies in the circumference of the above circle and the angle COD, being inscribed in a semicircle, is a right angle.

Also solved by I. V. Baco, G. Breit, H. H. Downing, William Herberg, I. Herman, J. A. McWilliams, C. E. Mange, H. L. Olson, A. Pelletier, H. R. Phalen, J. B. Reynolds, W. F. Rigge, W. G. Simon, and the Proposer.

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Edited by E. J. Moulton, Northwestern University, Evanston, Ill.

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Professor J. E. Rowe, of Pennsylvania State College, has resigned to accept an appointment as mathematics and dynamics expert in the ordnance bureau of the war department. He is serving as chief ballistician at the Aberdeen Proving Ground.

Dr. Gaston Julia (*1919*, 223) is maître de conférences de mathématiques at the Ecole des Hautes Etudes, Paris.

Dr. J. A. Brashear, manufacturer of astronomical and physical instruments at Pittsburgh, Pa., and member of many scientific societies, died on April 8, 1920 in his eightieth year.

Dr. Woldemar Voigt, professor of theoretical physics at the University of Göttingen since 1883, died December 13, 1919 at the age of sixty-nine years.

Vice-Admiral C. A. Campos Rodrigues, director of the obseravtory of Lisbon, died December 25, 1919 aged eighty-three years.

Professor Moritz Benedikt Cantor, who died at Heidelberg, April 10, 1920, was born at Mannheim, August 23, 1829. He received his doctorate sixty-nine years ago at the University of Heidelberg where he was appointed extraordinary professor in 1853 and honorary professor in 1877. Apart from his monumental *Vorlesungen über Geschichte der Mathematik*, 1880–1907, he was the author of: *Ueber ein wenig gebräuchliches Coordinatensystem* (diss.) 1851; *Grundzüge einer Elementar-Arithmetik*, 1855; *Mathematische Beiträge zum Kulturleben der Völker*, 1863; *Euklid und sein Jahrhundert*, 1867 (Italian translation, 1873); *Die römischen Agrimensoren und ihre Stellung in der Geschichte der Feldmesskunst, eine historisch-mathematische Untersuchung*, 1875; *Politische Arithmetik oder die Arithmetik des täglichen Lebens*, 1898; second edition, 1903.

Professor A. S. Eddington has been appointed President of the Section of Mathematics and Physics of the British Association for the Advancement of Science.

Professor Paul Appell, honorary dean of the Faculty of Sciences of the University of Paris has been appointed Rector of the Académie de Paris, in succession to the late Lucien Poincaré.

Dr. Emile Borel, recently professor of the theory of functions in the University of Paris, has, at his own request, been appointed professor of the calculus of probabilities and mathematical physics as successor to Professor Boussinesq, who has retired (*1919*, 419).

At the meeting of the American Mathematical Society in New York on April 24, the afternoon session was devoted to a symposium on relativity. Papers

were presented as follows: By Professor LEIGH PAGE, "The physical and philosophical significance of the principle of the theory of relativity and Einstein's theory of gravitation;" by Professor L. P. EISENHART, "Geometric aspects of the Einstein theory."

At the meetings of the Division of Physical Sciences of the National Research Council on April 28–29, 1920, the following motions with reference to mathematics were adopted:

1. That the MATHEMATICAL ASSOCIATION OF AMERICA be asked to nominate one member for the Division of Physcial Sciences of the National Research Council.

2. That the Division of Physical Sciences recommend to the National Research Council that the American Section of Mathematics, organized under the statutes adopted by the Division, be made the authorized agent of the Council in the negotiations leading to the organization of the proposed International Mathematical Union and become its representative in that international body when it shall have been organized. Further, the Division recommends that upon the formal organization of the International Mathematical Union the Council signify its adhesion and that the annual contribution be paid as soon as funds are available.

3. That the number of members at large be increased by one with the understanding that the first man elected to this membership should represent mathematics.

4. That the Division approve of the project for the publication of a Journal of Mathematical Abstracts and recommend the appointment of a committee to work out details and to take steps in consultation with the Executive Committee of the Division and the Finance Committee of the Council, looking toward the securing of the necessary funds.

5. That a committee be appointed to consider further and in detail the project for securing a revolving fund to assist in the publication of important scientific books, monographs and translations which are so unprofitable from a commercial standpoint as not to appeal to regular publishing houses.

6. That the Division approve the project for the establishment of one or more Research Fellowships in Mathematics, such Fellowships to be administered by the Division, and that the Division request the assistance of the Finance Committee of the Council in securing funds for this purpose.

The Division considered also the recommendations of its Committee on Coöperation in Mathematical Projects (consisting of the representatives of the American Mathematical Society in the Division [see *1920*, 239] and Professors E. R. HEDRICK, and R. C. ARCHIBALD) regarding committees on two special branches of mathematics offering advantages in coöperative research, and the formation of these committees was left to the chairman.

The American Section of the International Mathematical Union referred to above is, according to its statutes, composed of: the president and secretary of the American Mathematical Society and its three representatives on the National Research Council, four other members selected by the American Mathematical Society, three members selected by the Mathematical Association of America, and one member each by the National Academy of Sciences, the American Astronomical Society, the American Physical Society, and the American Association for the Advancement of Science.

Additional announcements of mathematics courses in Summer Sessions (cf. *1920*, 192–194; 235–236) are as follows:

University of California, June 21–July 31. By Professor D. N. LEHMER: Theory of numbers; Special advanced study and research. By Professor R. G. D.

RICHARDSON: Plane trigonometry; Theory of functions of a complex variable. By Professor G. E. F. SHERWOOD: Graphic algebra; solid geometry. By Dr. F. R. MORRIS: Integral calculus. By Professor W. E. MILNE: Plane analytical geometry. Each class, except the one in "study and research," meets five times a week.

Cornell University, July 3–August 13. Elementary courses in solid geometry, advanced algebra, trigonometry, analytic geometry, and calculus will be given. The following advanced courses are offered: By Professor W. B. CARVER: Teachers' course, selected topics in algebra and geometry. By Professor D. C. GILLESPIE: Projective geometry. By Professor C. F. CRAIG: Analysis. In addition, opportunity for directed reading and research is offered in various fields by Professors F. R. SHARPE, W. B. CARVER, D. C. GILLESPIE, W. A. HURWITZ, F. W. OWENS, and C. F. CRAIG.

Johns Hopkins University, July 6–August 13: By Dr. TERESA COHEN, Analytic geometry, and a choice of work from (*a*) Geometrical transformations, (*b*) Elliptic functions, (*c*) Finite groups.

University of Maine, June 28–August 6. By Professors J. N. HART, H. R. WILLARD, and M. O. TRIPP: Teachers' course in algebra, Teachers' course in geometry, Solid geometry, Plane Trigonometry, College algebra, Analytic geometry, Differential and integral calculus, Advanced analytic geometry, Advanced calculus, Theory of functions.

Massachusetts Institute of Technology, June 21–July 26. By Professor F. S. WOODS: Analytic geometry. By Professors H. B. PHILLIPS and F. L. HITCHCOCK, and Mr. R. D. DOUGLASS: Elementary calculus, and Theoretical mechanics (introductory course). By C. L. E. MOORE: Theoretical mechanics (introductory course). By L. H. RICE: Elements of Differential equations, and Elementary calculus. By S. D. ZELDIN, Elementary calculus.

University of Texas, First Term, June 8–July 20: By Professor E. R. HEDRICK (University of Missouri), Advanced calculus, analytic geometry (straight line and circle). By Professor W. H. ROEVER (Washington University), Descriptive geometry, trigonometry. By Professor H. Y. BENEDICT, Differential calculus, analytic geometry (conic sections). By Professor H. J. ETTLINGER, Fundamentals in elementary mathematics, trigonometry. By Dr. P. M. BATCHELDER, Modern algebra, elementary algebra. By Dr. GOLDIE P. HORTON, Solid analytic geometry, elementary algebra. By Professor E. L. DODD, Least squares, life insurance. Second Term, July 20–August 31: By Professor P. R. RIDER (Washington University), Differential calculus (second part), analytic geometry (conic sections). By Professor C. D. RICE, Advanced mechanics, integral calculus. By Miss MARY DECHERD, Modern algebra (second part), analytic geometry (straight line and circle). By Mr. J. E. BURNAM, Trigonometry, solid geometry.

The American Mathematical Monthly

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Fifth Annual Meeting, December, 1920.

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KENTUCKY, Centre College, Danville, April 17
MARYLAND–DISTRICT OF COLUMBIA–VIRGINIA, Goucher College, Baltimore, Md., May 15; Annapolis, Md., December
MINNESOTA, St. Catherine's College, St. Paul, June 5
MISSOURI, Kansas City, November 12–13
OHIO, Ohio State Univ., Columbus, April 2
ROCKY MOUNTAIN, Colorado College, Colorado Springs, April 2

EDITORIAL CORRESPONDENCE AND BOOKS FOR REVIEW should be addressed to the EDITOR-IN-CHIEF, R. C. ARCHIBALD, Brown University, Providence, R. I.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 27 King Street, Oberlin, Ohio.

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DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

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THE APRIL MEETING OF THE KANSAS SECTION.

The sixth regular meeting of the Kansas Section, postponed from 1918 on account of war conditions, was held at the Kansas State Agricultural College, Manhattan, Kansas, Saturday, April 3, 1920. The chairman, Professor W. A. Harshbarger, presided at both the morning and afternoon sessions. In the absence of the secretary Professor E. B. Stouffer acted as temporary secretary.

The attendance was nineteen, including the following fifteen members of the Association: C. H. Ashton, Lucy T. Dougherty, Ottilia W. Dueker, W. H. Garrett, W. A. Harshbarger, Emma Hyde, S. Lefschetz, T. Lindquist, U. G. Mitchell, Mary W. Newson, B. L. Remick, E. B. Stouffer, W. T. Stratton, J. J. Wheeler, A. E. White.

The following officers were elected: Chairman, Professor W. H. GARRETT, Baker University; Vice-Chairman, Professor J. A. G. SHIRK, State Manual Training Normal School; Secretary, Professor E. B. STOUFFER, University of Kansas. It was decided to hold the next meeting at Topeka in the autumn in connection with the State Teachers Association. The visitors were entertained at a luncheon which was very kindly provided by the members of the department of Mathematics of the Kansas State Agricultural College.

The following six papers were read, the first three at the morning session and the remaining three at the afternoon session:

(1) "Content of the freshman algebra course," by Professor U. G. MITCHELL;

(2) "The bridge between high school and college mathematics," by Professor MARY W. NEWSON;

(3) "A comparative study of the algebraic preparation of the college freshman," by Professor W. H. GARRETT;

(4) "Geometrical representation of the nature of an essential singular point," by Professor C. H. ASHTON;

(5) "The dual of duality," by Professor E. B. STOUFFER;

(6) "Report on tests on college freshmen," by Professor W. T. STRATTON.

The discussion on the first paper was led by Professors Ottilia W. Dueker and A. E. White. There was also a general discussion on this paper and on the paper given by Professor Garrett. Abstracts of the papers follow below, the numbers corresponding to the numbers in the list of titles:

1. In this paper Professor Mitchell presented a list of topics, with a distribution of the number of class-periods suggested for each, to make up a five hour course for one semester designed for students who present but one year of high school algebra for entrance. With certain modifications and omissions the same could be used for a three hour course for one semester designed for students presenting one and one half years of high school algebra for entrance. The selection of material as presented was based upon a logical development of the subject-matter. It was pointed out, however, that the chief concern was not

5. Professor Stouffer showed that an easy method of introducing point-duality into projective geometry is by considering it as the space dual of plane-duality. Such a method is in line with the desire to form the duals of every concept introduced. Numerous illustrations of the different kinds of duality were shown.

6. Professor Stratton presented briefly the results of the freshmen engineering tests in arithmetic and algebra which had been conducted under the direction of Dr. J. C. Peterson of the Department of Education of the Kansas State Agricultural College. While the tests have not been carried far enough to justify any definite conclusions, yet there seems to be a close correlation between the test grades and the ability of the students to do their college work successfully.

H. E. JORDAN, *Secretary-Treasurer.*

THE APRIL MEETING OF THE KENTUCKY SECTION.

The fourth annual meeting of the Kentucky Section was held at Centre College, Danville, Kentucky, on April 17; Professor C. G. CROOKS of Centre College was chairman of the meeting.

There were ten present including the following seven members of the Association; P. P. Boyd, C. G. Crooks, J. M. Davis, H. H. Downing, E. L. Rees, C. H. Richardson, G. W. Smith. The section was the guest of the retiring chairman at an excellent dinner. For the ensuing year Professor H. H. DOWNING was elected chairman, and Dr. G. W. SMITH, secretary-treasurer.

The following six papers were read;

(1) "Calculus ideas before Newton and Leibnitz," by Professor P. P. BOYD;

(2) "Some problems concerning the catenary," by Professor H. H. DOWNING;

(3) "A solution of Euler's equation," by Professor C. H. RICHARDSON;

(4) "Some functions of a single nilpotent number," by Dr. G. W. SMITH;

(5) "An early treatment of the quadratic transformation," by J. OSBORN, Graduate Student, University of Kentucky. (By invitation.)

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2. Professor Downing made brief mention of the general theory in which it is shown that if $y = f(x)$ is the equation of a curve joining two fixed points and $I = \int F(x, y, y')dx$, then for I to be a minimum (or maximum) $f(x)$ must be a solution of the Euler differential equations. Also that the Weierstrass condition, the Legendre condition and the Jacobi condition relating to conjugate points, must be satisfied by $f(x)$.

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For the integral, $I = \int_a^b y\sqrt{1 + y'^2}\,dx$, which is the integral appearing in the expression for the surface obtained by revolving the curve $y = f(x)$ about the X-axis, he pointed out that the catenary is the solution of the Euler equations. The other conditions were applied and the results discussed. MacNeish's discussion of the Goldschmidt discontinuous solution was presented.

3. Professor Richardson proved the following theorem analytically, and found thereby a solution of Euler's equation

$$\frac{dt'}{\sqrt{R(t)}} + \frac{dt''}{\sqrt{R(t)}} = 0 \tag{1}$$

where $R(t) = At^4 + 2Bt^3 + Ct^2 + 2Dt + E$: *If* Σ *is a conic represented parametrically by the equations*

$$x = \frac{a_1t^2 + b_1t + c_1}{a_3t^2 + b_3t + c_3}, \qquad y = \frac{a_2t^2 + b_2t + c_2}{a_3t^2 + b_3t + c_3}$$

and if Σ' *is a second conic cutting* Σ *in four points of which the arguments are roots of an equation of the fourth degree* $R(t) = 0$, *any tangent whatever of* Σ' *intersects* Σ *in two points of which the arguments* t' *and* t'' *and their differentials when the tangent varies are connected by the relation* (1). Hence, in order to write a solution to this equation we find the condition that the line joining two points of Σ whose arguments are t' and t'' shall be tangent to Σ', the general conic which cuts Σ in four points whose arguments are roots of $R(t) = 0$. The relation obtained between t' and t'' contains an arbitrary constant and is a solution of the given equation.

4. After proving as does Peirce that in every linear associative algebra there is at least one idempotent or one nilpotent number Dr. Smith considered polynomials of a single nilpotent number j, $A(j) = \sum a_n j^n$ $(n = 0, 1, 2, 3, \cdots, \mu - 1)$ where $j^\mu = 0$ but $j^{\mu-1} \neq 0$. If $A(j)\cdot B(j) = 0$ and $a_0 \neq 0$, then $B(j)$ vanishes entirely. However if the first h of the coefficients of $A(j)$ are zero then the last h coefficients of $B(j)$ are arbitrary. $A(j)/B(j) = C(j)$ will exist (though part may be arbitrary) provided $A(j)$ does not start with a lower power of j than does $B(j)$. If $a_0 = a_1 = a_2 = \cdots = a_{kn-1} = 0$, but $a_{kn} \neq 0$ $(k = 0, 1, 2, \cdots)$ then $[A(j)]^n$ may be uniquely determined for any positive value of n.

5. Mr. Osborn briefly reviewed the first work done on quadratic transformations then discussed Steiner's transformation by means of the simple hyperboloid.

6. Professor Rees in a semi-humorous way pointed out some of the inconsistencies, the fallacies, and the improprieties of mathematics and their nomenclature, presenting thereby many of the things which so trouble the beginner.

GUY W. SMITH, *Secretary-Treasurer.*

FIFTH ANNUAL MEETING OF THE OHIO SECTION.

The fifth annual meeting of the Ohio Section of the Mathematical Association of America was held at the Ohio State University, Columbus, on the afternoon and evening of April 2, 1920, in connection with the meetings of the Ohio College Association. The widely advertised program attracted a large attendance not only of mathematicians, but also of physicists, other scientists and educators. The attendance included the following thirty-five members of the Association:

R. B. Allen, W. E. Anderson, G. N. Armstrong, C. L. Arnold, C. B. Austin, Grace M. Bareis, L. A. Bauer, W. S. Beckwith, R. D. Bohannan, R. L. Borger, W. D. Cairns, V. B. Caris, E. H. Clarke, O. L. Dustheimer, T. M. Focke, Harris Hancock, William Hoover, H. W. Kuhn, A. C. Lunn, C. N. Mills, C. N. Moore, C. C. Morris, A. D. Pitcher, J. B. Preston, S. E. Rasor, C. Lois Rea, Hortense Rickard, W. G. Simon, K. D. Swartzel, T. Elmer Trott, J. H. Weaver, R. B. Wildermuth, F. B. Wiley, D. T. Wilson, and E. I. Yowell.

Professor R. L. Borger occupied the chair, being relieved for an interval by Professor S. E. Rasor. Many of those in attendance at the afternoon meetings attended the dinner at the Ohio Union with the Ohio College Association, the dinner being followed by addresses of interest to college teachers. This gathering adjourned at 8.30 to attend Dr. Bauer's lecture of the evening program.

At the business meeting the Secretary reported that the membership of the Section is seventy, two more than last year, nine names having been removed from the roll and eleven added. Besides there are eight institutional members. The following officers were elected: Chairman, Professor S. E. Rasor, Ohio State University; Secretary-Treasurer, Professor G. N. Armstrong, Ohio Wesleyan University; Third member of the executive committee, Professor A. D. Pitcher, Western Reserve University.

The following program was carried through:

(1) Address of the chairman: "Some geometric methods for curve tracing" by Professor R. L. Borger;

(2) "The theory of relativity" by Professor A. C. Lunn;

(3) Symposium on Number (2) and related questions by (*a*) F. C. Blake, Professor of Physics, Ohio State University; (*b*) Dr. C. W. Chamberlain, President of Denison University; (*c*) D. C. Miller, Professor of Physics, Case School of Applied Science; (*d*) Members of the Section and visitors.

(4) "The deflection of light observed during the solar eclipse of May 29, 1919, and its bearing upon the Einstein theory of gravitation" (illustrated lecture) by Dr. L. A. Bauer;

(5) Informal social and round table meeting at the Ohio Union following the evening lecture. Topic: "Freshman mathematics to meet the changing high school mathematics as presented for entrance to college." Discussion led by Mr. H. M. Beatty, Ohio State University.

Lantern slides were used to illustrate the apparatus and methods. Incidents of historical interest were related and particular attention was called to a persistent occurrence at the last experiments of a small displacement of the fringes, far less than the theory calls for, which has never been satisfactorily explained.

4. This lecture was fully reported in *Science,* March 26, 1920, pp. 301–311.

5. In the round table discussion, Mr. Beatty summarized the replies to questionnaires mailed out to about fifty of the leading high schools of Ohio. These reveal a tendency to minimize the amount of mathematics required for graduation. One unit each of algebra and geometry is required. In most cases one half unit each of advanced algebra and soild gometry is offered as an elective, but in many cases is not elected by the pupil. There is a tendency for more pupils to enter college deficient in a half-unit or more of mathematics. There was a feeling expressed that the same care in selecting teachers of mathematics was not exercised nor the same respect accorded mathematics as was done in former years. The opinion seemed to prevail that there was no more reason for discouragement over results in mathematics than in other subjects.

G. N. ARMSTRONG, *Secretary-Treasurer.*

THE APRIL MEETING OF THE ROCKY MOUNTAIN SECTION.

The fourth regular meeting of the Rocky Mountain Section was held at Colorado College, Colorado Springs, Colorado, April 2, 3. There were two sessions, presided over by Professor C. H. Sisam.

The attendance was twenty-five, including the following fourteen members of the Association: I. M. DeLong, J. C. Fitterer, W. H. Hill, H. A. Howe, Claribel Kendall, G. H. Light, J. Q. McNatt, S. L. Macdonald, O. A. Randolph, H. E. Russell, C. H. Sisam, C. S. Sperry, C. E. Stromquist, J. W. Woodrow.

The officers appointed for the meeting to be held at Denver in 1921 are: Chairman, H. A. HOWE, Denver University; Vice-chairman, W. H. HILL, Greeley High School; Secretary-Treasurer, G. H. LIGHT, Univ. of Colorado.

The following eight papers were read:

(1) "Some physical correlations in a group of one hundred S. A. T. C. men" by Professor J. C. FITTERER;

(2) "Families of curves whose evolutes are similar curves" by Professor G. H. LIGHT;

(3) "Grades for different placings of ears of corn" by Professor W. V. LOVITT;

(4) "Ionization in the mercury arc" by Professor J. W. WOODROW;

(5) "Discussion of the cycloidal curve" by Mr. J. Q. MCNATT;

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(8) "On ruled surfaces whose asymptotic curves are cubics" by Professor C. H. SISAM.

1. Professor Fitterer presented correlation tables which were computed between stature and stride, stride and weight, weight and stature. The correlation coefficient in the first was 0.29, in the second very nearly zero, and in the third 0.55. The average age was 19.3 years, average weight was 136 pounds, average stature was 5′ 8″, average stride was 5.7 ft. A hypsobaric coefficient (weight in pounds per foot of stature) was also found, which averaged 24 pounds per foot.

2. Professor Light's paper appears elsewhere in this issue.

3. Numerical grades were given by Professor Lovitt for different placings of any number of ears. The results are determinate, though arbitrary. The results are in use and are giving satisfaction with competent corn judges.

4. It was assumed by Professor Woodrow that (*a*) An electron, on the average, loses all of its translatory energy at each impact; (*b*) The molecules are capable of storing up energy, *i.e.*, after the energy is received, it is radiated by electromagnetic waves at a rate which is proportional to the instantaneous energy, and additional increments of energy can be added by successive impacts of different electrons with the same molecule; (*c*) The molecule can also receive energy which has been radiated from the surrounding molecules and which is proportional to the fourth power of the temperature of the gas or vapor. From these assumptions, the following equation was obtained

$$X = \frac{H}{I} p^{1/3}(K - p^{2/3}).$$

Where X is the electric force, I is the current, p is the pressure of the gas, and H and K are constants.

5. Mr. McNatt gave methods of constructing the cycloid and its evolute.

6. Given the linear differential equation of order five

$$y^{(5)} + 5p_1y^{(4)} + 10p_2y^{(3)} + 10p_3y^{(2)} + 5p_4y^{(1)} + p_5y = 0,$$

Professor Lovitt found invariants and covariants for the transformation $y = \lambda(x)\bar{y}$, $\xi = \xi(x)$. Some geometric interpretations were given.

7. Professor Stromquist suggested the following course for the first year of high school: (*a*) Tabulation and graphing of functions; (*b*) Meaning of positive and negative exponents, applying the four rules; (*c*) Square root of arithmetical numbers; (*d*) Logarithms, based on exponents; (*e*) Slide rule. The course has been successful in the Laramie High School.

8. Professor Sisam classified completely, and discussed the properties of, the ruled surfaces whose asymptotic curves are gauche cubics.

G. H. Light, *Secretary-Treasurer.*

MATHEMATICS AND LIFE INSURANCE.[1]

By PERCY C. H. PAPPS,[2] Newark, N. J.

For an actuary to prepare an address for an association of mathematical teachers is in many respects more difficult than to prepare a paper for a society of actuaries. The thought has occurred to me that it might be interesting if I endeavor to show to what extent mathematics is used in actuarial work. I know that there is a very general idea that an actuary must be a skilled mathematician. In fact, it may not be too much to say that in many instances it is taken for granted that a skilled mathematician must of necessity make an expert actuary. Except for unusual computations, an elementary knowledge of mathematics is all that is required to understand the actuarial formulæ used in daily work. Some of the simpler formulæ and the manner in which they are derived will first be described and I will then outline some of the applications of the higher mathematics to life insurance work.

One of the most common fallacies which we meet is that the expectation of life is used in actuarial calculations. So far as I am aware, the only use which is ever made of the expectation of life is in comparing the effects of mortality under different mortality tables. It is rather generally supposed that in order to arrive at the single premium for an insurance of $1,000, payable upon the death of a man age 35, we take the expectation of life at age 35 and ascertain what sum of money now invested will at the end of the term of years equal $1,000 at the assumed rate of interest. A single premium thus computed is smaller than the true single premium. This can be proved mathematically, but it can be readily proved by general reasoning. In the first place the expectation of life of a man age 35 is the mathematical mean of the number of years lived by a large number of men now of age 35. According to the "American Experience Table of Mortality," the expectation of life at age 35 is about thirty-two years, and it may be supposed that in the computations there is one life that lived twenty-two years, and one which lived forty-two years, the mean of these two lives being thirty-two years. At 3% interest $521.89 is the present value of $1,000 due at the end of twenty-two years, and $288.96 is the present value of $1,000 due at the end of forty-two years. $810.85 is therefore the amount that would be required to be placed at interest at 3% in order to pay the two claims of $1,000 each at the end of twenty-two and forty-two years respectively. If in place of using the actual assumed duration for these two lives we used the expectation of life, which is thirty-two years, we would find that $388.34 is the present value of $1,000 at the end of thirty-two years, or $776.68 is the present value of $2,000 payable at the end of thirty-two years. The present value of these two cases when the expectation of

[1] Read before the Association of Mathematics Teachers of New Jersey, November 20, 1915. Two sections enclosed between asterisks * were added in proof sheets.

[2] Mathematician for the Mutual Benefit Life Insurance Company.

life is used is therefore $34.17 less than the correct amount. As a matter of fact, the expectation of life could only be correctly used if we assumed interest at the rate of 0%. When we assume any interest we are dealing with a geometrical rather than an arithmetical progression, and as stated before, the expectation of life is an arithmetical mean.

To find the single premium for $1,000 of insurance on a life now age 35, we require to know the rate of interest which it is safe to assume will be earned by the company for many years to come, and it is also necessary that we should have a table showing the rates of mortality likely to be experienced in the future, or, to put it perhaps more correctly, a table of mortality showing rates which will be not less than those likely to be experienced in the future. Before proceeding further it may be of interest briefly to outline the method employed in compiling mortality tables.

In the first place, by ascertaining for each age of life how many lives, how many policies, or how much insurance, as the case may be, was exposed to the risk of death, and how many lives, policies, or how much insurance was actually cancelled by death, we can ascertain for each age the rate of mortality. Even with a large amount of data, these mortality rates will exhibit accidental fluctuations, so that the data as first ascertained must be suitably graduated. It is not sufficient to ascertain the rate of mortality at each age, for it is necessary that we should have a table to show us out of a certain number living at one age how many will be living at some future age. This is done by taking an arbitrary number, such as 100,000 lives, at age ten, let us say, and by the rate of mortality at age ten we find out the number dying between ages ten and eleven. Subtracting this from 100,000, we have the number living at age eleven. Proceeding in this manner, we have a table showing the number living and the number dying at each age. We are then in a position to proceed to the calculation of single premiums and annuities. It may be mentioned in passing that the rate of mortality is not the only function or the usual function used in the graduation of a mortality table. It would be possible, for example, to start with a fixed radix and use the ungraduated rates to find the number living at each age, and then graduate the figures thus ascertained.

To find the single premium for an insurance of unity on a life now of age 35, we have theoretically to ascertain the cost of providing insurance between ages 35 and 36, between ages 36 and 37, and so on, until we come to the last age shown in the mortality table. In other words, we have a large number of computations to make, the sum of which will give us the single premium required. From our life table we know the number dying between ages 35 and 36. It is assumed that the claims will be paid at the end of the year. If we discount the amount required to the beginning of the year and divide this by the number of lives shown by the mortality table to be living at age 35, we would have the cost of insurance on a single life between ages 35 and 36. We can ascertain the number dying between ages 36 and 37, discount for two years the amount required to pay these claims, divide by the number living at age 35, and we have the cost of providing

the insurance between ages 36 and 37. This is the way in which the computations are made in theory. As a matter of fact, by means of what are known as commutation tables the calculations are made very expeditiously.

The commutation tables consist of several columns. If we take the number dying between ages 35 and 36 and multiply this by the present value of a unit payable thirty-six years hence, we will find the value of the C column for age 35. If we multiply the number living at age 35 by the present value of a unit due at the end of thirty-five years, we will have the value of the D column for age 35. If we then divide the value of the C column by the value of the D column for age 35, we will have the probability of a life age 35 dying between ages 35 and 36, discounted for one year. We may start from the bottom of the C column and form what is known as the M column by continuous addition of the values in the C column. Opposite age 35 in this column we will then have the sum of the values for the C column from age 35 to the end of the table. If this value in the M column is divided by the value for age 35 in the D column, we will immediately have the single premium for an insurance of unity at age 35. A similar summation of the D column gives what is known as the N column, and dividing any value in the N column by the corresponding value in the D column gives the value of a life annuity for the age selected. If, for example, we wished to obtain the value of an annuity running for ten years on a life age 35, we could take the sum of the values in the D column for ages 35 to 44 inclusive, and divide this by the value in the D column at age 35. It is a simpler matter, however, to subtract from the value in the N column at age 35 the value in the N column at age 45, which will immediately give us the sum of the ten values in the D column.

It has been explained that the single premium for an insurance at age 35 is merely the sum of the several costs of providing insurance payable in the event of a life dying in every year of life, and it is of course impossible that the event can happen more than once. In the same way, the value of an annuity of unity a year is the sum of the present values of the probabilities of a life living to each age to the end of the mortality table.

There are comparatively few who can afford to buy their insurance by means of a single cash payment, and a large proportion of the policies sold are what are known as Ordinary Life policies, where the premiums are payable during the continuance of the contract. Knowing the single premiums and life annuities, it is a simple matter to ascertain the annual premium required. It may be supposed that a company sells $1,000 of insurance and that the policyholder sells to the company a life annuity of a certain amount. The present value of the one must be equal to the present value of the other. If, therefore, we divide the single premium by the value of a life annuity of unity, the quotient will be the annual premium required. Under a Twenty Payment Life policy the premiums are limited to twenty years, so that the annual premium is ascertained by dividing the single premium by the present value of an annuity of unity running for twenty years. It may be mentioned that under an ordinary annuity the payments to the annuitant are made at the end of each year; under a life insurance

contract the premiums are paid at the beginning of each year. Care must be taken, therefore, that in calculating the value of annual premiums we use an annuity providing for the payments at the beginning of each year. On account of this difference, there are two forms of the N column of the commutation tables, which are likely to confuse the student until the difference is recognized.

The calculation of single and annual premiums for endowment insurances offers no difficulties. It must be remembered that a Twenty-Year Endowment insurance is made up of two parts. We have Twenty-Year Term insurance and what is known as a Twenty-Year Pure Endowment. Under the term insurance the claim is paid if the life dies during the twenty years, otherwise the contract becomes valueless at the end of twenty years. Under the Pure Endowment the contract is valueless in the event of the death of the insured, but is paid in full if the insured survives the twenty years. The single premium for the term insurance is ascertained for age 35 by taking the sum of twenty values in the C column from age 35 to 54 inclusive, and dividing by the value in the D column for age 35. The sum of the values in the C column is of course ascertained by subtracting the value in the M column at age 55 from the value in the same column for age 35. The single premium for the pure endowment at age 35 is ascertained by dividing the value in the D column for age 55 by the value in the same column for age 35.

We now come to the question of reserve. We have in the business of life insurance a very good illustration of the old saying, that a little knowledge is a dangerous thing. If the functions of the reserve on a life insurance policy had been well understood, this country would have been saved the enormous dissatisfaction and hardships caused by the failure of assessment insurance and fraternal orders. These concerns operated on what is known as the "reserve in the pocket" plan. That is, the members were supposed to keep the reserve in their own pockets. This will be referred to later. The reserve on a life insurance policy may be explained in several different ways. It has already been shown that at the inception of a contract of life insurance on the Ordinary Life plan, the single premium is exactly equal to the present value of the annual premiums. Consider the conditions of affairs when the policy has been in force one year. The value of the insurance granted by the company is greater than it was at the inception of the contract, for the insured is one year older, and consequently one year nearer his death. On the other hand, the value of the future premiums which the insured has agreed to pay to the company has decreased for the reason that the insured is one year older and certainly has one less premium to pay. The company's liability represented by the present value of the insurance granted is therefore in excess of the company's asset, which is represented by the present value of the annual premiums yet to be received. Therefore, in order that the company may not be insolvent, it must have on hand a sum equal to the difference between the value of the insurance granted and the value of the premiums yet to be received. This is technically known as the reserve.

The question may arise as to where the reserve comes from. It is readily seen that as a man grows older the yearly cost of providing insurance, provided a

new contract is taken each year, must steadily increase. Under an Ordinary Life policy it does not increase for the reason that the cost is equalized. The first premiums are more than sufficient to pay for carrying the insurance, and it is the balance that goes into the reserve. Furthermore, in the event of death the company is called upon to provide also the difference between the face of the contract and the reserve. As the life grows older, the reserve steadily increases, so that the amount at risk steadily decreases. It thus comes about that in spite of the increasing age of the insured, the annual premium plus the interest on the reserve is always sufficient to pay for the yearly cost of insurance and provide for the necessary increase in the reserve. At age 96, which is the limit of life according to the American Table of Mortality, the entire reserve exactly equals the face of the policy.

The assessment plan of insurance is correct in theory, for there is no theoretical reason why a policyholder should not pay each year just the mere cost of providing his insurance. The objections are practical ones. The net premium at 3 per cent. interest for one year's insurance of $1,000 at age 35 is $8.68. At age 75 it is $91.62. If a man were so fortunate, or unfortunate, as to live ten years more, he would have to pay for one year's insurance $228.69. Many assessment societies have attempted to provide insurance at level rates at an amount more than sufficient to cover the cost at the young ages, but without holding any reserve which would enable them to continue that cost at the older ages. They have claimed that they left the reserve in their members' pockets. If that was the case it was permissible for them to allow the members to withdraw and retain the reserve in their pockets. The great mistake which they made was that when members died they forgot all about the reserve being in the members' pockets, and paid the full amount of the certificates to the beneficiaries. It is quite possible, if not probable, that many fraternal societies which have been wound up on account of insolvency, would be today in existence if they had paid the beneficiaries merely the difference between the face of the certificates and the amount which the deceased members had retained as reserve in their pockets.

From what has been said you will readily see that a knowledge of elementary algebra up to and including the progressions is sufficient to enable anyone to grasp the ordinary formulæ necessary for calculating premiums and reserves. We will now pass on to consider a few cases where some knowledge of the higher mathematics is necessary. From what has been said it will readily be seen that the calculation of a single premium, for example, where we have merely the mortality table without any commutation columns would be an exceedingly laborious undertaking. Let us suppose that we wish to calculate the cost of an annuity on the life of a woman after the death of her husband. Now in the first place, we know that the mortality on the life of an annuitant is very different from the mortality on the life of one who is insured. It is necessary, therefore, to suppose that our prospective annuitant will experience a mortality according to an annuity table, while her husband will be one whose mortality we will have to measure by a table on which our premium rates for life insurance are

based. We are therefore forced to consider a computation which will involve two mortality tables as well as a rate of interest. In such a case the integral calculus is of inestimable benefit. By means of the calculus we can use a formula of approximate summation which will give results surprisingly close to an exact calculation. The use of one of these formulæ can be best described by taking a simple example of a single premium for life insurance on a single life. It has already been shown how by exact calculation we can ascertain the present value of a life age 35 dying in the first, second, third years, etc., from the present time, and add all these values together. By means of the approximate summation formulæ we can ascertain the present value of the probability of a life dying in the eighth year, the twenty-fourth, fortieth, forty-eighth and fifty-sixth years; multiplying each of these values by proper coefficients and taking eight times the sum of the results will give us a practically correct result. It is true that the approximate summation formulæ are based on the integral calculus, and to understand the derivation of the formulæ a knowledge of the differential and integral calculus is required; yet, as a matter of fact, these approximate summation formulæ are in daily use by those who have no knowledge of either the differential or integral calculus, but who have been trained to apply the formulæ.

* Starting with the well known Maclaurin's Theorem

$$u_x = u_0 + x\frac{du_0}{dx} + \frac{x^2}{\underline{|2}}\,\frac{d^2u_0}{dx^2} + \frac{x^3}{\underline{|3}}\cdot\frac{d^3u_0}{dx^3} + \text{etc.}$$

and integrating we get

$$\int u_x dx = xu_0 + \frac{x^2}{\underline{|2}}\,\frac{du_0}{dx} + \frac{x^3}{\underline{|3}}\,\frac{d^2u_0}{dx^2} + \frac{x^4}{\underline{|4}}\,\frac{d^3u_0}{dx^3} + \text{etc.}$$

Taking the integral between the limits $-n$ and $+n$

$$\int_{-n}^{n} u_x dx = 2nu_0 + \frac{2n^3}{\underline{|3}}\,\frac{d^2u_0}{dx^2} + \text{etc.} \tag{A}$$

All but the first two terms on the right hand side of the above equation may be ignored and it may be assumed that

$$\int_{-n}^{n} u_x dx = au_0 + b(u_{-n} + u_n). \tag{B}$$

Expanding by Maclaurin's Theorem this becomes

$$\int_{-n}^{n} u_x dx = au_0 + b\left(2u_0 + \frac{2n^2}{\underline{|2}}\,\frac{d^2u_0}{dx^2} + \text{etc.}\right). \tag{C}$$

Equating the coefficients of expressions (A) and (C) it will be seen that $a+2b=2n$ and $bn^2 = n^3/3$. From this it follows that $a = 4n/3$ and $b = n/3$. Substituting these values in (B) we get

$$\int_{-n}^{n} u_x dx = \frac{n}{3}(u_{-n} + 4u_0 + u_n)$$

or

$$\int_0^{2n} u_x dx = \frac{n}{3}(u_0 + 4u_n + u_{2n}).$$

From this it follows that

$$\int_0^{\infty} u_x dx = \int_0^{2n} u_x dx + \int_{2n}^{4n} u_x dx + \int_{4n}^{6n} u_x dx + \text{etc.}$$

$$= \frac{n}{3}\{(u_0 + 4u_n + u_{2n}) + (u_{2n} + 4u_{3n} + u_{4n}) + \text{etc.}\}$$

$$= \frac{n}{3}\{u_0 + 2(u_{2n} + u_{4n} + u_{6n} + \text{etc.}) + 4(u_n + u_{3n} + u_{5n} + \text{etc.})\}.$$

In the example illustrating this formula given in the *Institute of Actuaries' Text Book*, n has been given the value 15 in ascertaining the value of a continuous life annuity at age 30. Instead of computing the probabilities of living every duration from one to sixty-five years only four values are used, namely, $v^{15}l_{45}/l_{30}$, $v^{30}l_{60}/l_{30}$, $v^{45}l_{75}/l_{30}$ and $v^{60}l_{90}/l_{30}$ where l_{30}, for example, represents the number living at age 30 according to the mortality table used in the computations and v^{15} the present value of a unit payable fifteen years hence. The value of the annuity obtained by using the formula is 20.3899 while the true value is 20.3919; an error of only .002.

By taking more terms in formulæ (A) and (B), other summation formulæ may be derived. The one referred to above is thus derived. The formula is

$$\int_0^{\infty} u_x dx = n(.28u_0 + 1.62u_n + 2.2u_{3n} + 1.62u_{5n} + .56u_{6n} + 1.62u_{7n}).$$

In applying this formula n is so taken that $7n$ falls just within or just beyond the limit of life shown by the mortality table.*

In the graduation of mortality tables many methods are employed. We may take the sum of several values, assume that the average of these values is applicable to the central age of the group, plot the points, and by drawing a smooth curve through the points a series of graduated values may be obtained. The graduation of a set of values may also be accomplished by means of a summation method. For example, if we take three times the sum of three consecutive values from ten times the central value, the result will be a very rough graduation. By taking these values and summing each five adjacent values, repeating this operation three times and taking eight one thousandths of the final result a smoothly graduated series of values will result, unless the ungraduated values are very uneven. An extension of this principle enables us to derive formulæ where the graduated value at any age will be made up of suitable portions of the values extending as far as fourteen terms on each side of the central value. A graduation of a mortality table may also be effected by assuming that a certain mathematical law will hold. One well-known law involves four arbitrary constants, which must

ity, able to meet the public, and compose a decent letter. He must have the mathematical ability to enable him to pass his actuarial examinations and qualify as an actuary, but so far as his practical work is concerned, in the training of his actuarial department for the regular routine work of the office he can forget if he will practically all of his higher mathematics. When an actuary is called upon, as is frequently the case, to take his part in the executive work of the company, it is necessary that he should have some knowledge of life insurance law, the principles of accounting, and a general knowledge of investments and finance. From what I have said I believe that you will agree with me when I say that an actuary is not necessarily an expert mathematician, and that an expert mathematician will not necessarily make a good actuary.

NOTE ON THE ROOTS OF THE DERIVATIVE OF A POLYNOMIAL.

By W. H. ECHOLS, University of Virginia.

(Read before the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America, May 15, 1920.)

The problem of determining regions to which are confined the roots of the derivatives of functions has received a great deal of attention. Lucas established the fact[1], by a mechanical proof, that the roots of the derivative of any polynomial are confined to the smallest convex region enclosing the roots of the polynomial. Maxime Bôcher gave a geometrical demonstration of this.[2] He noted that the roots of the derivative of any cubic equation are the foci of the ellipse tangent at their mid-points to the sides of the triangle of the roots of the cubic; and remarked that it might be possible to associate the roots of the derivative of any polynomial with the foci of a higher plane curve. B. Z. Linfield gave a beautiful demonstration, in a paper read at the meeting, in St. Louis last December, of the American Mathematical Society (*Bulletin*, p. 264), that the roots of the derivative of any polynomial of degree n were the foci of a curve of class $n - 1$ touching at their mid-points the segments joining, two and two, the roots of the polynomial. Recently J. L. W. V. Jensen stated,[3] without demonstration,

[1] F. Lucas, "Géométrie des polynômes," *Journal de l'Ecole Polytechnique*, 1879, cahier 46, tome 28, pp. 1–33.—EDITOR.

[2] M. Bôcher, "Some propositions concerning the geometric representation of imaginaries," *Annals of Mathematics*, March, 1893, vol. 7, pp. 70–72.—EDITOR.

[3] In a letter to the writer Dr. J. L. Walsh, of Harvard, says: "Jensen originally stated his theorem without proof, in *Acta Mathematica*, vol. 36 (1912), p. 190. Apparently the theorem remained unnoticed until Professor D. R. Curtiss called attention to it in a paper presented at a meeting of the American Mathematical Society; his abstract was published in the current volume of the *Bulletin*, pp. 61–62. Professor Curtiss presented further results in April (see the June *Bulletin*, p. 392). I also have some further results which I presented to the Society in December; the abstract of which is in the March *Bulletin*, p. 259. My paper has been accepted for publication in the *Annals of Mathematics*; it contains a proof of Jensen's theorem based on mechanical considerations. Your own proof is surely different from mine (in form but not in substance). I do not know whether it is different from Curtiss's. I was not aware an algebraic proof could be given so simply. No proof of Jensen's theorem has yet been published."

ity, able to meet the public, and compose a decent letter. He must have the mathematical ability to enable him to pass his actuarial examinations and qualify as an actuary, but so far as his practical work is concerned, in the training of his actuarial department for the regular routine work of the office he can forget if he will practically all of his higher mathematics. When an actuary is called upon, as is frequently the case, to take his part in the executive work of the company, it is necessary that he should have some knowledge of life insurance law, the principles of accounting, and a general knowledge of investments and finance. From what I have said I believe that you will agree with me when I say that an actuary is not necessarily an expert mathematician, and that an expert mathematician will not necessarily make a good actuary.

NOTE ON THE ROOTS OF THE DERIVATIVE OF A POLYNOMIAL.

By W. H. ECHOLS, University of Virginia.

(Read before the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America, May 15, 1920.)

The problem of determining regions to which are confined the roots of the derivatives of functions has received a great deal of attention. Lucas established the fact[1], by a mechanical proof, that the roots of the derivative of any polynomial are confined to the smallest convex region enclosing the roots of the polynomial. Maxime Bôcher gave a geometrical demonstration of this.[2] He noted that the roots of the derivative of any cubic equation are the foci of the ellipse tangent at their mid-points to the sides of the triangle of the roots of the cubic; and remarked that it might be possible to associate the roots of the derivative of any polynomial with the foci of a higher plane curve. B. Z. Linfield gave a beautiful demonstration, in a paper read at the meeting, in St. Louis last December, of the American Mathematical Society (*Bulletin*, p. 264), that the roots of the derivative of any polynomial of degree n were the foci of a curve of class $n - 1$ touching at their mid-points the segments joining, two and two, the roots of the polynomial. Recently J. L. W. V. Jensen stated,[3] without demonstration,

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[1] F. Lucas, "Géométrie des polynômes," *Journal de l'Ecole Polytechnique*, 1879, cahier 46, tome 28, pp. 1–33.—EDITOR.

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that the complex roots of the derivative of a polynomial with real coefficients lie inside circles having the segments joining the pairs of conjugate roots of the polynomial as diameters. It is the purpose of this note to present a complete demonstration establishing the truth of this statement.

THEOREM. If $f(z)$ is a polynomial whose coefficients are real numbers, then the imaginary roots of the derivative $f'(z)$ lie on or within circles whose diameters are the segments joining the pairs of conjugate imaginary roots of $f(z)$.

Proof. Let

$$f(z) \equiv L_1^{p_1}L_2^{p_2} \cdots L_m^{p_m} \cdot Q_1^{q_1}Q_2^{q_2} \cdots Q_n^{q_n},$$

wherein p_r and q_r are positive numbers, and

$$L_r \equiv z - a_r, \qquad Q_r \equiv (z - b_r)^2 + c_r^2,$$

a_r, b_r, c_r real numbers. Taking the logarithm and differentiating, the derivative is

$$f'(z) = f(z) \cdot \Sigma \left(\frac{p_r}{z - a_r} + 2q_r \frac{z - b_r}{Q_r} \right).$$

Realizing the denominators ($z \equiv x + iy$)

$$\frac{p_r}{z - a_r} = \frac{p_r}{x - a_r + iy} = p_r \frac{(x - a_r) - iy}{(x - a_r)^2 + y^2},$$

and

$$\frac{z - b_r}{Q_r} = (x - b_r) \frac{(x - b_r)^2 + y^2 + c_r^2}{D_r} - iy \frac{(x - b_r)^2 + y^2 - c_r^2}{D_r},$$

where, for brevity,

$$D_r \equiv \{(x - b_r)^2 - y^2 + c_r^2\}^2 + 4(x - b_r)^2 y^2.$$

Hence, equating to zero the real and imaginary components of the sigma factor of $f'(z)$, we have the necessary and sufficient conditions for those roots of $f'(z)$ which are not common to $f(z)$,

$$\Sigma \left\{ p_r \frac{x - a_r}{(x - a_r)^2 + y^2} + 2q_r(x - b_r) \frac{(x - b_r)^2 + y^2 + c_r^2}{D_r} \right\} = 0,$$

$$y\Sigma \left\{ \frac{p_r}{(x - a_r)^2 + y^2} + 2q_r \frac{(x - b_r)^2 + y^2 - c_r^2}{D_r} \right\} = 0.$$

The real roots are furnished by $y = 0$ and the first condition. For $y \neq 0$ it is impossible for the second condition to vanish unless the point x, y is inside one of the circles

$$(x - b_r)^2 + y^2 = c_r^2.$$

The multiple roots of the polynomial are of course also roots of the derivative and these are on the circles. This justifies the statement.

May 1, 1920.

QUESTIONS AND DISCUSSIONS.

EDITED BY W. A. HURWITZ, Cornell University, Ithaca, N. Y.

REPLIES.

34. Given the mixed integral and functional equation [*1917*, 134, 341; 1920, 114]

$$\int_{x=0}^{x=h} f(x)dx = \frac{h}{6}\left[f(0) + 4f\left(\frac{h}{2}\right) + f(h)\right],$$

to determine the function $f(x)$. This equation is of rather fundamental practical value as it has to do with the most general solid whose volume is given by the prismatoid formula.

I. REPLY BY ELIJAH SWIFT, University of Vermont.

Assume that $f(x)$ is developable in a power series in the neighborhood of the origin

$$f(x) = A_0 + A_1x + \cdots + A_nx^n + \cdots.$$

Substituting this value for $f(x)$ and integrating we have

$$A_0h + A_1\frac{h^2}{2} + \cdots + A_n\frac{h^{n+1}}{n+1} + \cdots = A_0h + A_1\frac{h^2}{2} + \cdots + \frac{h}{6}A_n\left[4\cdot\frac{h^n}{2^n} + h^n\right] + \cdots;$$

or transposing,

$$\sum_{n=2}^{\infty} A_nh^{n+1}\left\{\frac{1}{n+1} - \frac{1}{6}\cdot\frac{4}{2^n} - \frac{1}{6}\right\} = 0.$$

If this is to hold for all values of h near 0, the coefficients of the terms of the series must all vanish. Obviously this is the case for $n = 2$ and $n = 3$. This leads to the equation

$$A_n\left(\frac{1}{n+1} - \frac{1}{6\cdot 2^{n-2}} - \frac{1}{6}\right) = 0.$$

Since

$$\frac{1}{n+1} - \frac{1}{6\cdot 2^{n-2}} - \frac{1}{6} < \frac{1}{n+1} - \frac{1}{6},$$

it is clear that for $n \geqq 5$, the coëfficient of h^{n+1} cannot vanish unless $A_n = 0$. For $n = 4$, direct calculation gives the same result. Consequently *if the function $f(x)$ is analytic, it must be a polynomial of degree not larger than* 3.

II. REPLY BY A. A. BENNETT, Washington, D. C.

The problem as stated is ambiguous. The question depends upon whether h is to be some fixed value, as might be implied by the notation, or is a variable for all values of which the functional relation is to hold.

If h is constant, the solution obviously admits of an infinite number of independent parameters since only the definite integral and the value of the function at three points are involved. It is not difficult to select even an infinite number of values for which the above relation holds, while the function still retains an infinite number of independent parameters. In view of the application suggested, the following formulation of the problem may be of interest: To determine the character of $f(x)$ subject to the condition that the above relation holds for all values of h between 0 and H, (> 0), inclusive, and that $f(x)$ shall be defined in this interval as made up of a finite number of continuously joining analytic pieces—these pieces to be analytic even at the end points of the intervals in which they are used.

If h lies in the first interval (starting toward H from 0), it is at once verified that $f(x)$ is a polynomial of degree not greater than 3, but not otherwise restricted, so far as this interval is concerned. Expansion in series and comparison of undetermined coefficients suffices for this proof. Let a be the end point of this first interval. Let h be in the second interval, near a, so that $h/2$

In view of the previous remarks, it seems reasonable to adopt as a goal with reference to this question the proof that if the equation holds for some range of the variable h, the function $f(x)$ can be only a polynomial of degree $\leqq 3$, under restrictions as light as possible—*e.g.*, that $f(x)$ should be continuous and possess a stated number (as small as it can be made) of derivatives.

DISCUSSIONS.

In the study of a geometric locus, it is frequently useful to have an analytic form of representation which shall exhibit the properties of the locus as regards its size and shape, without reference to its position. Such a form of representation, in the case of a plane curve, is furnished by the *intrinsic equation* of the curve —the functional relation which subsists between the length of arc measured from a fixed point of the curve to a variable point, and the radius of curvature at the variable point. Professor Light makes use of the intrinsic equation to study curves whose evolutes are similar to themselves. He obtains only the cases already known—the logarithmic spiral, and the cycloidal curves, and shows that no other examples exist whose intrinsic equations are of the special type $AR^n + BS^m - C = 0$. It would be of interest to know whether any other curves whatever can have this property.

In this department for May appeared a derivation of the formulæ for the tangents of the half angles of a triangle from the law of sines, by Professor Baudin, In the present number Professor Bohannan gives a method of obtaining tangent, sine and cosine of the half-angles by a geometric proof based on the use of inscribed and escribed circles.

Professor Poor recommends the more extensive use of directed lines and the method of projection in connection with elementary analytic geometry. In fact the use of such notions in the class-room is perhaps more prevalent than their appearance in text-books might indicate. Even in text-books however they are to be found here and there. The theorems stated by Professor Poor are now generally used in trigonometry to prove the formulæ for sines and cosines of sums and differences of angles. In some texts on analytic geometry they form the only mode of approach to transformation of coördinates by rotation; and they are frequently used in just the fashion suggested by Professor Poor to obtain the normal form of the equation of a straight line.

I. Note on Curves Whose Evolutes are Similar Curves.

By G. H. Light, University of Colorado.

It is known[1] that the evolutes of certain curves are similar curves. It is the purpose of this paper to give the general conditions that must be satisfied by

[1] Euler: "Investigatio curvarum quae evolutae sui similes producunt," *Comm. Petrop.* vol. 12, 1750, pp. 19–23.

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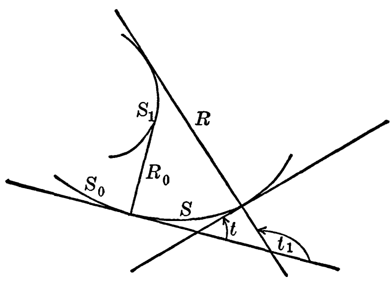
the base curve when this property holds true. Let the intrinsic equation of the base curve be $F(R, S) = 0$. Then, if subscript 1 applies to the evolute (see figure)



$$S_1 = R - R_0, \qquad t_1 = t + 90°.$$

Hence

$$R_1 = dS_1/dt_1 = dR/dt = R\,dR/dS.$$

Therefore, the equation of the evolute of $F(R, S) = 0$ is to be found by eliminating S and R from the three equations

$$F(R, S) = 0, \qquad R_1 = R\,dR/dS, \qquad S_1 = R - R_0. \tag{1}$$

If now the first of the three equations is solved for S, one obtains $S = S(R)$. Then, since

$$dR/dS = -F_S(R, S)/F_R(R, S),$$

the second equation of (1) states that

$$R_1 = -RF_S(R, S)/F_R(R, S).$$

Finally, use of the third of the equations of (1) gives

$$R_1 = -(S_1 + R_0)F_S(S_1 + R_0, S(S_1 + R_0))/F_R(S_1 + R_0, S(S_1 + R_0)).$$

which may be rewritten in the more symmetric form

$$R_1F_R + (S_1 + R_0)F_S = 0, \tag{2}$$

where it is, of course, understood that the arguments R and S in F_S and F_R have been replaced by $S_1 + R_0$ and $S(S_1 + R_0)$ respectively. Equation (2) is the form of the evolute most suitable for further use. Consider now the class of curves whose intrinsic equations are of the form

$$F(R, S) = AR^n + BS^m - C = 0, \tag{3}$$

where A, B and C are constants.

The relations between m and n and their exact values will be determined when their evolutes are similar curves.

Solving equation (3) for S, and remembering that $R = S_1 + R_0$, gives

$$S = \left(\frac{C - AR^n}{B}\right)^{1/m} = \left[\frac{C - A(S_1 + R_0)^n}{B}\right]^{1/m}$$

Hence

$$F_S = mBS^{m-1} = mB\left[\frac{C - A(S_1 + R_0)^n}{B}\right]^{(m-1)/m}$$

and

$$F_R = nAR^{n-1} = nA(S_1 + R_0)^{n-1};$$

Use of the defining equations of the evolute gives

$$F_R = nR^{n-1} = n(S_1 + R_0)^{n-1},$$

$$F_S = -mCS^{m-1} = -mC^{1/m}(S_1 + R_0)^{n(m-1)/m}.$$

From these values of F_S and F_R the equation of the evolute is found, from (2), to be

$$nR_1(S_1 + R_0)^{n-1} - mC^{1/m}(S_1 + R_0)(S_1 + R_0)^{n(m-1)/m} = 0.$$

An easy simplification of this equation reduces it to

$$R_1^m = C\left(\frac{m}{n}\right)^m (S_1 + R_0)^{2m-n}. \tag{6}$$

If this last equation is to represent a curve similar to the base curve (5), then it is again necessary—first, that $R_0 = 0$, *i.e.*, that a cusp exist on the base curve; and second, that the exponents correspond. This gives $m = n$. Hence[1] the result is obtained;

All curves whose intrinsic equations can be written $R^n = CS^m$ have evolutes similar to themselves only when $R_0 = 0$ and when $m = n$. This gives the logarithmic spiral $R^m = CS^m$.

II. Functions of Half-Angles of a Triangle.

By R. D. Bohannan, Ohio State University.

In the accompanying figure I is the center of the circle inscribed in the triangle ABC, and E is the center of the escribed circle touching the side BC and the prolongations of AB and AC. IH, IK, EF, EG are radii of these circles drawn to some of the points of tangency. If $BC = a$, $AC = b$, $AB = c$, $a + b + c = 2s$, then

$$AH = s - a; \qquad AG = s; \qquad CK = s - c; \qquad BK = s - b.$$

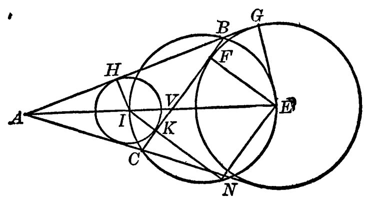
A circle on IE as diameter passes through C, B. If IK is prolonged to meet this circle in N, KN is equal to FE.

$$\tan\frac{A}{2} = \frac{IH}{AH} = \frac{EG}{AG}.$$

Hence

$$\tan^2\frac{A}{2} = \frac{IH}{AH}\cdot\frac{EG}{AG} = \frac{IH\cdot EG}{s(s-a)}$$

$$= \frac{IK\cdot FE}{s(s-a)} = \frac{IK\cdot KN}{s(s-a)}$$



$$= \frac{CK\cdot KB}{s\cdot(s-a)} = \frac{(s-b)(s-c)}{s(s-a)}.$$

[1] The cases excluded by the assumptions $m \neq 0$, $n \neq 0$, $C \neq 0$ in the course of the proof can be shown as before to yield nothing new.—Editor.

Let V be the intersection of AE and BC. Then

$$\sin^2 \frac{A}{2} = \frac{IH}{AI} \cdot \frac{EG}{AE}, \qquad \cos^2 \frac{A}{2} = \frac{AH}{AI} \cdot \frac{AG}{AE}.$$

Since BI and CE (not drawn in the figure) are angle-bisectors,

$$\frac{AI}{IV} = \frac{AB}{BV}, \qquad \frac{AE}{VE} = \frac{AC}{VC}.$$

But

$$IV \cdot VE = BV \cdot VC.$$

Hence

$$AI \cdot AE = AB \cdot AC = bc.$$

Therefore

$$\sin^2 \frac{A}{2} = \frac{(s-b)(s-c)}{bc}, \qquad \cos^2 \frac{A}{2} = \frac{s(s-a)}{bc}.$$

III. The Use of the Vector in Analytical Geometry.

By Vincent C. Poor, University of Michigan.

In a first course in analytical geometry it is necessary to adhere rather closely to the text, whatever the text. This is true not because of the nature of the subject matter but because of the mathematical immaturity of the students electing the subject. Important innovations thus furnish one excuse for another textbook.

In many of the textbooks on analytical geometry the directed line is not mentioned at all. This is deplorable from the point of view of the physicist, for the geometric interpretation of many physical quantities leads to simplicity in expression and clearness in comprehension. Aside from this need the subject of analytical geometry may, in my opinion, be much more easily and directly presented if a more extended use of the vector be made.

The ground work for this is to be found in some of our textbooks, *e.g.*, Woods and Bailey, *A Course in Mathematics*, Vol. I; Ziwet and Hopkins, *Analytic Geometry*. In their study of directed lines we find the equivalents of the following theorems:

Theorem I. *The projection of a line segment on another line is equal to the length of the line segment into the cosine of their included angle.*

Theorem II. *The projection of a broken line on another line is equal to the projection of the join of its end points.*

In a number of the books the fundamental theorem of the geometry of segments is deduced, namely

Theorem III. *Given three points, O, P, Q, on a directed line, then*

$$PQ = OQ - OP,$$

in magnitude and sense.

If O is taken as the origin and the coördinates x_1 and x_2 be assigned to the

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If O is taken as the origin and the coördinates x_1 and x_2 be assigned to the

the equation of the line through the two points (x_2, y_2) and (x_3, y_3) be written in determinantal form, the distance h from the point (x_1, y_1) to the line is

$$h = \frac{\begin{vmatrix} x_1, & y_1, & 1 \\ x_2, & y_2, & 1 \\ x_3, & y_3, & 1 \end{vmatrix}}{\sqrt{(x_3 - x_2)^2 + (y_3 - y_2)^2}}.$$

Since the denominator is the base of the triangle determined by the three points, the determinantal form for the area will be evident. The transformation equations are as easily deduced by the direct application of Theorems I and II. The application of these three theorems to three dimensions is equally successful.

I do not at all mean to imply that these general proofs should be used to the exclusion of the time honored method of taking a figure in the first quadrant and deducing the result geometrically. These geometrical exercises may well be relegated to the problem lists. But the general method is the simpler, and the knowledge of the vector requisite for its use is not beyond the college freshman. For many students the vector idea can be introduced none too early. Anyone contemplating a new text on analytical geometry should certainly weigh these possibilities.

RECENT PUBLICATIONS.

REVIEWS.

Mathematical Logic.

A Survey of Symbolic Logic. By C. I. Lewis. Berkeley, University of California Press. 1918. Royal 8vo. 6 + 409 pp. Price $4.00.

Molière's M. Jourdain was very much surprised when told that he had been using prose all his life. Equally astonished are many present-day mathematicians when informed that they have been using 'logical prose'—propositional functions, the Zermelo axiom, and the like—for a correspondingly long period.

What is this logical prose of which the mathematical and the logical world at large have been, till quite recently, so blissfully ignorant? It is the principles of modern deductive logic, known also as symbolic or mathematical logic. Though Professor Lewis prefers the term *symbolic,* Russell and his school seem to have established almost irrevocably the name *mathematical* logic. And Professor Lewis's book is a survey of the history of the various stages in the discovery of the principles of deductive logic.

What are these principles? Everyone has heard of the famous 'Laws of Thought'—the Laws of Identity, Contradiction, and Excluded Middle. Assuming that these laws, considered as principles of *logic* (not of thought), are necessary, are they also sufficient? Obviously not; for the principle of the Syllogism is just as necessary to logical procedure as are these laws. Are the four principles sufficient? How shall we decide? We can study the problem empirically.

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A Survey of Symbolic Logic. By C. I. Lewis. Berkeley, University of California Press. 1918. Royal 8vo. 6 + 409 pp. Price $4.00.

Molière's M. Jourdain was very much surprised when told that he had been using prose all his life. Equally astonished are many present-day mathematicians when informed that they have been using 'logical prose'—propositional functions, the Zermelo axiom, and the like—for a correspondingly long period.

What is this logical prose of which the mathematical and the logical world at large have been, till quite recently, so blissfully ignorant? It is the principles of modern deductive logic, known also as symbolic or mathematical logic. Though Professor Lewis prefers the term *symbolic,* Russell and his school seem to have established almost irrevocably the name *mathematical* logic. And Professor Lewis's book is a survey of the history of the various stages in the discovery of the principles of deductive logic.

What are these principles? Everyone has heard of the famous 'Laws of Thought'—the Laws of Identity, Contradiction, and Excluded Middle. Assuming that these laws, considered as principles of *logic* (not of thought), are necessary, are they also sufficient? Obviously not; for the principle of the Syllogism is just as necessary to logical procedure as are these laws. Are the four principles sufficient? How shall we decide? We can study the problem empirically.

We can make a careful and detailed investigation of some of the important proofs of mathematical theorems—for instance, in function theory—and after a step-by-step analysis we may find that these proofs employ various principles in addition to the four above mentioned. Such is, for example, the principle that whenever we have three propositions p, q, and r, and we know that

(1) If p is true, then q implies r, we have a right to replace (1) by

(2) If p and q are both true, then r is true. (This principle is called by Peano the principle of Importation, since it enables us to 'import' q into the hypothesis of the main implication.) In mathematical proofs we can no more dispense with the principles of Importation than we can with the Syllogism or with the principle of Contradiction. No more can we dispense with the 'obvious' principle that whenever any two propositions p and q are both true, then propositions q and p are both true, *i.e.*, the commutativity of the logical operation called 'propositional conjunction.' And similarly with a host of other principles used implicitly by every logician and mathematician.

May not some of these principles be derivable from some of the others? Certainly. The long train of researches, culminating in the mathematical logician's 'Bible'—Whitehead and Russell's volumes of *Principia Mathematica*—has demonstrated the fact that just as the entire system of euclidean geometry can be compressed deductively into a small number of unproved geometric propositions based on a small number of undefined geometric terms, so the set of propositions that constitute the system of deductive logic can be compressed into a surprisingly small number of fundamental logical propositions based on a small number of fundamental logical terms.

But the Rome of mathematical logic was not built in a day. And the history of the long mathematical logical journey from Leibniz to Russell is a history of the triumph of the human intellect in every way as marvellous as the triumphs of the intellect in the realms of natural, as opposed to mental, experimentation.

The volume under review attempts to trace this history through its various stages. In Chapter I Professor Lewis summarizes the development of mathematical logic from Leibniz through De Morgan, Boole, Jevons, and Peirce. In Chapter II he presents the formal principles of the algebra of logic, that is, the algebra as founded by Boole and improved by Schroeder. This algebra, many of the laws of which are identical with those of number-algebras, has a 'function theory' that may well interest the student of mathematics from the standpoint of 'comparative algebra'. In Chapter III we are introduced to the well-known interpretations of this Boole-Schroeder algebra, in terms of logical classes, propositions, and relations, and also, geometrically, in terms of planar regions.[1] Ch. IV is devoted in part to an elementary and extremely successful

[1] Although Whitehead (*A Treatise on Universal Algebra*, vol. I, p. 35) characterizes this algebra as "the only known member of the non-numerical genus of Universal Algebra," mathematical readers may be interested in the following purely numerical interpretation. Let us call any positive integer, > 1, whose prime factors occur but once, a *Boolean* integer. Examples: 6, 30, 70. Choose any Boolean integer, B. Let the elements, $a, b, c, \cdots$, of this algebra be the 2^n factors of B (including 1 and B). Let $a \times b =$ H. C. F. (a, b). Then $a + b =$ L. C. M. (a, b), "0" $= 1$, "1" $= B$, "$-a$" $= B \div a$, and $a \subset b = a$ is a factor of b.'

sion proposed, on account of their statistical interests, but apparently they did not consider this revision of this book the time or place to do so.

In general the book is substantially improved and although it does not blaze out any new paths, seems well adapted to the needs of those who hold to the established system of a rather careful and extensive course in college algebra before analytic geometry and calculus are begun.

R. W. BURGESS.

BROWN UNIVERSITY,
February, 1920.

Lectures on the Theory of Plane Curves delivered to Post-Graduate Students in the University of Calcutta. By SURENDRAMOHAN GANGULI. 2 parts. Calcutta, University of Calcutta, 1919. 8vo. Part 1, 10 + 1–138b pp.; Part 2, 13 + 139–350 pp. + 17 plates.

Preface: "The present volume . . . is intended as an introductory course suitable for advanced students of geometry and assumes scarcely any further knowledge of analysis on the part of the reader than is to be found in most of the ordinary text-books on differential calculus and on analytical geometry. Throughout the whole work I have endeavored to give prominence to geometrical methods, as it appears that geometry, in judicious combination with analysis, is likely to simplify otherwise tedious and lengthy investigations. With this end in view, Professor Sylvester's Theory of residuation has been introduced at the outset and occasional application of the principle has been found very useful. I have carefully avoided complicated forms of equations but at times they have been found indispensable.

In teaching the subject constant recourse has been had to the classic treatises of Salmon and Clebsch, and the works of Basset, Scott, and others, have been frequently consulted. My obligations to these authors, which are probably much greater than I am aware of, are gratefully acknowledged. I am indebted on almost every page to the great work of Salmon on *Higher Plane Curves* and it is impossible to record in detail my obligations to this inspiring writer. The English edition of the great work of Salmon has been long out of print and it has been found necessary to publish a new text-book based on modern methods, with a view to remove the inconvenience experienced by the English-knowing students of the University."

Contents—Chapter I: Introduction, 1–10; II: Theory of plane curves, 11–32; III: Singular points on curves, 33–50; IV: Polar curves, 51–65; V: Covariant curves—the hessian, 66–81; VI: Polar reciprocal curves, 82–95; VII: Foci of curves, 96–103; VIII: The analytical triangle—asymptotes, 104–125; IX: System of curves, 126–138; X: Curves of the third order—cubic curves, 139–157; XI: Harmonic properties of cubic curves, 158–168; XII: Canonical forms, 169–203; XIII: Unicursal cubics, 204–215; XIV: Special cubics, 216–231; XV: Invariants and covariants of cubic curves, 232–243; XVI: Curves of the fourth order—quartic curves, 244–266; XVII: Trinodal quartics, 267–275; XVIII: Bicircular quartics, 276–305; XIX: Circular cubics as degenerate bicircular quartics, 306–315; XX: Special quartic curves, 316–328; Appendices, 329–345.

NOTES.

In *Historical Portraits* 1700–1850. *The Lives by C. R. L. Fletcher. The Portraits chosen by Emery Walker* (Part I, 1780–1800, Oxford, Clarendon Press, 1919) there are portraits of Newton and Halley. "Sir Isaac Newton," 29–33; opposite page 30 is a reproduction of the painting by John Vanderbank, at Trinity College, Cambridge. Edmund Halley, 33–36; opposite page 34 is a reproduction of the painting by Thomas Murray at Queen's College, Oxford.

The opening article (16 pages) of the second volume (1920) of *Norsk Matematisk Tidsskrift*, the organ of the Norwegian Mathematical Society, is a lecture on Evariste Galois delivered by the late Ludvig Sylow.

For 1920 the first A-number of *Matematisk Tidsskrift*, the organ of the Mathematical Society of Copenhagen, contains a six-page sketch by C. Juel of Hieronymus Georg Zeuthen (1839–1920) and an interesting portrait of him taken in 1880; in the first B-number is an article by T. Bonnesen on ancient and modern theories of irrationality.

Bulletin de la Société Mathématique de Grèce I, 1—*Δελτίον τῆς Ἑλληνικῆς Μαθηματικῆς Ἑταιρείας*, Τόμος Α′, Τεῦχος Α′, was published at Athens (Π. Α. *Πετράκου*) in May, 1919. The second number published in December completed the volume of 186 pages, rather more than a third of which is in French, the rest being in Greek. The four parts of each number are devoted to (*a*) the proceedings of the Society; (*b*) mathematical papers of a purely scientific nature; (*c*) papers in their essence philosophic, didactic, etc.; (*d*) miscellaneous mathematical news. The editors are G. Rémoundos, P. Zervos, N. Sakellarios and K. Lambiris. The scientific papers in the first number are entitled: "Formules fondamentales relatives aux courbes d'un couple de surfaces," "Les séries divergentes par le calcul des probabilités," "Sur l'équivalence des systèmes d'équations différentielles," "Sur quelques rémarques relatives aux théories de l'intégration des systèmes en involution du second ordre," "Sur les formes bilinéaires," and "*Περὶ μηχανικῶν ἀναλλοιώτων*."

A sumptuous volume by Mr. JAY HAMBIDGE entitled *Dynamic Symmetry, the Greek Vase*, was published last May by the Yale University Press (178 line-cuts; 161 pages, small folio; price $6.00). "Dynamic symmetry deals with commensurable areas which represent the projection of solids. The symmetry of man and plant is dynamic; the symmetry of the entire fabric of classic art, including buildings, statuary and the products of all of the crafts, is dynamic. The symmetry of art since classic times is *static*.

"So revolutionary are the discoveries made by Mr. Hambidge, so tremendous will be their effect on the fundamental rules of artistic expression, that the world of art is roused to a high pitch of interest. The principles of 'dynamic symmetry',[1] have now been adopted by many craftsmen, designers and a number of important advertising illustrators." These principles are based upon golden section and the logarithmic spiral form. Professor R. C. ARCHIBALD contributes "Notes on the logarithmic spiral, golden section and the Fibonacci series," pages 146–157,—an extensive elaboration of his notes which appeared in this MONTHLY *1918*, 189–193, 232–238.

ARTICLES IN CURRENT PERIODICALS.

AMERICAN JOURNAL OF MATHEMATICS, volume 42, no. 1, January, 1920 [published March, 1920]: "Groups of order $2^{m'}$ which contain a relatively large number of operators of order 2" by G. A. Miller, 1–10; "The Green's function for a plane contour" by H. D. Frary, 11–25; "On the solution of certain types of linear differential equations in infinitely many variables" by W. G. Simon, 27–46; "Periodic orbits on a surface of revolution" by D. Buchanan, 47–75.

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Brown Alumni Monthly, Brown University, volume 20, May, 1920: "Henry Parker Manning and the development of mathematics at Brown" by R. C. Archibald, 183–185.

Bulletin des Sciences Mathématiques, volume 44, January, 1920: Review by E. Picard of "Mémoire sur certains nombres invariants qui se présentent dans la théorie des multiplicités algébriques" by S. Lefschetz, 5–7 [From *Comptes Rendus*, volume 169, see *1920*, 143]; Review by P. Drouin of P. Boutroux's *Les Principes de l'analyse mathématique*, tome 2 (Paris, 1919), 16–20.

Bulletin of the American Mathematical Society, volume 26, no. 6, March, 1920: "The twenty-sixth annual meeting of the American Mathematical Society" by F. N. Cole, 241–259; "The St. Louis meeting of the American Mathematical Society" by O. D. Kellogg and A. Dresden, 260–273; "Poncelet polygons in higher space" by A. A. Bennett, 274–275; "On the rectifiability of a twisted cubic" by Mary F. Curtis, 275–277; "Note on linear differential equations of the fourth order whose solutions satisfy a homogeneous quadratic identity" by C. N. Reynolds, Jr., 277–280; "An acknowledgment of priority" by A. A. Bennett, 280–281; "Dickson's *History of the Theory of Numbers*" by D. N. Lehmer, 281; "Notes," 281–285; "New publications," 285–288.

Bulletin of the Calcutta Mathematical Society, volume 10, no. 3, December, 1919: "On a special square matrix of order six" by C. E. Cullis, 127–140; "On the formation of optical images by a diffracting boundary" by B. C. Das, 141–150; "On Joachimsthal's attraction problem" by S. C. Dhar, 151–156; "On the potentials of heterogeneous incomplete ellipsoids and elliptic discs" by N. Sen, 157–178; "On the wave-equation in ellipsoidal coördinates" by S. Banerji, 179–186; "On the numerical calculation of the roots of the equations $P_n^m(\mu) = 0$ and $\dfrac{dP_n^m(\mu)}{d\mu} = 0$ regarded as equations in n," Part 2, by B. Pal, 187–194.

Hibbert Journal, London, volume 18, no. 3, April, 1920: "Euclid, Newton, and Einstein" by C. D. Broad, 425–458 [Last paragraph: "I have now fulfilled my promise to the best of my ability. We have seen what exactly Einstein's theory is and how it is related to Euclidean geometry and to Newtonian mechanics. The connection with the former is not really very intimate, and Einstein himself makes very little play with it. The connection with the latter is all-important. Einstein's discovery synthesizes Newton's two great principles—the laws of motion and the law of gravitation. It removes the obscurity that has always hung over the former, by working out the relativity of motion to the bitter end, whilst it generalizes and slightly corrects the latter and accounts for its peculiar position among all the other laws of nature. Such work can only be done by a man of the highest scientific genius, and we have no right and no need to enhance his greatness by decrying the immortal achievements of his predecessors. It is enough that we can, without the slightest flattery or hyperbole, class Einstein with Newton, and say of the former what is written on the tomb of the latter:—'Sibi gratulentur homines tale tantumque exstitisse humani generis decus.'"]

Journal of Education, Boston, volume 90, December 25, 1919: "Concrete geometry for seventh grade" by W. H. Fletcher, 654–657.

Mathematical Gazette, volume 10, March, 1920: The annual meeting of the Mathematical Association, 17–19; "Gleanings far and near," 19, 29, 34; "Geometry teaching: the next step" by C. Godfrey, 20–24; "Convention and duplexity in elementary mathematics" by E. H. Neville, 25–26; "The position of common logarithms in mathematical training," by H. M. Cook, 27–28 [followed by discussion, 28–29]; "The teaching of mechanics to beginners" by R. C. Fawdry, 30–34; "The graphical treatment of differential equations" (continued) by S. Brodetsky, 35–38; "Coördinate geometry in schools" by W. J. Dobbs, 39–41; "Obituary, G. W. Palmer," 42 [Mr. Palmer's death was recorded in the Monthly, *1920*, 43. Quotations: "We have heard it said that he well deserved to be called the 'Father of Arithmetic' in English education. . . . By the stress we have laid on his contributions to the literature of one branch of elementary mathematics, we do not wish to imply that his work was thereby limited. It may be said, indeed, that he inaugurated a new era in the teaching of mathematics at Christ's Hospital]; "Mathematical Notes" by G. H. Bryan ('A formal geometrical construction for the solution of the sound ranging problem'), A. O. P. ('A curiosity'), T. Carleman (Chances in winning a game at lawn tennis), and W. E. H. Berwick ('The four fours'), 43 [The curiosity is: $\frac{18534}{9267} \times \frac{17469}{5823} = \frac{34182}{5697}$. Here the set of digits occurs in each fraction, each digit once and only once]; Reviews and notices, books received, contents of journals, etc., index to volume 9, 44–48 + 14 pp.

Messenger of Mathematics, volume 49, no. 3, July, 1919: "Factorisation of N & $N' = (x^n \mp y^n) \div (x \mp y)$, &c. [when $x - y = 1$]" (continued) by A. Cunningham, 33–36; "The eliminant of two binary quantics with determinantal coefficients" by T. Muir, 37–41; "On certain plane configurations of points and lines" by W. Burnside, 41–43; "A property of groups of even order" by W. Burnside, 43; "On the solution of a cubic equation" by A. Lodge, 44–48. No. 4, August: "On the solution of a cubic equation" (continued) by A. Lodge, 48–51; "On uniform Diophantine approximation" by H. T. J. Norton, 51–57; "Standard relation of Legendre's functions" by R. Hargreaves, 58–62; "Note on the mth compound of a determinant of the $(2m)$th order" by T. Muir, 62–64.

Nature, volume 105, March 18, 1920: "Mathematics: pure and applied" by S. Brodetsky, 64–67 [reviews of F. Slate's *The Fundamental Equations of Dynamics and its Main Co-ordinate Systems Vectorially Treated and Illustrated from Rigid Dynamics* (Berkeley, 1918), of L. Silberstein's *Projective Vector Algebra: An Algebra of Vectors Independent of the Axioms of Congruence and of Parallels* (London, 1919), of E. S. Andrews's *Elements of Graphic Dynamics* (London, 1919), of C. Davison's *Differential Calculus for Colleges and Secondary Schools* (London, 1919) and of J. Milne's *The Analytical Geometry of the Straight Line and the Circle* (London, 1919)]; "Some methods of approximate integration and of computing areas" by A. C. Percival, 70–71; "Time-reckoning of the North American Indians," 75; "The gyrostatic compass" by S. G. Brown, 77–80.—March 25: "Aeronautical research" 95–97 [Review of L. Bairstow's *Applied Aerodynamics* (London, 1920).]—April 1; "Some methods of approximate integration and of computing areas" by R. A. P. Rogers, 138 ["The formulæ which Mr. Percival gives in *Nature* for March 18 for approximate integration are well known, but there are one or two points in connection with them which are frequently overlooked, especially by writers of books on mathematics for engineers . . ."]; "Gravitational deflection of high-speed particles" by H. G. Forder, 138 ["The result mentioned by Mr. Leigh Page and verified by Prof. Eddington (*Nature*, March 11, p. 37), that the gravitational effect on a particle travelling radially is a repulsion if the speed exceeds $1\sqrt{3}$ times the light-velocity, is given by Hilbert in the *Göttinger Nachrichten* for 1917. The same paper contains interesting remarks on the path of a particle or light-pulse moving spirally round the gravitation centre."]—April 8: "Recent mathematical books" by J. M., 162–163 [review of Karpinski, Benedict, and Calhoun's *Unified Mathematics* (Boston, 1918), C. H. P. Mayo's *Elementary Calculus* (London, 1919), J. W. Angles's *Mensuration for marine and mechanical engineers* (London, 1919), W. G. Borchardt's *School Mechanics*, Part 1. *School Statics* (London, 1919).]—April 15: "Matrices" by G. B. M., 191–192 [review of C. E. Cullis's *Matrices and Determinoids* (Cambridge, 1918)]; "A dynamical specification of the motion of Mercury" by G. W. Walker, 198–199.—April 22: "Gravitational deflection of high-speed particles" by L. Page, 233.—April 29: "Critical mathematics" by G. B. M., 256–267 [review of P. Boutroux's *Les principes de l'analyse mathématique; exposé historique et critique* (Paris, 1919)]; "Artillery science" by G. Greenhill, 268–270; "Courses on the history of science," 279.—May 6: "Euclid's Elements" by G. B. M., 288–289 [review of T. L. Heath's *Euclid in Greek, Book I* (Cambridge, 1920)]; "Leonardo de Vinci" by E. McCurdy, 307–309.—June 17: Review of W. W. Smith's *A Theory of the Mechanism of Survival: The Fourth Dimension and its Applications* (London, Kegan, Paul, Trench, 1920), 484; "A new method for approximate evaluation of definite integrals between finite limits" by T. Y. Baker, 486; "S. Ramanujan, F.R.S." by G. H. Hardy, 494–495 [Quotations: "I first heard of Ramanujan in 1913. The first letter which he sent me was certainly the most remarkable that I have ever received. There was a short personal introduction written, as he told me later, by a friend. The body of the letter consisted of the enunciations of a hundred or more mathematical theorems. Some of the formulæ were familiar, and others seemed scarcely possible to believe. A few (concerning the distribution of primes) could be said to be definitely false. There were no proofs, and the explanations were often inadequate. . . . Whatever reservations had to be made, one thing was obvious, that the writer was a mathematician of the highest quality, a man of altogether exceptional originality and power. . . . Ramanujan's activities lay primarily in fields known only to a small minority even among pure mathematicians—the applications of elliptic functions to the theory of numbers, the theory of continued fractions, and perhaps above all the theory of partitions. His insight into formulæ was quite amazing, and altogether beyond anything I have met with in any European mathematician. It is perhaps useless to speculate as to his history had he been introduced to modern ideas and methods at sixteen instead of at twenty-six, It is not extravagant to suppose that he might have become the greatest mathematician of his time. What he did actually is wonderful enough. Twenty years hence, when the researches which his work has suggested have been completed, it will probably seem a good deal more wonderful than it does to-day."]

Nouvelles Annales de Mathématiques, volume 78, December, 1919: "Avis" by les rédacteurs, 441–443 [Quotations: ". . . Nous avons ainsi été conduits à concevoir une véritable réorganisation de ce journal, dans le sens que nous allons indiquer. Chaque numéro des *Nouvelles Annales* contiendra, en principe, des *Mémoires originaux,* des *Exercices de Licence et d'Agrégation,* une *Chronique du mouvement mathématique,* des *Enonces et des solutions de questions* . . . Dans la *chronique* . . . nous publierons des nouvelles intéressant le monde des mathématiciens, telles que nominations, distinctions, ouvertures de cours importants, des résumés plus ou moins détaillés de découvertes récentes (sans qu'il s'agisse d'un dépouillement systématique des périodiques, travail pour lequel il exist des publications auquelles les *Nouvelles Annales* ne prétendent pas se substituer); des analyses bibliographiques, etc."]; "Sur les équations de Didon" by P. Humbert, 443–451; "Sur le cercle de Miquel" by F. Girault, 452–456; "Sur les surfaces tétraédrales symétriques" by C. Servais, 456–468; Questions and solutions, 468–472; Index, 473–480.—Volume 79, January, 1920: "Exposé élémentaire d'une théorie rigoureuse des liaisons finies unilatérales" by E. Delassus, 1–12; "Simple remarque sur la cyclide de Dupin" by M. d'Ocagne, 13–14; "Lieux des foyers ordinaires des courbes algébriques d'un faisceau tangentiel ou ponctuel" by T. Lemoyne, 14–17; Licence questions 17–30; Chronique, 31–35; Questions and solutions, 35–40.

Pedagogical Seminary, volume 26, no. 4, December, 1919: "Relation of initial ability to the extent of improvement in certain mathematical traits" by F. M. Phillips, 330–355.

Philosophical Transactions of the Royal Society of London, series A, vol. 220, no. A579, April 27, 1920: "A determination of the deflection of light by the sun's gravitational field, from observations made at the total eclipse of May 29, 1919" by F. W. Dyson, A. S. Eddington, and C. Davidson, 291–333 + plate 1.

Proceedings of the American Academy of Arts and Sciences, volume 55, no. 3, March, 1920: "Contribution to the general kinetics of material transformations" by A. J. Lotka, 135–154.—No. 4, March: Rotations in space of even dimensions" by H. B. Phillips and C. L. E. Moore, 155–188.

Proceedings of the National Academy of Sciences, volume 6, no. 2, February, 1920: "Groups generated by two operators, S_1, S_2, which satisfy the conditions $S_1^m = S_2^n$, $(S_1S_2) = 1$, $S_1S_2 = S_2S_1$" by G. A. Miller, 70–73; "The larger opportunities for research on the relations of solar and terrestrial radiation" by C. G. Abbott.—March: "Note on geometrical products" by C. L. E. Moore and H. B. Phillips, 155–158.

Proceedings of the Royal Society, volume 97, no. A683, April 15, 1920: "A new apparatus for drawing conic curves" by A. F. Dufton, 199–201 [Quotations: "1. The attention of mathematicians has been attracted to the mechanical description of conic sections since the discovery of the curves by Menaechmus, but in the numerous mechanisms which have been invented only partial success has been attained.

"In an early conograph, the invention of which is ascribed to the Arabs, the curve is the actual intersection of the surface upon which it is drawn with a straight line generating a cone. Instruments of this kind were designed at the end of the sixteenth century for use in the construction of sun dials.

"Newton in his 'Principia' (Lib. I, Prob. XIV) discusses the drawing of conic sections and describes a mechanical method of plotting them. Two angles are rotated about their vertices and the intersection of one pair of arms is kept upon a fixed straight line. The locus of intersection of the other pair is a conic section.

"Sylvester (*Proc. Roy. Inst.*, vol. 7, p. 179 (1873–75)) showed that a conic can be drawn by means of an apparatus of thirteen links. His method fails to draw the curve at the vertex but is simpler than that of Peaucellier (ibid.), which involves the use of fifteen links besides a cross-piece rigidly attached to one of them. Both these methods depend upon the principle of inversion.

"A conograph based upon the constancy of the anharmonic ratio subtended at the tracing point by four fixed points on the curve was invented by Willy Jürges (*Zeitschrift für Math. und Physik,* vol. 38, p. 350 (1893)). In this instrument the use of eight sliding constraints makes smooth work difficult.

"2. With the apparatus described in this paper, the conic is drawn as the polar reciprocal of a circle. . . . In the remarkable precision of even a roughly made instrument, in the tracing of the curve at one sweep and in the application to all conics from circle to straight line, the apparatus offer a satisfactory solution to a very ancient problem".

University of California Publications in Mathematics, volume 1, no. 12, April 12, 1920: "A set of five postulates for Boolean algebras in terms of the operation 'exception'" by J. S. Taylor, 241–248.

Zeitschrift für mathematischen und naturwissenschaftlichen Unterricht, volume 50, nos. 11–12, published January 10, 1920: "Zur elementaren Behandlung von Exponentialfunktion und Logarithmus" by A. Loewy, 330–339; "Bücherbesprechungen," 340–347—Volume 51, no. 1, published February 5: "Die Simsonsche Gerade" by R. Henke, 1–12; Lektorate für Mathematik, ein Vorschlag zur Erweiterung des mathematischen Hochschulunterrichts" by A. Rohrberg, 13–17; "Kleine Mitteilungen," 18–24.

AMERICAN DOCTORAL DISSERTATIONS.

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UNDERGRADUATE MATHEMATICS CLUBS.

Edited by U. G. Mitchell, University of Kansas, Lawrence.

NOTES AND SUGGESTIONS.

If our readers know of the formation or existence of undergraduate mathematics clubs which have not been mentioned in this department we would be glad to have them send us full information concerning their organization and activities. We hope, too, that clubs which plan their programs for the year soon after the opening of college will send us copies without waiting for a request from the editor. The issuing of a printed folder in October or November giving information about the club as well as times, places and programs for meetings during the year seems to contribute decidedly to the success of the work and the number of clubs which are adopting the plan is increasing. A sample of such a folder will be sent by the editor of the department to any one requesting it.

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In response to our request published sometime ago for suggestions which might be used in social meetings, Mr. John W. Arnold, Treasurer of the Undergraduate Mathematics Club at the University of Illinois, sends the following account of the annual social meeting of their club held Friday evening, March 12, 1920.

The company of about eighty persons was numbered and tagged upon arrival and later divided by means of the last digit of the number into five groups—followers of Newton, Descartes, Euclid, Leibniz, and Pythagoras—who arranged

University of California Publications in Mathematics, volume 1, no. 12, April 12, 1920: "A set of five postulates for Boolean algebras in terms of the operation 'exception'" by J. S. Taylor, 241–248.

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themselves around separate tables and vied with each other for the honors of the evening. As a beginning, the several parties, by turns, presented charades representing mathematical terms which were guessed with more or less difficulty by the other groups. Music followed and then some time was spent at work on a motley assortment of odd, unusual, and unique problems of a generally baffling nature. All the old and a number of new tricks, puzzles and paradoxes were brought out and debated within the groups, solutions being presented to "headquarters" for verification. For the remainder of the evening the gathering puzzled over mathematical anagrams. "Geometrical refreshments" were served in the form of gelid spheres inscribed in right circular cones in quantities approaching ∞.

CLUB ACTIVITIES.

Mathematics Club of the University of Nebraska, Lincoln. [*1918*, 313, 459; *1919*, 263–264.]

The officers of the club for the year 1919–20 were: President, Josiah A. Brooks '20; vice-president, Stella Abraham '21; secretary, Benjamin F. Margolin '21.

October 27, 1919: "The classification of real numbers" by Professor William C. Brenke.

November 13: "Curious configurations in geometry" by Professor Tracy A. Pierce; "Binomial coefficients" by Randolph T. Major '22.

January 16, 1920: "The Rhind mathematical papyrus" by Professor Albert L. Candy; "Formulæ used in artillery" by George S. Madsen '22; "How long a strip of paper would be required to write out the value of $2^{2^{36}}$, counting 10 digits per inch?", solution by Thomas G. Bowman '22.

February 12: "Horner's method of approximating roots" by Thomas G. Bowman '22; "The quadratrix" by Stella Abraham '21.

March 11: Social meeting. Professor Mayer G. Gaba gave a humorous reading "The new education in mathematics" and Miss Esther Daily furnished music. Mathematical games were played and refreshments served.

Pi Mu Epsilon, Mathematical Fraternity, Ohio State Chapter, Columbus, O.

This chapter formed a tentative organization in June and received its charter in October of 1919.

Readers of the Monthly will recall the account already given (*1918*, 271) of the Pi Mu Epsilon mathematical fraternity organized (1913) and incorporated (1914) at Syracuse, N. Y. The Ohio State chapter holds its charter from Syracuse and, like the parent chapter, it "aims to promote mathematics and scholarship." The high standards for eligibility to membership inaugurated at Syracuse have been maintained at the Ohio State University.

The active membership now numbers 39, of whom 18 are faculty members and graduate students and 21 are senior undergraduates. The list of charter members included every member of the mathematics department faculty.

The founding of the honorary mathematical fraternity was welcomed with much enthusiasm by the advanced students of mathematics. Departmental fraternities, both active and purely honorary, are numerous at Ohio State and they appear to meet a real need. Any college favored with an active chapter of Sigma Xi will readily understand the advantages of coöperative effort. Pi Mu Epsilon hopes to accomplish for the more limited field of mathematics what Sigma Xi is doing in the larger field of science. Ohio State has every intention of being an active chapter.

The officers for the year 1919–20 were: Director, Professor C. L. Arnold; vice-director, Professor Grace M. Bareis; secretary, Clarice S. Hobensack Gr.; treasurer, J. R. Anderson '20.

The following papers have been presented: "Differential symbolism in connection with series" by Professor R. D. Bohannan; "Three remarkable fractions appearing in a parameter study of certain curves of order ∞^3" by Professor C. L. Arnold; "A graphical presentation of the locus of the perihelion and aphelion points of certain satellites" by Harry F. Kohl Gr.; "Functions having preassigned zeros and infinities" by Mabel M. Madden '20; "A generalization of the law of signs" by Clarice S. Hobensack Gr.

Ohio State has, also, a strong undergraduate mathematics club which meets twice a month with an average attendance of over fifty. The officers of this club are all undergraduates and its papers and discussions lie in the field of interest of its younger members. Speaking of the two organizations our correspondent, Professor C. L. Arnold, says:

"There is a need for both the club and the fraternity. They coöperate rather than conflict. Mathematics is an elective in all the general courses. The rôle of the club is to foster interest early in the student's elective course. Pi Mu Epsilon hopes to recognize and honor exceptional work at the end of the course. Other departments do this. Surely the mathematician deserves and values an equal recognition."

THE PENTAGRAM, University of Texas, Austin. [*1918*, 273–274; *1919*, 364.]

The following programs have been given during 1919–20.

October 22, 1919: "Bees" by Professor M. B. Porter.

November 5: "Some ruler constructions" by Edison H. Thomas '20.

November 25: Social meeting at the home of Professor J. W. Calhoun.

December 3: "Mathematics and mental development" by Mary F. Decherd, Instructor in mathematics.

January 14, 1920: "Bonds" by Henry H. Hammer Gr.

January 28: "Some definite integrals by actual summation" by Joseph E. Burman Gr.

February 11: "Line coördinates" by Martha Randall '20.

March 3: "Codes and ciphers" by Renke G. Lubben '21; "Egyptian hieroglyphics" by Professor M. B. Porter.

March 10: Party at the home of Miss Goldie P. Horton, Instructor in pure mathematics, Misses Horton and Decherd hostesses.

The reason for this is that the circle ordinarily used to construct these roots is symmetrical with the X axis and thus we find the center by the simple bisection and do not have to draw this circle to find the one root that we need from this equation. In exactly the same manner M is the intersection of the X-axis and the circle with center B and radius BE. Hence we have the larger root of the equation (2),

$$OM = y_1.$$

Now erect a perpendicular to the X axis at M and lay off

$$MH = OF.$$

Thus the coördinates of the point H are (y_1, y_4). It is evident that the circle on EH as a diameter intersects the X axis so that OS and OR are the roots of equation (1) and since OR is the larger,

$$OR = 2 \cos \frac{2\pi}{17}.$$

Having twice the cosine of the angle it is an easy matter to find the required angle. Bisect OR in T and draw the unit circle VUE. Construct the angle, $\varphi = 2\pi/17$ from the known cosine. Of course UV subtending the angle φ is the required side of the regular 17-gon.

Gauss and the Regular Polygon of Seventeen Sides.

By R. C. Archibald, Brown University.

Recent publications call attention to new material[1] in connection with the history of the construction of the regular polygon of seventeen sides. The discovery that this construction could be effected with ruler and compasses only, was one of which Gauss was vastly proud[2] throughout his life and also, according to Sartorious von Waltershausen,[3] the one which decided him to dedicate his life to the study of mathematics. As it is recorded of Archimedes that he desired a sphere inscribed in a cylinder to be engraved on his tombstone, and similarly with Ludolph van Ceulen as to the value of π to 35 places of decimals, and with

[1] *C. F. Gauss als Geometer* von P. Stäckel (*Materialien für eine wissenschaftliche Biographie von Gauss*, Heft V), Leipzig, 1918 (see pp. 78, 96); *Carl Friedrich Gauss Werke*, Band 10, Abteilung 1, Leipzig, 1917 (see pp. 3–4, 120–126, 487 and the facsimile of Gauss's notebook 1796–1814).

[2] The very first entry in his notebook "1796 Mart. 30–1814 Jul. 9" is: "1796. Principia quibus innitur sectio circuli ac divisibilitas eiusdem geometrica in septemdecim partes etc. Mart. 30. Brunsv[igae]." Again, in his own copy of his *Disquisitiones Arithmeticae* he wrote the following note in the margin beside article 365: "Circulum in 17 partes divisibilem esse geometrice, deteximus 1796 Mart. 30." And finally, on page 77 of "Scheda Af," begun in 1801, Gauss brought together a number of dates which were of importance to him. The first four of these were: 1. Jan., 1801, Ceres discovered; 28. March, 1802 Pallas discovered; 19. Feb., 1803 Pallas rediscovered; 30. March, 1796, "Construction des 17 Ecks."

[3] *Gauss zum Gedächtnis*, Leipzig, 1856, p. 16: "Diese Entdeckung, welche Gauss bis zum Ende seines Lebens sehr hoch schätzte, ist es vornehmlich gewesen, welche seinem Leben eine bestimmte Richtung gegeben hat, denn von jenem Tage an war er fest entschlossen, nur der Mathematik sein Leben zu widmen."

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Jacques Bernoulli as to a logarithmic spiral, so also, according to Weber,[1] Gauss requested that the regular polygon of seventeen sides should be engraved on his tombstone.[2]

In a letter to Gerling dated January 6, 1819, Gauss elaborates the thought underlying the theory of polygons in his *Disquisitiones Arithmeticæ* (*D.A.*), Lipsiæ, 1801, through special reference to the polygon of seventeen sides; then after a paragraph generalizing the result to suitable prime numbers $p = 3, 5, 17, 257, 65537, \ldots$ he continues:

"The history of this discovery has up to the present nowhere been publicly alluded to by me; I can give it very exactly, however. The day was the March 29, 1796,[3] and chance had absolutely nothing to do with it. Before this, indeed during the winter of 1796 (my first semester in Göttingen), I had already discovered everything related to the separation of the roots of the equation

$$\frac{x^p - 1}{x - 1} = 0$$

into *two* groups, on which the beautiful theorem on the lower part of page 637[4] depends, without making note of the day. After intensive consideration of the relation of all the roots to one another on arithmetical grounds, I succeeded during a holiday in Braunschweig, on the morning of the day alluded to (before I had got out of bed), in viewing this relation in the clearest way, so that I could immediately make special application to the 17-side and to the numerical verification. Of course still other investigations of the seventh section of the *D.A.* were added later. I announced this discovery in the *Literaturzeitung* of Jena where my advertisement was published in May or June 1796.[5] The printing of my *Disq. Arith.* began in April 1798, continued slowly and was several times entirely stopped (because the printer moved away from Braunschweig, whence from sheet *R* on another type is used) and was completed in the summer of 1801.

"In the year 1798 or 1799 v. Zimmerman told something of my research to a certain Hugene or Huguenot (a Prussian officer) passing through Braunschweig. On his request I gave him a little paper, which he kept, containing the complete theory of the 17-side (about as above, but much more in detail). This person had the impudence afterwards to publish a work,[6] which I have not myself seen,

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[1] *Encyclopädie der elementaren Algebra und Analysis* bearbeitet von H. Weber. Zweite Auflage, Leipzig, 1906, p. 362.

[2] This request was not granted, as it was in each of the other cases mentioned.

[3] At this time Gauss was eighteen years of age, but was nineteen on the following April 30.

[4] Gauss, *Werke*, Band I, p. 443.

[5] *Intelligenzblatt* of the *Allgemeine Literatur-Zeitung*, Nr. 66, 1 Junius, 1796, col. 554. Two mistakes in connection with this reference are made by Klein, "Gauss' Wissenschaftliches Tagebuch 1796–1814 mit Anmerkungen herausgegehen" (*Mathematische Annalen*, Band 57, 1903); his reproduction of Gauss's advertisement is also inexact. Gauss's "Tagebuch" is reprinted in facsimile in *Gauss Werke*, Band 10. The last sentence of Gauss's advertisement is as follows: "Diese Entdeckung ist eigentlich nur ein Corollarium einer noch nicht ganz vollendeten Theorie von grösserem Umfange, und sie soll, sobald diese ihre Vollendung erhalten hat, dem Publicum vorgelegt werden."

[6] *Mathematische Beyträge Zur Weiteren Ausbildung angehender Geometer*, von dem K. Preuss. Hauptmann im Feld-Artillerie Corps v. Huguenin, Königsberg, bey Goebbels & Unger, 1803.

While Gauss refers to two earlier synthetic constructions of Paucker,[1] as the only previous ones which he knew had been publicly discussed, he remarks that that of Erchinger is "different and carried through more in the spirit of pure geometry."

Stäckel reviews the discussions of the division of a circle into n equal parts, by means of imaginary quantities and the solution of the equation $x^n - 1 = 0$, by Cotes,[2] De Moivre,[3] Euler,[4] and Vandermonde,[5] and the development of their ideas by Gauss in connection with the problem of constructing a regular inscribed polygon.

PROBLEMS FOR SOLUTION.

[N.B. The editorial work of this department would be greatly facilitated if, on sending in problems, the proposers would also enclose their solutions—*when they have them.* If a problem proposed is not original the proposer is requested *invariably* to state the fact and to give an exact reference to the source.]

2843. Proposed by E. H. MOORE, University of Chicago.

Show that the maximum of the absolute value of

$$2(a + ib)(x + iy) + i(a + ib)(z + iw) + i(c + id)(x + iy),$$

where $i = \sqrt{-1}$ and a, b, c, d, x, y, z, w are real numbers for which

$$a^2 + b^2 + c^2 + d^2 = x^2 + y^2 + z^2 + w^2 = 1,$$

is $1 + \sqrt{2}$. Study the locus of point-pairs $P = (a, b, c, d)$, $Q = (x, y, z, w)$ of the unit-sphere in real four-space for which this absolute value assumes its maximum value.

2844. Proposed by J. L. RILEY, Stephenville, Texas.

Decompose into simple fractions the number $\frac{6099380351}{1271808720}$ (Gauss, *Disq. Arith.*, *Werke*, vol. 1, pp. 386–387).

2845. Proposed by E. L. POST, Princeton University.

Prove that if y_x is a solution of the functional equation

$$y_x = \frac{y_{x+1}^2}{x} + y_{x+1}$$

for positive integral values of x with $y_x > 0$, then

$$\lim_{x \to \infty} y_x \log x = 1.$$

2846.

Find the entire volume within the surface $x^{1/2} + y^{1/2} + z^{1/2} = a^{1/2}$. (W. A. Granville, *Elements of Differential and Integral Calculus*, revised ed., 1911, p. 420.)

This equation, rationalized, is the equation of Steiner's quartic surface, every tangent plane to which cuts it in two conics. (Cf. Salmon-Rogers, *Analytic Geometry of Three Dimensions*, 5th ed., vol. 2, 1915, pp. 171, 201, 207, 213f. Also C. M. Jessop, *Quartic Surfaces* 1916, Chapter 7.)

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[1] (1) "Geometrische Verzeichnung des regelmässigen 17-Ecks und 257-Ecks in d. Kreis," *Jahresverhandl. d. kurländischen Gesellschaft für Litteratur und Kunst*, Mitau, Band 2, 1822. (2) *Die ebene Geometrie der geraden Linie und des Kreises.* Königsberg, 1823, p. 187. Paucker is also the author of: (3) *De divisione geometrica peripheriae circuli in XVII partes æquales*, Königsberg, 1817.

[2] R. Cotes, *Harmonia mensurarum, sive analysis et synthesis per rationum et angulorum mensuras promota.* Cambridge, 1722.

[3] A. De Moivre, *Miscellanea analytica*, London, 1730.

[4] L. Euler, *Introductio in analysin*, Lausanne, 1748, especially t. 1, cap 8: De quantitatibus transcendentibus ex circulo ortis.

[5] C. A. Vandermonde, "Remarques sur les problèmes de situation," *Histoire de l'Acad.*, année 1771, Paris 1774; *Mémoires*, p. 566. "Sur la résolution des équations," p. 365.

2847. Proposed by B. F. FINKEL, Drury College.

Convert $+\sqrt{R^2 - x^2}$ into a Fourier series.

2848. Proposed by the late L. G. WELD.

A particle is attracted by a finite, uniform, material right line. Define its path, considering: (*a*) that the path is the envelope of the instantaneous lines of resultant attraction, as when the particle moves in a highly viscous medium (*i.e.*, without inertia); (*b*) that the particle moves freely (with inertia).

2849. Proposed by S. A. COREY, Des Moines, Iowa.

In the *Annals of Mathematics*, for April, 1911, Professor Byerly has given a method of approximately representing $f(x)$ in terms of $F_1(x)$, $F_2(x)$, $\cdots$, $F_r(x)$, in an interval $x_0 \leqq x \leqq x_1$. If $F_n(x) = x^n$ and if the mth derivative of $f(x)$ is zero when $x = 0$ for $m > r$, it is evident that Byerly's development becomes exact, and, therefore, identical with Maclaurin's development in the interval $0 \leqq x \leqq x_1$, provided $f(0) = 0$. If $f(0) \neq 0$, it is necessary to replace $f(x)$ by $f(x) - f(0)$ in Byerly's development. The coefficient of x^n in this development is D_n/D, where the D_n and D are certain determinants with elements of the type

$$A_n = \frac{x_1^{2n+1}}{2n+1}, \qquad B_{s,t} = \frac{x_1^{s+t+1}}{s+t+1}, \qquad C_n = \int_0^{x_1} f(x)x^n dx.$$

Prove that as r becomes infinite, D_n/D approaches $f^{[n]}(0)/n!$, the corresponding coefficient in the Maclaurin development, whenever $f(x)$ is analytic, and that any value of x within the range of convergence of the Maclaurin development may be substituted for x_1 without altering the value of D_n/D.

SOLUTIONS OF PROBLEMS.

416 (Algebra) [1914, 156; 1919, 312] Proposed by C. E. FLANAGAN, Wheeling, W. Va.

The sides of a given rectangle are a and b in which a rectangle is to be inscribed one of whose sides is c. Find the other side, using Euler's rule for quartics.

I. Solution and Discussion by Otto Dunkel, Washington University.

Let the given rectangle be $RSS'R'$ such that $RS = R'S' = a$ and $RR' = SS' = b$, and let $PP'Q'Q$ be a rectangle inscribed in it such that $QP = Q'P' = c$ and $PP' = QQ' = x$. Suppose further that the vertex P lies on RS, the vertex P' on SS' and so on in order. Let $SP = m$ and $SP' = n$; then from the four similar triangles in the figure, pairs of which are equal, we have at once

$$(1) \qquad \frac{n}{x} = \frac{a-m}{c}, \qquad \frac{m}{x} = \frac{b-n}{c}, \qquad n^2 + m^2 = x^2,$$

and from the first two equations follow

$$(2) \qquad \begin{aligned} n + m &= \frac{(a+b)x}{c+x}, & n - m &= \frac{(a-b)x}{c-x}, \\ n(c^2 - x^2) &= (ac - bx)x, & m(c^2 - x^2) &= (bc - ax)x. \end{aligned}$$

Squaring and adding the equations in the first line of (2) and using the third equation in (1) we obtain

$$(3) \qquad 2 - \left(\frac{a+b}{c+x}\right)^2 - \left(\frac{a-b}{c-x}\right)^2 = 0.$$

It will be seen that the form of the equation above is convenient for separating the roots. On clearing of fractions it becomes

$$(3') \qquad x^4 - (a^2 + b^2 + 2c^2)x^2 + 4abcx - c^2(a^2 + b^2 - c^2) = 0.$$

The other side x of the inscribed rectangle must satisfy this equation, but in order to say that a given root of this equation determines the remaining side of an *inscribed* rectangle it must be shown that this root gives suitable values of m and n for such a rectangle.

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The sides of a given rectangle are a and b in which a rectangle is to be inscribed one of whose sides is c. Find the other side, using Euler's rule for quartics.

I. Solution and Discussion by Otto Dunkel, Washington University.

Let the given rectangle be $RSS'R'$ such that $RS = R'S' = a$ and $RR' = SS' = b$, and let $PP'Q'Q$ be a rectangle inscribed in it such that $QP = Q'P' = c$ and $PP' = QQ' = x$. Suppose further that the vertex P lies on RS, the vertex P' on SS' and so on in order. Let $SP = m$ and $SP' = n$; then from the four similar triangles in the figure, pairs of which are equal, we have at once

$$\frac{n}{x} = \frac{a-m}{c}, \qquad \frac{m}{x} = \frac{b-n}{c}, \qquad n^2 + m^2 = x^2, \tag{1}$$

and from the first two equations follow

$$\begin{gathered} n + m = \frac{(a+b)x}{c+x}, \qquad n - m = \frac{(a-b)x}{c-x}, \\ n(c^2 - x^2) = (ac - bx)x, \qquad m(c^2 - x^2) = (bc - ax)x. \end{gathered} \tag{2}$$

Squaring and adding the equations in the first line of (2) and using the third equation in (1) we obtain

$$2 - \left(\frac{a+b}{c+x}\right)^2 - \left(\frac{a-b}{c-x}\right)^2 = 0. \tag{3}$$

It will be seen that the form of the equation above is convenient for separating the roots. On clearing of fractions it becomes

$$x^4 - (a^2 + b^2 + 2c^2)x^2 + 4abcx - c^2(a^2 + b^2 - c^2) = 0. \tag{3'}$$

The other side x of the inscribed rectangle must satisfy this equation, but in order to say that a given root of this equation determines the remaining side of an *inscribed* rectangle it must be shown that this root gives suitable values of m and n for such a rectangle.

From either (1) or (2) it follows that, if $x = c$, $a = b$. The case $a = b$ is easily disposed of, and the following results may be verified by the above equations. If $x \neq c$, $m = n$ and roots are $x_1 = \sqrt{2}\cdot a - c$, $x_2 = -(\sqrt{2}\cdot a + c)$. If $c < \sqrt{2}\cdot a$, x_1 gives the other side of an inscribed rectangle, which becomes a square if $c = a/\sqrt{2}$ and hence $m = n = a/2$. If $c > \sqrt{2}\cdot a$, then m, n and x_1 are negative and there is, of course, no inscribed rectangle, but there is a rectangle satisfying the analytical conditions, P and P' lying, respectively, on the prolongations of RS and of $S'S$. Also x_2 gives the side of a rectangle with P and P' on the prolongations of SR and SS'. For the double root of (3′), $x = c$, it follows that $n + m = a$. For each value of c such that $a \geqq c \geqq a/\sqrt{2}$, neither m nor n is negative and there is an inscribed square. If $c > a$, m and n have opposite signs and the square is not inscribed but the vertices P and P' are on the prolongations of SR and of $S'S$, respectively. The last two figures are the familiar figures in the proofs of the Pythagorean Theorem.

It will now be assumed that $a > b > 0$. If $c^2 \geqq a^2 + b^2$ it will be obvious that there are no inscribed rectangles. The roots of the equation, which are all real, are easily separated by the method given below for the case $c > a$. It is also easy to determine the positions of the four rectangles. In what follows it will be assumed that $c^2 < a^2 + b^2$.

The equation (3′) may be solved by any of the usual methods and there is no difficulty in writing out the expression for the roots other than that of the length and complication of the final result, and besides the result would be neither interesting nor useful. Important functions of the coefficients will be given and by the aid of these the roots may be obtained. Using the notation in Burnside and Panton's *Theory of Equations*, Vol. 1, (1904), page 121, we find for the reducing cubic of (3′)

$$4\theta^3 - I\theta + J = 0 \tag{4}$$

the following values for the quantities involved

$$H = -\frac{1}{6}(a^2 + b^2 + 2c^2), \qquad I = \frac{1}{12}(a^2 + b^2 - 4c^2)^2, \tag{5}$$
$$J = \frac{1}{216}[(a^2 + b^2 - 4c^2)^3 + 54c^2(a^2 - b^2)^2].$$

The substitution $\theta = t + H$ gives Euler's cubic. The derivative of the left side of (4) vanishes for $\theta = \pm(a^2 + b^2 - 4c^2)/12$ and for these values of θ the left side takes on the extreme values $c^2(a^2 - b^2)^2/4$ and $[(a^2 + b^2 - 4c^2)^3 + 27c^2(a^2 - b^2)^2]/108$ and the product of these values gives $-\Delta/27$, where Δ is the discriminant (it may also be computed from $\Delta = I^3 - 27J^2$). Thus we have

$$\Delta = -\frac{1}{16}c^2(a^2 - b^2)^2[(a^2 + b^2 - 4c^2)^3 + 27c^2(a^2 - b^2)^2]. \tag{6}$$

It is now necessary to examine the last factor in Δ. Considering c^2 as the independent variable, it is seen that this factor has the maximum and minimum values $27a^2(a^2 - b^2)^2/2$ and $27b^2(a^2 - b^2)^2/2$ and hence vanishes only once for real values, passing from positive to negative values. Calling this root c_0^2 it is found to have the value

$$c_0^2 = \frac{a^2 + b^2}{4} + \tfrac{3}{8}(a^2 - b^2)^{2/3}[(a + b)^{2/3} + (a - b)^{2/3}] = \tfrac{1}{8}[(a + b)^{2/3} + (a - b)^{2/3}]^3. \tag{7}$$

If c^2 is greater than c_0^2 the roots of (3′) are all real or all imaginary, but since the equation has always one positive and one negative root the roots must be all real in this case. If c^2 is less than c_0^2 there are two imaginary roots in addition to the positive and negative root. For the value c_0^2 there are two equal roots.

In order to determine the number of inscribed rectangles for a given c several cases will be considered, the simplest of which seems to be the one for which $c > a$. This case may be treated without the use of the discriminant Δ. Setting $f(x)$ for the left side of (3), we have

$$f(0) = 2\left[1 - \frac{a^2 + b^2}{c^2}\right] < 0, \qquad f(b) = 2 - \left(\frac{a + b}{c + b}\right)^2 - \left(\frac{a - b}{c - b}^2\right) > 0, \tag{8}$$
$$f\left(\frac{bc}{a}\right) = 2\left[1 - \frac{a^2}{c^2}\right] > 0, \quad f(c) = -\infty, \quad f\left(\frac{ac}{b}\right) = 2\left[1 - \frac{b^2}{c^2}\right] > 0.$$

which is easily verified to agree with the previous result. The corresponding double root is then found to be

$$x_0 = \frac{c_0^{1/3}}{2}\left[(a+b)^{2/3} - (a-b)^{2/3}\right] = \sqrt{\frac{a^2+b^2-c_0^2}{3}} = \frac{6abc_0}{a^2+b^2+8c_0^2}. \tag{11}$$

The remaining two roots are obtained by adding to the negative of the double root

$$\pm\sqrt{\frac{a^2+b^2+8c_0^2}{3}}. \tag{12}$$

The maximum value of $f(x)$ at $x = (x_0/c_0)c$ will be found to be

$$2\left[1 - \frac{c_0^2}{c^2}\right], \tag{13}$$

and it gives the same results as the discriminant and in a much more evident form. This shows that the first two positive roots are separated by $(x_0/c_0)c$.

In Osgood's *Calculus*, page 404, is given a solution of the case $a = 15$, $b = 10$, $c = 1$. Here $c_0 = 13.741$, $x_0 = 6.738$, and since $a^2/b^2 = 2.25 > 6\sqrt{3} - 9$ this example falls in II2 and there is only one inscribed rectangle. If $c = 13$ there is no inscribed rectangle. If $c = 14$ there are two inscribed rectangles, one having the side 5.+ and the other the side 8.+.

II. Note on the Preceding by H. P. Manning, Brown University.

There is no particular distinction between the sides c and x of the inscribed rectangle. We can call them x and y, and equation (3) or (3′) will then be represented by a curve of the fourth degree in which many of the results of the above discussion appear graphically.

Moreover, m and n satisfy the equation $m^2 - n^2 = am - bn$, and so are represented by the points of an equilateral hyperbola, which passes through the vertices of the given rectangle if we lay off m and n from its lower left-hand corner.

273 (Number Theory) [1917, 427]. Proposed by V. M. SPUNAR, Chicago, Ill.

The ratio of the chances that all numbers ending in 1 or 9 and those ending in 3 or 7 are composite is 3 : 2.[1]

Note by Norman Anning, University of Maine.

Since

$$1 \times 1 \equiv 3 \times 7 \equiv 9 \times 9 \equiv 1 \pmod{10}, \qquad 1 \times 3 \equiv 7 \times 9 \equiv 3 \pmod{10},$$

$$1 \times 7 \equiv 3 \times 9 \equiv 7 \pmod{10}, \qquad 1 \times 9 \equiv 3 \times 3 \equiv 7 \times 7 \equiv 9 \pmod{10},$$

the conclusion might be drawn that in the long run as many primes end in 1 as in 9 and as many end in 3 as in 7 and that there would be more of the latter than of the former. A census of primes taken over a considerable range supports these statements but does not point towards the ratio "3 : 2." Counting cases up to 3,200, a number chosen at random, shows the following results:

110 primes end in 1, 113 in 3, 116 in 7 and 111 in 9.

Since these numbers are so nearly in equilibrium and since primes are so perfectly lawless no statement could be hazarded about the distribution in a larger interval.

2728 [1918, 397]. Proposed by NORMAN ANNING, University of Maine.

A material triangle of uniform density and thickness is of such a shape that when suspended from the vertices in succession, the lower sides have slopes of 1 : 1, $1\frac{1}{2}$: 1, and 3 : 1. Construct the triangle given that the shortest side is 10 inches.

By definition, an a : 1 slope makes an angle with the vertical whose tangent is a.

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[1] The enunciation of the problem is not clear. The chance that all numbers ending in 1 or 9 are composite numbers is zero. — EDITORS.

SOLUTION BY THE PROPOSER.

As this triangle is designed to show the slope of earth embankments the word 'slope' is defined as it is understood by the civil engineer: the ratio of horizontal to vertical distance.

Let AD, BE, CF be the medians of the required triangle, ABC, then

$$2 \cot ADB = \cot C - \cot B, \tag{1}$$

$$2 \cot BEC = \cot A - \cot C, \tag{2}$$

$$2 \cot CFA = \cot B - \cot A, \tag{3}$$

and hence $\cot ADB + \cot BEC + \cot CFA = 0$. If we take $\cot ADB = \frac{2}{3}$ then from the assigned values: (a) $\cot BEC = -1$, $\cot CFA = \frac{1}{3}$; (b) $\cot BEC = \frac{1}{3}$, $\cot CFA = -1$. Considering case (a), these values inserted in (1), (2), (3) give two independent equations, which are to be combined with the identity,

$$\cot B \cot C + \cot C \cot A + \cot A \cot B = 1. \tag{4}$$

Putting, from (2) and (3), $\cot B = \cot A + \frac{2}{3}$, $\cot C = \cot A + 2$, equation (4) becomes

$$9 \cot^2 A + 16 \cot A + 1 = 0;$$

whence,

$$\cot A = \frac{-8 \pm \sqrt{55}}{9}; \qquad \cot B = \frac{-2 \pm \sqrt{55}}{9}; \qquad \text{and} \qquad \cot C = \frac{10 \pm \sqrt{55}}{9}.$$

Since a triangle can have only one obtuse angle only one cotangent can be negative. Consequently, the $(-)$ before the radical does not lead to a solution. Using the $(+)$ sign,

$$A = 93° \, 42' \, 42'', \qquad B = 58° \, 57' \, 37'', \qquad C = 27° \, 19' \, 41''.$$

When $c = 10.00$, $a = 21.74$ and $b = 18.66$.

Case (b) can be treated in the same way and we get

$$\cot A = \frac{8 + \sqrt{55}}{9}, \qquad \cot B = \frac{-10 + \sqrt{55}}{9}, \qquad \cot C = \frac{2 + \sqrt{55}}{9}.$$

2745 [1919, 37]. Proposed by G. I. HOPKINS, Manchester, N. H.

A recent English publication contains the following method of constructing a regular polygon of 17 sides: Draw the radius CB perpendicular to the diameter AQ of the circle whose center is B. On BC lay off BD equal to one-fourth of BC. On BA, lay off BE and draw DE making angle BDE one fourth of angle BDA. On BQ lay off BF and draw DF, making angle FDE 45°. On AF as diameter, construct semi-circle FHA intersecting CB in H. With E as center and EH as radius construct semi-circle LHK intersecting CB in H. At the points L and K draw the ordinates NL and MK. Bisect the arc NM and let P be the point of bisection. Then the chord $NP(= MP)$ is a side of the regular polygon of 17 sides. Is the method of construction correct?

I. SOLUTION BY C. H. CHEPMELL, Hove, England.

The abscissas BK, BL, and their ordinates to M and N, make the angles MBA, NBA equal to $10\pi/17$, and $6\pi/17$ respectively. Consequently the difference of these two angles, the angle MBN, is equal to $4\pi/17$. This justifies the claim of the construction.

But tables of all the trigonometrical functions of $n\pi/17$ are not readily available; and it may be more satisfactory if we outline the connection between the numerical value of NM and the numerical value of some function given in a standard work.

Taking the radius of the circle ACQ as unity, we find

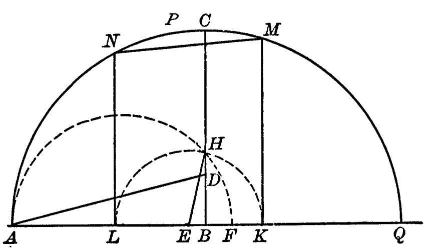
$$BE = \frac{-1 - \sqrt{17} + \sqrt{34 + 2\sqrt{17}}}{16},$$

$$BF = \frac{+1 - \sqrt{17} + \sqrt{34 - 2\sqrt{17}}}{16},$$

$$EH = \frac{\sqrt{2(\sqrt{17} - 3)(2\sqrt{17} + \sqrt{34 + 2\sqrt{17}})}}{16} = \frac{\beta}{16},$$

$$AK = \frac{+17 + \sqrt{17} - \sqrt{34 + 2\sqrt{17}} + \beta}{16},$$

$$AL = \frac{+17 + \sqrt{17} - \sqrt{34 + 2\sqrt{17}} - \beta}{16},$$



$$KQ = \frac{+15 - \sqrt{17} + \sqrt{34 + 2\sqrt{17}} - \beta}{16},$$

$$LQ = \frac{+15 - \sqrt{17} + \sqrt{34 + 2\sqrt{17}} + \beta}{16},$$

$$LN = \sqrt{AL \cdot LQ} = \tfrac{1}{8}\sqrt{+34 + 2\sqrt{17} + 2\sqrt{34 + 2\sqrt{17}} - 2\beta'},$$

$$KM = \sqrt{AK \cdot KQ} = \tfrac{1}{8}\sqrt{+34 + 2\sqrt{17} + 2\sqrt{34 + 2\sqrt{17}} + 2\beta'},$$

$$[\beta' = \sqrt{2(\sqrt{17} - 3)(2\sqrt{17} - \sqrt{34 + 2\sqrt{17}})}; \quad \text{and} \quad \beta \times (-1 - \sqrt{17} + \sqrt{34 + 2\sqrt{17}}) = 4 \cdot \beta'],$$

$$LN \times KM = \tfrac{4}{64}\sqrt{2(\sqrt{17} + 3)(2\sqrt{17} + \sqrt{34 - 2\sqrt{17}})} = \frac{4\alpha}{64},$$

$$\begin{aligned} NM^2 &= LK^2 + (KM - LN)^2 \\ &= 4 \cdot EH^2 + KM^2 + LN^2 - \frac{8\alpha}{64} \\ &= \frac{+136 - 8\sqrt{17} + 8\sqrt{34 - 2\sqrt{17}} - 8\alpha}{64}, \end{aligned}$$

$$[\alpha' = \sqrt{2(\sqrt{17} + 3)(2\sqrt{17} - \sqrt{34 - 2\sqrt{17}})}; \quad \text{and} \quad 4\alpha = \alpha' \times (-1 + \sqrt{17} + \sqrt{34 - 2\sqrt{17}})],$$

$$\therefore \quad NM^2 = \frac{+136 - 8\sqrt{17} + 8\sqrt{34 - 2\sqrt{17}} - 2(-1 + \sqrt{17} + \sqrt{34 - 2\sqrt{17}})\alpha'}{64}.$$

And, in the circle with radius equal to unity, NM represents the value $2 \sin (2\pi/17) \times 1$; and therefore

$$\begin{aligned} 4\cos^2 \frac{2\pi}{17} &= 4 - NM^2 \\ &= \frac{+120 + 8\sqrt{17} - 8\sqrt{34 - 2\sqrt{17}} + 2(-1 + \sqrt{17} + \sqrt{34 - 2\sqrt{17}})\alpha'}{64} \\ &= \left[\frac{-1 + \sqrt{17} + \sqrt{34 - 2\sqrt{17}} + \alpha'}{8}\right]^2 \end{aligned}$$

and

$$2\cos\frac{2\pi}{17} = \frac{-1 + \sqrt{17} + \sqrt{34 - 2\sqrt{17}} + \sqrt{2(\sqrt{17} + 3)(2\sqrt{17} - \sqrt{34 - 2\sqrt{17}})}}{8}.$$

And this value of $2 \cos (2\pi/17)$ will be found to agree with that given in Klein's *Famous Problems in Elementary Geometry* (Beman & Smith), though our α' is there written in a different form.

II. Historical Note by R. C. Archibald, Brown University.

This method of construction is due to H. W. Richmond, *Quarterly Journal of Mathematics*, Volume 26, 1893, pp. 206–207; and *Mathematische Annalen*, Volume 67, 1909, pp. 460–461. It is reproduced on page 34 of H. P. Hudson's *Ruler and Compasses*, London, 1916.

Various constructions of the regular polygon of seventeen sides were reviewed by R. Goldenring in his *Die elementargeometrischen Konstruktionen des regelmässigen Siebzehnecks* (Leipzig, 1915), but many omissions in this professedly complete survey were noted by the writer in the *Bulletin of the American Mathematical Society*, vol. 22, 239-246. The first solution in an English publication, given by Lowry in 1819,[1] was reproduced in this *Monthly*[2] in 1899 and 1914. Other solutions and historical notes are set forth in the articles printed above, pages 322–326.

2767 [1919, 171]. Proposed by W. W. JOHNSON, U. S. Naval Academy.

Let the complex quantities p, q, and r satisfy the relation $p^2 + q^2 + r^2 = 0$; prove that the corresponding vectors OP, OQ, and OR are such that if any two of them are taken as conjugate semi-diameters of an ellipse, the third lies on the minor axis, and its length is the distance from the center to either focus.

Solution by A. Pelletier, Montreal, Can.

Let $(x^2/a^2) + (y^2/b^2) = 1$, be the equation of the ellipse having OP and OQ for conjugate semi-diameters ($2a$ and $2b$ being the axes, and $a \geqq b$). If α, α', α'' are the respective arguments of

[1] *The Mathematical Repository*, new series, vol. 4, p. 160; Lowry's proof occupies pages 160-168.
[2] Volume 6, p. 239 and volume 21, p. 252.

OP, OQ, and OR, we have

$$OP^2(\cos 2\alpha + i \sin 2\alpha) + OQ^2(\cos 2\alpha' + i \sin 2\alpha') = - OR^2(\cos 2\alpha'' + i \sin 2\alpha''),$$

from datum; hence,

$$OP^2 \cos 2\alpha + OQ^2 \cos 2\alpha' = - OR^2 \cos 2\alpha'' \tag{1}$$

and

$$OP^2 \sin 2\alpha + OQ^2 \sin 2\alpha' = - OR^2 \sin 2\alpha''. \tag{2}$$

Now P and Q being points on the ellipse, we have from known properties,

$$OP^2 \cos^2 \alpha + OQ^2 \cos^2 \alpha' = a^2, \qquad OP^2 \sin^2 \alpha + OQ^2 \sin^2 \alpha' = b^2;$$

hence, $OP^2 \cos 2\alpha + OQ^2 \cos 2\alpha' = a^2 - b^2$, and (1) becomes

$$- OR^2 \cos 2\alpha'' = a^2 - b^2. \tag{3}$$

Also, from known properties concerning the ends of conjugate diameters,

$$OP^2 \sin 2\alpha = - OQ^2 \sin 2\alpha';$$

hence, (2) becomes

$$- OR^2 \sin 2\alpha'' = 0. \tag{4}$$

It follows from (3) and (4), that $2\alpha'' = 180°$ or $540°$, and $OR^2 = a^2 - b^2$, that is, $OR = \sqrt{a^2 - b^2}$, the distance from the center to focus, and $\alpha'' = 90°$ or $270°$, which shows that OR lies on the minor axis.

Also solved by H. Halperin, A. M. Harding, and H. L. Olson.

2780 [1919, 311]. Proposed by ELMER LATSHAW, West Philadelphia, Pa.

A quadrilateral whose sides are a, $2a$, $3a$, $4a$ is inscribed in a circle. Find the radius of the circle.

I. Solution by H. S. Uhler, Yale University.

The interest in this problem may be enhanced by giving a perfectly general solution. Let the sides of any convex inscriptible quadrilateral be denoted by a_1, a_2, a_3, a_4. A diagonal c may be drawn dividing the quadrilateral into two non-overlapping triangles the sides of which are a_1, a_2, c and a_3, a_4, c, respectively. If the angle between a_1 and a_2 be symbolized by C, the angle between a_3 and a_4 must be $180° - C$. Accordingly

$$c^2 = a_1^2 + a_2^2 - 2a_1a_2 \cos C,$$

$$c^2 = a_3^2 + a_4^2 + 2a_3a_4 \cos C.$$

Eliminating $2\cos C$ we find

$$c^2 = \frac{(a_1a_3 + a_2a_4)(a_2a_3 + a_4a_1)}{a_1a_2 + a_3a_4}. \tag{1}$$

The area of a plane triangle having the sides a_1, a_2, c is given by either member of the following equation

$$\frac{a_1a_2c}{4R} = \sqrt{s(s - a_1)(s - a_2)(s - c)}, \tag{2}$$

where $2s = a_1 + a_2 + c$, and R denotes the radius of the circumscribed circle.

Substituting the trinomial value of s in equation (2) we obtain

$$\frac{a_1a_2c}{R} = \sqrt{[(a_1 + a_2)^2 - c^2][c^2 - (a_1 - a_2)^2]}. \tag{3}$$

Replacing c in equation (3) by expression (1) we eventually find that

$$R = \frac{\sqrt{(a_1a_2 + a_3a_4)(a_1a_3 + a_4a_2)(a_2a_3 + a_4a_1)}}{\sqrt{(a_2 + a_3 + a_4 - a_1)(a_3 + a_4 + a_1 - a_2)(a_4 + a_1 + a_2 - a_3)(a_1 + a_2 + a_3 - a_4)}}, \tag{4}$$

or

$$R = \frac{1}{4K}\sqrt{(a_1a_2 + a_3a_4)(a_1a_3 + a_4a_2)(a_2a_3 + a_4a_1)}. \tag{5}$$

where if $2S = a_1 + a_2 + a_3 + a_4$, $K = \sqrt{(S - a_1)(S - a_2)(S - a_3)(S - a_4)}$ = area of quadrilateral.

The denominator of formula (4) brings out the geometrically-evident fact that each side of the quadrilateral must not exceed the sum of the remaining three sides. When the quadrilateral degenerates into a straight line, formulas (4) and (5) give $R = \infty$, as they should. These formulæ also show explicitly that the order or succession of the sides has no effect on the value of R, a fact which is obvious geometrically since the sum of the arcs subtended by the four sides of the quadrilateral equals the entire circumference.

The answer to the given problem may be obtained at once from formula (5) by substituting a, $2a$, $3a$, $4a$, $5a$ for a_1, a_2, a_3, a_4, S respectively. It is

$$R = \frac{a\sqrt{385}}{4\sqrt{6}} = (2\cdot 002602\cdots)a.$$

II. Solution by Bing Chin Wong, Berkeley, Calif.

Let $ABCD$ be the polygon with sides $AB = a$, $BC = 2a$, $CD = 3a$, $DA = 4a$ inscribed in the circle with O as center. Join O to A, B, C, D. Then

$$\angle AOB + \angle BOC + \angle COD + \angle DOA = 2\alpha + 2\beta + 2\gamma + 2\delta = 2\pi,$$

or

$$\alpha + \beta + \gamma + \delta = \pi.$$

Then

$$\cos(\alpha + \beta) + \cos(\gamma + \delta) = 0,$$

or

$$\cos\alpha\cos\beta - \sin\alpha\sin\beta + \cos\gamma\cos\delta - \sin\gamma\sin\delta = 0. \qquad \text{(I)}$$

Let r be the radius of the circle. We obtain from the figure,

$$\sin\alpha = \frac{a}{2r}, \qquad \sin\beta = \frac{a}{r}, \qquad \sin\gamma = \frac{3a}{2r}, \qquad \sin\delta = \frac{2a}{r};$$

and

$$\cos\alpha = \frac{\sqrt{4r^2 - a^2}}{2r}, \qquad \cos\beta = \frac{\sqrt{r^2 - a^2}}{r}, \qquad \cos\gamma = \frac{\sqrt{4r^2 - 9a^2}}{2r}, \qquad \cos\delta = \frac{\sqrt{r^2 - 4a^2}}{r}.$$

Substituting these values in (I) and multiplying by $2r^2$, we have

$$\sqrt{4r^2 - a^2}\sqrt{r^2 - a^2} + \sqrt{4r^2 - 9a^2}\sqrt{r^2 - 4a^2} = 7a^2.$$

Squaring, collecting terms, and dividing by 2, we have

$$\sqrt{(4r^2 - a^2)(r^2 - a^2)(r^2 - 4a^2)(4r^2 - 9a^2)} = 6a^4 + 15a^2r^2 - 4r^4.$$

Squaring again and collecting terms, we have

$$96a^4r^4 = 385a^6r^2, \qquad \text{or} \qquad r^2 = 385a^2/96,$$

and, therefore,

$$r = a\sqrt{385/96} = a\sqrt{2310/24} = 2.0026a.$$

Also solved by H. C. Bradley, H. N. Carleton, S. A. Corey, Laura Guggenbuhl, T. F. Noismann, H. L. Olson, A. Pelletier, J. B. Reynolds, and the Proposer.

2781 [1919, 311]. Proposed by J. L. Riley, Stephensville, Texas.

Show that the asymptotic lines on a pseudospherical surface are curves of constant torsion.[1]

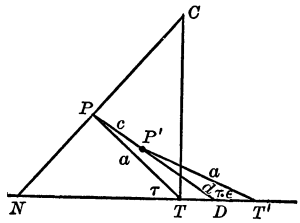
Solution by Otto Dunkel, Washington University.

The Theorem of Euneper states that the square of the torsion of an asymptotic line at any point of a surface is equal to the total curvature of the surface (with sign changed) at the point (Eisenhart, *Differential Geometry*, p. 140). Since it is *known* that the total curvature of a pseudospherical surface is constant, it follows at once that the torsion is constant.

[1] This problem is given as an example in Eisenhart's *Differential Geometry*, p. 290.—Editors.

The pseudosphere is obtained by rotating the tractrix about its asymptote. That its total curvature is constant may be proved as follows: In any surface of revolution the principal radii of curvature are the radius of curvature of the meridian section and the length of the normal at the given point (Goursat-Hedrick, vol. 1, p. 505). As is well known the length of the tangent for the tractrix is a constant a; in fact, this is usually given as its definition. Let P and P' be two neighboring points on a tractrix, $PT = P'T' = a$ the tangents at these points, and τ and $\tau - \epsilon$, their *acute* inclinations to the axis. Produce $PP' = c$ to cut the axis in D and set $P'D = d$; then

$$\frac{c+d}{d} = \frac{a \sin \tau}{a \sin (\tau - \epsilon)} \quad \text{or} \quad \frac{c}{d} = \frac{\sin \tau - \sin (\tau - \epsilon)}{\sin (\tau - \epsilon)}.$$



Thus

$$\frac{c}{\epsilon} = d\,\frac{\cos\left(\tau - \dfrac{\epsilon}{2}\right)}{\sin (\tau - \epsilon)}\,\frac{\sin \dfrac{\epsilon}{2}}{\dfrac{\epsilon}{2}}.$$

As ϵ approaches zero, d approaches a and c/ϵ approaches the radius of curvature R. Hence $R = a \cot \tau$, and this shows that the line joining the center of curvature C with T is perpendicular to the axis. If PN is the normal at P, the right triangle CTN gives $CP \times PN = a^2$. (See Goursat-Hedrick, vol. 1, p. 441 for an analytical proof.) Thus the total curvature at every point of the pseudosphere is $-1/a^2$ and the torsion of an asymptotic line is $1/a$.

Also solved by J. B. REYNOLDS.

2782 [1919, 311]. Proposed by WARREN WEAVER, University of Wisconsin.

A great number, n, of jackstraws are jumbled up in such a way that any one is as likely to have one direction as another. Show that the probable number that make an angle lying between θ_1 and θ_2 as measured from any given direction is equal to $\dfrac{n(\cos \theta_1 - \cos \theta_2)}{2}$.

SOLUTION BY H. L. OLSON, University of Wisconsin.

Let us imagine the jackstraws all to have been translated in space so as to have *corresponding* ends coincident, and let us consider a sphere of unit radius with its center at this common end-point. The jackstraws satisfying the specified condition will then intersect the sphere in the points of a zone whose bases are circles with angular radii θ_1 and θ_2, respectively. Since the area of this zone is $2\pi(\cos \theta_1 - \cos \theta_2)$ (if $\theta_1 < \theta_2$) and the area of the entire sphere is 4π, the probable number of jackstraws making angles between θ_1 and θ_2 with a given line is $\dfrac{n(\cos \theta_1 - \cos \theta_2)}{2}$.

2789 [1919, 414]. Proposed by KURT LAVES, University of Chicago.

Given a quadrilateral $ABCD$ for which $|AC - BC| > |AD - BD|$[1] $(AC + BC < AD + BD)$ to construct, by means of the ruler and compass only, the pair of tangents from D to the hyperbola (ellipse) for which A and B are the foci and C a point on the hyperbola (ellipse).

SOLUTION BY A. PELLETIER, Montreal, Can.

Ellipse: D is outside the curve and the solution is always possible. With B as a center and a radius of length $AC + BC$ describe the circle x; with D as a center and DA as a radius, describe the circle y. The circles x and y intersect at M and N. The perpendiculars erected on AM, AN, at the mid-points of these lines are the required tangents, as is well known.

Hyperbola: The construction is the same as above, the radius of x being $|AC - BC|$.

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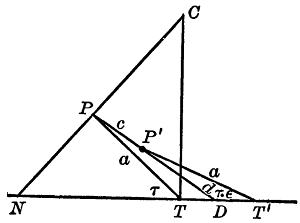

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NOTES AND NEWS.

EDITED BY E. J. MOULTON, Northwestern University, Evanston, Ill.

At Elmira College, Miss MARY C. SUFFA, instructor in mathematics at Beloit College, has been appointed head of the department of mathematics and Miss FRANCES W. WRIGHT, of Brown University, has been appointed instructor.

At Queen's University, Assistant Professor C. F. GUMMER has been promoted to be an associate professor, and Lecturer Dr. NORMAN MILLER to be an assistant professor. Mr. A. WOODS, a tutor in mathematics during 1919–20, has been appointed lecturer at Western University, London, Ontario.

Dr. J. W. CAMPBELL, associate professor of mathematics and astronomy at the University of Iowa, has been appointed assistant professor of mathematics at the University of Alberta.

Science states that Dr. J. P. MUSSELMAN, of Washington University, has been appointed associate in mathematics at the John Hopkins University. Dr. Musselman is the national president of the Gamma Alpha Graduate Scientific Fraternity.

Professor OSWALD VEBLEN has been elected the representative of mathematics in the Executive Committee of the Division of Physical Sciences of the National Research Council.

Professor IDA BARNEY, of Meredith College, has been appointed assistant professor of mathematics at Smith College.

At Oberlin College, Associate Professor W. D. CAIRNS has been promoted to be professor of mathematics and head of the department in place of Professor FREDERICK ANDEREGG who has retired after thirty-three years of service as a teacher.

Professor D. R. CURTISS has been appointed lecturer at Harvard University for the second semester of 1920–21. He will give courses in "The analytical theory of heat and problems in elastic vibration" and "Functions defined by linear differential equations of the second order."

Miss ZOE FERGUSON, of the Central High School and Junior College, St. Joseph, Mo., has accepted a position in the Crane Technical High School and Junior College of Chicago.

Dr. L. R. FORD, of Harvard University, has been appointed assistant professor of mathematics at the Rice Institute.

Professor T. R. HOLLCROFT was appointed in charge of the department of mathematics at Wells College a year ago in place of Miss ANNA L. VAN BENSCHOTEN who retired on account of ill health and is now living in California.

At the University of Pennsylvania, Dr. J. R. KLINE, associate at the University of Illinois, [*1919*, 322] has been appointed assistant professor of mathematics.

At the University of Texas, Assistant Professor R. L. MOORE, of the University of Pennsylvania, has been appointed associate professor of pure mathematics, and Dr. JESSIE M. JACOBS [*1920*, 92] instructor in pure mathematics.

Messrs. F. H. MURRAY and J. L. WALSH have been appointed Sheldon Fellows by Harvard University and expect to spend the coming year studying mathematics in Paris.

Miss GERTRUDE SMITH, of Vassar College, has been promoted to an assistant professorship of mathematics.

Assistant professor LOUISA M. WEBSTER, of Hunter College, has been promoted to an associate professorship. She is vice-president of the Association of Mathematics Teachers of the Middle States and Maryland.

At Cambridge University, England, Mr. J. E. LITTLEWOOD, of Trinity College, has been appointed Cayley lecturer in mathematics and Mr. J. H. GRACE, of Peterhouse, has been reappointed University lecturer in mathematics.

Dr. M. G. HUMBERT, professor of analysis at the Ecole Polytechnique for twenty-five years, has resigned and been appointed honorary professor.

Dr. RENÉ GARNIER, chargé d'un cours in rational and applied mechanics at the University of Poitiers, has been appointed professor of rational and applied mechanics in place of Professor FRÉCHET who had been transferred to the University of Strasbourg.

Professor E. P. J. VESSIOT, of the University of Paris, has been appointed sub-director of the Ecole Normale Supérieure in place of Professor EMILE BOREL [*1920*, 280] who resigned after ten years of service, and has been appointed honorary director. Louis Pasteur, L. E. Bertin, director of French naval construction, and Jules Tannery have occupied this post in earlier years.

Professor C. CARATHÉODORY, of the University of Berlin, has been appointed professor of mathematics at the national university in Athens. Professor Carathéodory's place at Berlin has been filled by the appointment of Professor L. E. BROUWER, of the University of Amsterdam.

Other appointments in Germany are as follows: Dr. R. COURANT, privatdozent at the University of Göttingen to be ordinary professor of mathematics at the University of Münster.—Professor H. HAPPEL, of the University of Tübingen, to be ordinary professor of applied mathematics at the Technische Hochschule, Breslau.—Dr. O. HAUPT, privatdozent at the Technische Hochschule, Karlsruhe, to be ordinary professor at the University of Rostock.—Professor H. JUNG, of the University of Kiel, to be ordinary professor of mathematics at the University of Halle.—Professor R. v. MISES, of the Technische Hochschule, Dresden, to be ordinary professor of applied mathematics at the University of Berlin.—Dr. R. H. WEBER, extraordinary professor of physics at the University of Rostock, to be honorary professor there.

Dr. P. VAN GEER, professor at the University of Leyden since 1867, died at The Hague, October 3, 1919, aged seventy-eight years. He was the author of various books on geometrical topics, from his Leyden dissertation on geodetic lines of an ellipsoid (1862), to his foundations of synthetic geometry (1900).

Dr. M. HAID, professor of practical geometry and higher geodesy at the Technische Hochscule, Carlsruhe, died November 5, 1919.

Dr. P. G. STACKEL, professor at the University of Heidelberg since 1912, died December 12, 1919, aged fifty-seven years. His section of the *Encyklopädie* on "Elementare Dynamik der Punktsysteme und starren Körper," and his activities in connection with the editing of the works of the Bolyais and Euler will be recalled.

SRINIVASA RAMANUJAN, brilliant Indian mathematician of "astonishing individuality and power" (Hardy), and with "powers as remarkable in their way as those of any living mathematician" (Hardy), died at Chetput, Madras Presidency on April 26, 1920 at the early age of thirty-one. He was a fellow of Trinity College, Cambridge, and a research fellow of the University of Madras, and was elected the first Indian fellow of the Royal Society, London, when only twenty-eight years of age. A portrait and biographical sketch from which quotation was made, have been already referred to in this MONTHLY, *1920*, 74. Quotation from a sketch by Professor G. H. Hardy is made elsewhere in this issue (page 316). The report of Professor Hardy to the University of Madras on "Mr. S. Ramanujan's mathematical work in England," which was published in *The Journal of the Indian Mathematical Society* for February, 1917, surveyed about a dozen papers written after his arrival in England in April, 1914. Some others are referred to in a memoir by Hardy and Ramanujan in the 1919 volume (95A) of the *Proceedings of the Royal Society of London*. It will be recalled that chapter 3 in the second volume of P. A. MacMahon's *Combinatory Analysis* (1916) is entitled "Ramanujan's identities."

Dr. C. A. LAISANT, examiner for admission to the Ecole Polytechnique since 1909, died May 6, 1920, in his seventy-eighth year. He was a nephew of the

celebrated ophthalmologist Ange Guépin of Nantes. From 1876 to 1893 he was a member of the chamber of deputies in the French parliament. He was the author of about fifteen volumes of mathematical works and of a large number of mathematical papers published, for the most part, in the *Bulletin de la Société Mathématique de France,* and in *Comptes Rendus de l'Association francaise pour l'Avancement des Sciences.* His book entitled *La Mathématique: Philosophie Enseignement* (Paris, 1898; second edition revised, 1907), was translated into Russian (St. Petersburg, sixth edition, 1912) and into English (London, 1913); and his six volume *Recueil de Problèmes de Mathématiques* (1893–96) is a very useful bibliographic aid. Details concerning his collaboration with Lemoine in founding *L'Intermédiaire des Mathématiciens*[1] have been set forth by Professor D. E. Smith in this MONTHLY, *1896,* 32. As to Dr. Laisant's non-mathematical publications and political life, reference may be made to his biography in G. Vapereau's *Dictionnaire Universel des Contemporains,* sixième édition, Paris, 1893.

The American Association for the advancement of Science made the following grants for mathematical research in 1919: Three hundred dollars to Professor SOLOMON LEFSCHETZ of Kansas University to assist in the publication of his memoir on algebraic surfaces, which was awarded the Bordin prize of the Paris Academy of Sciences [*1920,* 143], one hundred dollars to Dr. OLIVE C. HAZLETT of Mount Holyoke in support of her work on the theory of hypercomplex numbers and variants.

At the annual meeting of the American Academy of Arts and Sciences in May, Professor J. S. HADAMARD was elected a foreign honorary member in the division of mathematics and astronomy. [Cf. *1920,* 239.]

In the Division of Physical Sciences of the National Research Council, Professor G. D. BIRKHOFF has been appointed the new member at large and Professor E. R. HEDRICK the representative of the Mathematical Association of America.

The following are the members of the American Section of the International Mathematical Union [*1920,* 282] for 1920: Professors FRANK MORLEY, F. N. COLE, H. S. WHITE, OSWALD VEBLEN, L. E. DICKSON, VIRGIL SNYDER, G. A. MILLER, G. D. BIRKHOFF, E. B. VAN VLECK representing the American Mathematical Society; H. E. SLAUGHT, E. V. HUNTINGTON, and D. E. SMITH, representing the Mathematical Association of America, and E. R. HEDRICK as representing the Association in the Division of Physical Sciences; M. W. HASKELL representing the American Association for the Advancement of Science, LEIGH PAIGE representing the American Physical Society, and E. H. MOORE representing the National Academy.—At this writing the American Astronomical Society has appointed no representative.—This Section has appointed Professors L. E. DICK-

[1] Laisant's own account of this may be found in his memoir of Lemoine in *L'Enseignement Mathématique,* vol. 14, 1912, p. 182.

SON and L. P. EISENHART to be the representatives of American mathematicians at the meetings of the Union, which convene at Brussels about September 20. These representatives have been authorized to appoint alternates on arriving at Brussels. Other American mathematicians who expect to attend the Congress which opens at Strasbourg on September 22 are SOLOMON LEFSCHETZ, J. L. WALSH, F. H. MURRAY, NORBERT WIENER, J. S. TAYLOR, THOMAS BUCK, and D. E. SMITH.

Reference has been made in earlier numbers of this MONTHLY [*1916*, 270–271; *1919*, 372], to generous provisions made by MITTAG-LEFFLER and his wife for the founding of a Mathematical Institute in Stockholm. It is now announced that Cornell University has received an anonymous gift from a professor and his wife of a trust fund for an Institute of pure and applied mathematics. The gift amounts to $50,000 and is to be held in trust for a hundred years and allowed to accumulate. At five per cent. interest compounded annually this accumulation would amount to about six and one half million dollars; at six per cent. to over three times this amount.

On the recommendation of the National Academy of Sciences the Barnard medal for meritorious service to science has been conferred by Columbia University on Professor ALBERT EINSTEIN, of Berlin, in recognition "of his highly original and fruitful development of the fundamental concepts of physics through application of mathematics."

The adjudicators of the Hopkins prize of the Cambridge Philosophical Society have made the following awards: for the period 1903–6 Dr. W. BURNSIDE, of Pembroke College, for investigations in mathematical science; for the period 1906–9 to Professor G. H. BRYAN, of Peterhouse, for investigations in mathematical physics, including aerodynamic stability; and for the period 1909–12 to Mr. T. R. WILSON, of Sidney Sussex College, for investigations in physics, including the paths of radio-active particles.

In March, 1918, mathematicians from Athens, Piraeus, and the Provinces, gathered at Athens and organized a Greek Mathematical Society. In 1918 there were 93 members and the following officers: Honorary Presidents, J. HATZIDAKIS and D. AIGINITIS; President, N. HATZIDAKIS; Vice-Presidents, G. REMOUNDOS and P. ZERVOS; Secretaries, N. SAKELLARIOS and A. ARVANITIS; Treasurer, G. AUTONOPOULOS. We have referred elsewhere in this issue to the Society's *Bulletin* [1920, 314].

We have already referred to the plan for making the University of Strasbourg next to Paris the greatest center in France for mathematical study [*1919*, 371]. The research courses for 1920–21 leading to the doctorate have been anonunced. For the first semester (November 1, 1920–February 28, 1921) they are in mathe-

matical physics by Professor BAUER, and in higher analysis by Professor FRÉCHET. During the second semester (March 1–June 30, 1921) these professors will continue their lectures, and in addition, Professor VILLAT will lecture on hydrodynamics, Professor PÉRÈS on differential geometry, Professor VALIRON on theory of functions. The degree which would be normally sought by the American student at Strasbourg is the "doctorat de l'Université de Strasbourg." The higher (state) degree, "doctorat ès sciences mathématiques," first granted in 1811, is the one almost invariably sought by the Frenchman. Only one American, Miss DOROTHEA KLUMPE, has ever received this degree from a French university (1893). Professor HARRIS HANCOCK is the only American who has received the French degree of "doctorat de l'université" in mathematics (1903). Before 1870 sixteen persons received state doctorates at the University of Strasbourg. Among them was J. G. DOSTOR author of the well known *Eléments de la Théorie des Déterminants* (1877, second edition, 1883).

The late Moritz Cantor's library [see *1920*, 280] is offered for sale by a Leipzig firm for one hundred and twenty nine thousand six hundred marks. It contains about 2000 volumes and 2500 pamphlets.—Darboux's library has been purchased by Hermann of Paris.

THE WORK OF THE NATIONAL COMMITTEE ON MATHEMATICAL REQUIREMENTS JUNE 10, 1920 [*1920*, 145–146]

The National Committee on Mathematical Requirements held a meeting in Chicago on April 23 and 24, 1920. The principal topic discussed at this meeting was the preliminary report on "Junior High School Mathematics" prepared for the Committee by Mr. J. A. Foberg. After detailed discussion and some amendment and revision, the report was adopted by the Committee and its publication as a preliminary report authorized. It will be published by the U. S. Bureau of Education as one of its secondary school circulars.

This report considers briefly the history of the Junior High School Movement, its purposes and organization; it proposes certain general principles to govern the organization of work in mathematics in the junior high school and considers in some detail the topics to be included. The Committee feels that much experimentation must still precede any attempt at standardization of the junior high school curriculum in mathematics. Its recommendations are intended to form the basis of study, discussion and class room experiment. To this end the coöperation of teachers and supervisors is earnestly solicited. The National Committee hopes to act as a clearing house for constructive criticism based on actual class room experience.

The following resolution was adopted:

Resolved: That the National Committee on Mathematical Requirements approves the junior high school form of organization and urges its general adoption in the conviction that it will secure greater efficiency in the teaching of mathematics.

Reports of progress were made by subcommittees on The Training of Teachers, Experimental Schools and Courses, Disciplinary Values and Transfer of Training, Elective Courses in Mathematics for High Schools, and Mental Tests. It is expected that preliminary reports on all of these topics will be ready for consideration by the Committee at its next meeting on September 2, 3, and 4. The attention of experimental schools throughout the country is called to the report on this subject being prepared for the Committee by Mr. Raleigh Schorling of the Lincoln School, New York City. Any experimental schools or schools giving experimental courses in mathematics who desire to be represented in this report should communicate with Mr. Schorling without delay, if they have not already done so. A subcommittee on the Standardization of Terminology and Symoblism with Professor D. E. Smith as chairman and a subcommittee on Junior College Mathematics with Mr. A. C. Olney as chairman were appointed J. W. Young, Raleigh Schorling, and W. F. Downey were authorized to take steps to initiate investigations into the mathematical elements entering into various industries, professions, vocations, etc.

A budget for the coming year based on the recent appropriation of the General Education Board of $25,000 for the use of the Committee in completing its work was adopted. It is hoped that the increase in the item allowed for traveling expenses in this budget will make it possible for representatives of the Committee to reach educational meetings in all sections of the country where such representatives are desired to discuss the various reports of the Committee. Nearly 70 organizations are at present actively coöperating with the Committee and it is hoped that many others will communicate with the chairman in the interest of furthering the nation-wide study and discussion which is already under way. J. W. Young, 24 Musgrove Building, Hanover, New Hampshire and J. A. Foberg, 3829 North Tripp Avenue, Chicago, Illinois were reëlected chairman and vice-chairman, respectively, of the Committee for the ensuing year.

J. W. YOUNG, *Chairman.*

NOTE.

Through failure of the mail in returning proof an error occured in the announcement of John Wiley and Sons in the May issue of this MONTHLY. This stands corrected in their announcement of the present issue.

The American Mathematical Monthly

OFFICIAL ORGAN OF

The Mathematical Association of America

Is the Only Journal of Collegiate Grade in The Mathematical Field in this Country

Most of its mathematical contributions can be read and understood by those who have not specialized in mathematics beyond the Calculus.

The Historical Papers include not only those of a research character, but also some in less technical form.

The Questions and Discussions, which are timely and interesting, cover a wide variety of topics.

Surveys of the contents of recent books and periodicals constitute a valuable guide to current mathematical literature.

The Topics in the department of Undergraduate Mathematical Clubs have excited wide interest both in this country and in Great Britain.

The Notes and News cover a wide range of interest and information both in this country and in foreign countries.

The Problems and Solutions hold the attention and activity of a large number of persons who are lovers of mathematics for its own sake.

There are other journals suited to the Secondary field, and there are still others of technical scientific character in the University field: but the MONTHLY is the only journal of Collegiate grade in America suited to the needs of the non-specialist in mathematics.

THE MATHEMATICAL ASSOCIATION OF AMERICA now has over eleven hundred individual and institutional members. There are already nine sections formed, representing twelve different states. The Association has held so far two national meetings per year, one in September and one in December. The sections hold one meeting and some cases two meetings each year. All meetings, both national and sectional, are reported in the Official Journal, and many of the papers presented at these meetings are published in full.

The slogan of the Association is to include in its membership every teacher of collegiate mathematics in America and to make such membership worth while. Application blanks for membership may be obtained from the Secretary, W. D. Cairns, 27 King Street, Oberlin, Ohio.

CONTENTS

EDITORIAL CORRESPONDENCE AND BOOKS FOR REVIEW should be addressed to the EDITOR-IN-CHIEF, R. C. ARCHIBALD, Brown University, Providence, R. I.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 27 King Street, Oberlin, Ohio.

Fifth Summer Meeting of the Association, Chicago, September 6, 1920;
Fifth Annual Meeting, December, 1920

The following are dates of Section meetings of the Association in 1920:

IOWA, Univ. of Iowa, Iowa City, May 1

KANSAS, State Agricultural College, Manhattan, April 3; Topeka, November

KENTUCKY, Centre College, Danville, April 17

MARYLAND–DISTRICT OF COLUMBIA–VIRGINIA, Goucher College, Baltimore, Md., May 15; Annapolis, Md., December

MINNESOTA, St. Catherine's College, St. Paul, June 5

MISSOURI, Kansas City, November 12–13

OHIO, Ohio State Univ., Columbus, April 2

ROCKY MOUNTAIN, Colorado College, Colorado Springs, April 2

Missing Numbers of the Monthly

Cash will be paid for certain single numbers as follows, up to a limited number of copies:

February, March, May or September, 1913; September, 1914; February, March, April or June, 1915; February or September, 1918—fifty cents; January, May or September, 1915—seventy-five cents.

Extra copies or volumes of any dates which members wish to contribute will be used to the best advantage of the Association

Address all communications to the Secretary, W. D. Cairns, 27 King Street, Oberlin, Ohio.

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DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

The American Mathematical Monthly, founded in 1894 by Benjamin F. Finkel, was published by him until 1913. From 1913 to 1916 it was owned and published by representatives of fourteen Universities and Colleges in the Middle West

VOLUME XXVII, 1920

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Rates for 1921: \$4.00 a year, Single Copies 45 cents to Members; \$5.00 a year, Single Copies 60 cents, to Others.

THE MAY MEETING OF THE MARYLAND-VIRGINIA-DISTRICT OF COLUMBIA SECTION.

The seventh regular meeting of the Maryland-Virginia-District of Columbia Section of the Mathematical Association of America was held at Goucher College, Baltimore, Md., on May 15, 1920. The meeting consisted of two sessions with Professor Ralph E. Root presiding.

The attendance was forty, including the following twenty-eight members of the Association: O. S. Adams, J. J. Arnaud, H. G. Avers, Clara L. Bacon, Sarah Beall, A. A. Bennett, G. A. Bingley, J. A. Bullard, G. R. Clements, A. Cohen, J. B. Coleman, G. H. Cresse, A. Dillingham, J. B. Eppes, W. M. Hamilton, P. E. Hemke, L. S. Hulburt, W. D. Lambert, A. E. Landry, Florence P. Lewis, J. J. Luck, F. D. Murnaghan, C. E. Norwood, L. J. Reed, R. E. Root, J. E. Rowe, T. McN. Simpson, Jr., G. F. Winslow, Jr.

A generous luncheon was served to the members and their guests by Goucher College. A committee to coöperate with the National Committee on Mathematical Requirements was appointed, consisting of Professor Abraham Cohen, chairman, Professor Clara L. Bacon, and Mr. Harry English. Professor L. S. HULBURT was elected chairman of the Section, O. S. ADAMS, secretary-treasurer and Professor G. R. CLEMENTS, member of the executive committee. The next meeting will be held at Annapolis, probably early in December.

The following papers were read:

(1) "Determination of longitude by the U. S. Coast and Geodetic Survey" (Illustrated) by Miss SARAH BEALL, Coast and Geodetic Survey;

(2) "Graphical methods applied to a curve of pursuit problem" by Mr. F. V. MORLEY, Johns Hopkins University;

(3) "Modular geometry" by Professor A. A. BENNETT, Technical Staff, Army Ordnance, Baltimore, Md.;

(4) "Some intrinsic geometric properties of plane curves and the related algebraic functions" by Professor J. B. COLEMAN, University of South Carolina;

(5) "Coördinate systems in modern ballistics" by Capt. R. S. HOAR, Aberdeen Proving Grounds.

(6) "Plane curves developed upon cylinders and cones" by Professor J. J. LUCK, University of Virginia;

(7) "Some maxima and minima in elementary geometry" by Dr. T. H. GRONWALL, Technical Staff, Army Ordnance;

(8) "The mathematical basis of the Einstein theory of relativity" by Dr. F. D. MURNAGHAN, Johns Hopkins University;

(9) "Singular curves of a plane pencil" by Professor C. C. BRAMBLE, U. S. Naval Academy;

(10) "An expression for the tangential velocity of a particle" by Dr. G. H. CRESSE, U. S. Naval Academy;

(11) "A note on the roots of the derivative of a polynomial" by Professor W. H. ECHOLS, University of Virginia.

The papers by Dr. Gronwall, Professor Bramble, Dr. Cresse and Professor Echols were read by title. Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. Miss Beall's paper was a brief discussion of the method of determining the true local time at each of two stations the difference of longitude of which is to be determined, of the methods which have been used by the Survey for the comparison of those times, and finally of the work which has been accomplished in determining the longitude of stations in the United States and her possessions, and the connection with the initial longitude station Greenwich, England.

2. A particular case of the curve of pursuit gives rise to the differential equation

$$\left(\frac{dy}{dx}\right)^2 + 4c\frac{dy}{dx} + 4\frac{c^2y}{1-x^2} = 0.$$

This is the case where the pursued point moves around a circle, the pursuer starting at the center and moving c times as fast; x is the distance from the center to the tangent to the curve of pursuit, and y the power of the pursuer with respect to the circle. The differential equation is not susceptible to ordinary methods. Mr. Morley shows that it may, however, be looked upon as defining the directions of a series of integral curves in the plane, treating x and y as Cartesian coördinates. In tracing the integral curves through the point (0, 1), and in finding the value of x when $y = 0$, it is convenient to supplement Runge's graphical methods by finding the flex and cusp loci of the integral curves. The curve of pursuit is a good test for such graphical methods, since the curve itself may be drawn accurately by a bracketing method. Comparison of the value of x obtained by graphical computation and from the curve itself showed an agreement within seven per cent., an error which can be reduced by careful work. This comparison shows graphical methods to be satisfactory and suggestive in such an equation as the above, while integration by series is likely to be troublesome at the critical points.

3. Professor Bennett's paper appears in full in the present issue of the MONTHLY.

4. Professor Coleman considered $y = f(x)$ as representing the equation of a curve, $f(x)$ and its derivatives being continuous. After applying the formulae for rotation of axes, the conditions upon which the resulting equation is independent of the angle of rotation give rise to invariants of different orders which were discussed.

5. The equations of motion of a projectile, referred to the horizontal and vertical at any point in its flight are:

$$\bar{x}'' = -E\bar{x}', \qquad \bar{y}'' = -E\bar{y}' - g, \qquad E = GH/C,$$

where G is a tabular function of velocity, H is a function of altitude and C is a

constant throughout any given trajectory; accents represent time-derivatives. As the earth is not flat, these equations must be altered so as to relate to a single coördinate system, in place of an infinite number of instantaneous systems. Captain Hoar pointed out that the "tangent method" uses cartesian axes, horizontal and vertical at the origin, *i.e.*, the gun. The "curved method" measures x from the gun along a circle concentric with the earth, and y vertically upward from this circle. In each method, the equations are as above, plus terms in $1/r$ and higher powers, r being the radius of the earth. In practice, these terms may be omitted, the slight resulting error being absorbed in the value chosen for C.

6. The parametric equations of the space curves into which plane curves are twisted when their plane is developed on any cylinder or on any right circular cone are derived in the paper by Professor Luck. The purpose of this is to afford to students beginning the study of space curves a means of obtaining, by elementary means, the equations of certain classes of space curves rough models of which are easily available. In the case of the cylinder it is shown that if the arc length is the parameter used for the plane curve the equations of the space curve may be written down with the same parameter. The inverse problem of finding the equation of the plane curve into which a curve on the cylinder will pass when the cylinder is unrolled on the plane is also considered.

8. In this paper Dr. Murnaghan discusses the foundation of Newtonian dynamics, especially of the definitions of absolute systems of reference and vectors, Einstein's law of inertia and his hypothesis as to the nature of space which is sometimes called his "law of gravitation," and an application to the motion of a particle in the neighborhood of a single body.

10. If $f(x, y) = 0$ is the plane path of a particle, then in the equation:

$$\frac{f_x''x'^2 + 2f_{xy}''x'y' + f_y'y'^2}{\sqrt{f_x'^2 + f_y'^2}} + \frac{f_x'x'' + f_y'y''}{\sqrt{f_x'^2 + f_y'^2}} = 0$$

the second fraction is equal to $\alpha \cos \theta$, where α is the magnitude of the total acceleration and θ is the angle between the normal and the direction of acceleration. If then *e.g.*, $f(x, y)$ is taken as $b^2x^2 + a^2y^2 - a^2b^2$, and the acceleration is directed toward a focus, Dr. Cresse shows that the classical expressions for α and v in planetary motion are read off. If for a parabola of higher degree,

$$f(x, y) \equiv y - (a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n) \equiv y - \phi(x),$$

and the acceleration is composed of g taken vertically and of α_T, the additional acceleration along the tangent, then

$$v = \frac{\sqrt{g}}{[-\varphi'(x)]^{1/2}}\sqrt{1 + \left(\frac{dy}{dx}\right)^2}, \qquad \alpha_T = v\frac{\sqrt{g}}{2}\frac{\varphi'''(x)}{[-\varphi''(x)]^{3/2}}.$$

11. Professor Echols's "Note" was published in the last issue of this MONTHLY.

O. S. ADAMS, *Secretary-Treasurer.*

THE APRIL MEETING OF THE IOWA SECTION.

The sixth regular meeting of the Iowa Section of the Mathematical Association of America was held at the University of Iowa, Iowa City, on Saturday morning, April 24, 1920. Twenty-five persons were in attendance, including the following twenty members:

R. P. Baker, J. W. Campbell, E. W. Chittenden, L. M. Coffin, Julia Calpitts, I. S. Condit, Marian Daniells, Fay Farnum, M. E. Graber, R. B. McClenon, F. M. McGaw, J. V. McKelvey, M. A. Nordgaard, J. F. Reilly, H. L. Rietz, Maria Roberts, B. F. Simonson, F. M. Weida, C. W. Wester, W. H. Wilson.

The following officers were elected for 1920–21: Chairman: Professor H. L. Rietz, Univ. of Iowa; Vice-chairman: Professor R. B. McClenon, Grinnell College; Secretary-Treasurer: B. F. Simonson, Upper Iowa University.

The following papers were read:

(1) "Notes from the history of indeterminate analysis" by Professor R. B. McClenon, Grinnell College;
(2) "The taxonomy of algebraic surfaces" by Professor R. P. Baker, State University;
(3) "What is number?" by Professor C. W. Wester, State Teachers College;
(4) "A special form of standardization trajectory" by Professor M. E. Graber, Morningside College;
(5) "Note on a generalization of a theorem of Baire" by Professor E. W. Chittenden, State University;
(6) "The integration of the indefinite integral in the first course" by Dr. W. H. Wilson, State University;
(7) "Limits in secondary mathematics" by Professor J. V. McKelvey, State College;
(8) "A problem in summation of series" by Professor J. F. Reilly, State University;
(9) "Geometric construction for the regular 17-gon" by Linn Smith,[1] student at Grinnell College. (Presented in his absence by Professor McClenon.)

Professor Rietz presented the critical situation in the Iowa colleges and universities of students entering without mathematics, a condition expected soon to arise because of a recent law admitting to the three state institutions, without examination, graduates of any approved secondary institution who present sixteen units. A general discussion followed.

L. M. Coffin, *Secretary-Treasurer.*

[1] Mr. Smith's paper was printed in the last issue of this Monthly.

THE AVERAGE READING VOCABULARY; AN APPLICATION OF BAYES'S THEOREM.

By WARREN WEAVER, University of Wisconsin.

The rule for the computation of *a posteriori* probabilities was first developed by an English clergyman, T. Bayes, and was published after his death in the *Philosophical Transactions* for 1763. The careless application of this rule has led to many paradoxical results,[1] in consequence of which some mathematicians would abandon the rule entirely. Among this number may be mentioned Mr. J. Bing, a Danish actuary, the late Dr. T. Thiele, and especially Professor Chrystal, whose advice is to "bury the laws of inverse probability decently out of sight." The problem herein stated and solved may be of interest since it clearly emphasizes the point the neglect of which has led to incorrect results, since it shows what great allowances may sometimes be made in the *a priori* probabilities of existence and still allow us to change our view in regard to a statistical result, and since it is an answer to a specific case of that interesting question, to what degree does new experimental evidence justify us in modifying previously held opinions.

1. **Statement of the theorem.** Some one of the mutually exclusive causes $A_1, A_2, \cdots A_n$ is to produce an event. When the result is not known (*i.e.*, before the event occurs) the existence probability for each cause is $\pi_1, \pi_2, \cdots \pi_n$ (*i.e.*, the *a priori* probability of the existence of each cause). The event in question occurs. The cause A_1, when it is known to act, gives the productive probability p_1, etc. Then the *a posteriori* probability that the cause A_r produced the event is

$$P_r = \frac{p_r\pi_r}{p_1\pi_1 + p_2\pi_2 + \cdots + p_r\pi_r + \cdots + p_n\pi_n}. \tag{1}$$

If the event in question is able to occur in two alternative ways one of which we call "successful," and the other of which "unsuccessful"; and if, further, the productive probability that the cause A_r produce the event successfully be ω_r, then if the event occurs successfully m times in k trials the *a posteriori* probability that the cause A_r has been the one to act is

$$P_r = \frac{\pi_r\omega_r^m(1-\omega_r)^{k-m}}{\Sigma\pi_i\omega_i^m(1-\omega_i)^{k-m}} \quad (i = 1, 2, \cdots n). \tag{2}$$

The theorem may assume a third form in problems of such a nature that the different causes A_r may be considered different stages of a continuously changing complex. In this case the quantities involved in the formula become definite integrals.[2]

[1] For example, Bing's Paradox: "If among a large group of S equally old persons we have observed no deaths during a full calendar year, then another person of the same age outside the group is certain to die inside the calendar year" quoted from *The Mathematical Theory of Probabilities*, by A. Fisher. Volume 1, New York, 1915, p. 75.

[2] A. Fisher, *l.c.*, p. 67.

To those not entirely familiar with the theorem an example may make the statement of it more clear. Suppose that we have an urn filled with black and white balls in unknown proportion, and that our *a priori* estimate of the existence probability that there are x white and $(b - x)$ black is π_x (b being the total number of balls in the urn). Suppose that we draw k balls from the urn, returning each, and find that of these m are white and $k - m$ black. What is then the most probable mixture in the urn? It will not be the one originally most probable, that is, the one for which π_x is a maximum; nor will it be the one suggested by the drawing,[2] but will obviously be some mixture intermediate between these two. It will be, in fact, a mixture of x^* white and $(b - x^*)$ black, where x^* is that value of x for which

$$P_x = \frac{\pi_x \left(\frac{x}{b}\right)^m \left(\frac{b-x}{b}\right)^{k-m}}{\Sigma \pi_{x'} \left(\frac{x'}{b}\right)^m \left(\frac{b-x'}{b}\right)^{k-m}} \quad (x' = 1, 2, \cdots n) \tag{3}$$

is a maximum. In case π_x is given, by experiment, judgment, or calculation, for a certain finite number of values of x, we might determine this value x^* by plotting to any scale whatsoever, and noting the value of x corresponding to the highest point on the curve. If we should later wish the vertical scale of this curve we could most easily determine it from the fact that the area under it must equal unity.

2. **Statement of the problem.** A test has been devised by the department of educational psychology at the University of Wisconsin to determine a person's reading vocabulary. The process consists of taking at random 200 words from the dictionary, and having a person decide with how many of the 200 words he is familiar—say 117. Then the value of this person's reading vocabulary is taken as

$$(117/200)\ 104{,}000$$

or about 61,000. (104,000 being the approximate number of words in the dictionary.)

The scheme has been found to give, as the result of about five hundred tests by university students, the value given above. This value is far in excess of previous estimates, the general opinion before this test being, according to Professor Starch, that the correct figure was in the neighborhood of 25,000. We wish to investigate whether this process gives us a sound basis for raising our previous estimate of 25,000 to 61,000, and if not, what our answer should be.

Since, as will appear later, the *a priori* probabilities of different estimates have an important effect upon the solution of the problem it is necessary to inquire as carefully as the nature of the existing information will permit into the basis and reliability of this estimate of 25,000 words. Unfortunately the information is vague, but it appears to be an average opinion of those who had been interested in the matter, rather than a definite statistical result. There may have been, to be sure, some numerical method, however unsatisfactory, by which the

[2] Unless, of course, these two mixtures happen to be the same.

estimates were arrived at; or it may be that an investigator with a great deal of experience along this line would venture an estimate based upon intuition alone. It is evident as a mere matter of common sense that if this estimate of 25,000 words were formed upon the basis of one application of this test itself, and the next application of the test furnished an estimate of 61,000, one would not be justified in completely discarding the old estimate for the new. It is obviously a matter of the comparative reliability of the new and old judgments, which comparison is made accurately by the theorem stated.

3. **Solution of the problem.** The actual method used in taking the sample was to take the first word on the kth page of a Webster's *Unabridged Dictionary*, k having such a value that the method would result in a sample of 200. There being so many words to pick from, it is evident that it is immaterial whether or not we consider that we return each word after its drawing: a consideration which in other cases might be important. It seems likely, on an intuitive basis, that if the same person performed the test several times with different samples, or if it were performed with several persons and the same sample, results would be obtained that would vary widely, especially since 200 seems a small sample from a group of 104,000. It should be emphasized therefore that the datum which we use is the average of over five hundred results from different persons. And some knowledge of the variability of these results is important in making an estimate of the stability of the average. The following frequency table gives us an estimate of the variability in the results obtained from a typical group of fifty students, using the same sample of words. It is on the basis of 100 words rather than 200 since, as a matter of proceedure in making the test, the whole list was split up into two lists of 100 each, and the score kept for each separately.

No. of Words Known.	No. of Students.	No. of Words Known.	No. of Students.
46	0	61	1
47	0	62	3
48	1	63	2
49	1	64	3
50	2	65	0
51	0	66	1
52	3	67	3
53	1	68	4
54	2	69	0
55	3	70	5
56	1	71	0
57	3	72	1
58	4	73	0
59	1	74	0
60	5	⋮	—
			50

The result shown is typical of all obtained, a mean variation of approximately five being found among all the students tested. We conclude therefore that 117 is an average of considerable stability, coming as it does from a group of over 500 tests which show a relatively small variability. Unfortunately the data are not available as to how many sets were used: the errors of sampling in the dictionary would be reduced in the approximate ratio of 1 to $\sqrt{n}$ where n is the number of sets used. However the accuracy of any one list as expressed by the mean variation between two lists of 100 words each is between two and three words. We are led then to this conclusion: that if we imagine the hypothetical case of "the average" university student examining a perfectly fair sample of 200 words it seems reasonable to assume that he would find among these 117 that he would know. And any conclusion we may draw from this hypothetical case will have a reliability of approximately $\sqrt{500} = 22.4$ times as much as it would have if it actually came from but a single trial. The fact that 200 is indeed a small sample will later appear in this, that the most probable mixture, even though it may be many times more probable than other mixtures not in the immediate neighborhood of the most probable one, is, as a matter of fact, a mixture whose probability is very small. This apparent paradox is often met with in problems involving large numbers.

A mere change in the wording of the problem makes the application of equation (2) evident. We have an urn filled with b ($= 104{,}000$) words in unknown proportion of "white" (known) words to "black" (unknown) words. From this urn we draw 200 and find 117 white and 83 black. In other words the event in question occurs 200 times in one of two alternative ways, it occurring 117 times in the way which we may call successful. The probability that the dictionary contains a mixture of x known and $b - x$ unknown words is then

$$P_x = \frac{\pi_x \left[\frac{x}{b}\right]^{117} \left[\frac{b-x}{b}\right]^{83}}{\sum_1^n \pi_x \left[\frac{x}{b}\right]^{117} \left[\frac{b-x}{b}\right]^{83}} \tag{4}$$

or

$$P_x = K\pi_x \left[\frac{x}{b}\right]^{117} \left[\frac{b-x}{b}\right]^{83}, K \text{ being a constant.} \tag{5}$$

We see at once that it is impossible to determine the value of x for which this expression is a maximum without a knowledge of the character of the term π_x. This is exactly the point at which errors often creep into applications of the theorem. It is often assumed from the logical principle of insufficient reason that π_x is a constant: in other words since we know too little to form a judgment it is assumed that the *a priori* probabilities of all causes are equal. The principle of insufficient reason leads, however, to notoriously paradoxical results.

For the problem here considered, however, while the theoretically correct result cannot be obtained without a knowledge of π_x, we can easily show that for

practical purposes our knowledge concerning it may be very limited and still sufficient.

Consider the curve

$$P'_x = \left[\frac{x}{b}\right]^{117}\left[\frac{b-x}{b}\right]^{83} \tag{6}$$

where $b = 104{,}000$. This curve has its maximum value when

$$x = (117/200)\,b = 61{,}000 \tag{7}$$

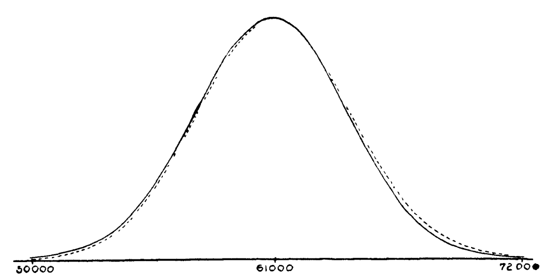
and the maximum value is equal to

$$P'_{61000} = 1.137 \times 10^{-59}. \tag{8}$$

However for $x = 25{,}000$

$$P'_{25000} = 2.207 \times 10^{-83}.$$

To obtain the value of x for which P_x of equation (5) is a maximum we have to multiply every ordinate of the curve P_x' by π_x (the existence probability of a mixture of x known and $b - x$ unknown words), and then take the value of x corresponding to the highest point on this new curve. Therefore unless the *a priori* existence probability of a mixture containing 25,000 known words exceeds the *a priori* existence probability of a mixture containing 61,000 known words in the ratio of 0.5×10^{24} to 1 it is evident that the *a posteriori* probability of a mixture characterized by $x = 61{,}000$ is greater than the *a posteriori* probability of a mixture in which x is only 25,000. In fact we have



$$\frac{P_{61000}}{P_{25000}} = \frac{K\pi_{61000} \times 1.137 \times 10^{-59}}{K\pi_{25000} \times 2.207 \times 10^{-83}} = 0.5 \times 10^{24}\frac{\pi_{61000}}{\pi_{25000}}, \tag{10}$$

which is greater than unity as long as

$$\pi_{25000} < 0.5 \times 10^{24} \times \pi_{61000}. \tag{11}$$

While this may convince us that the previous estimate of 25,000 is to be discarded it may not convince us that an estimate of, say, 50,000 is not as good a new estimate as that of 61,000 which the test indicates. Let us therefore consider the numerical magnitude of the ratio of the *a posteriori* probability of a mixture for which $x = 61{,}000$ to the *a posteriori* probability of a mixture for which $x = 50{,}000$. We have

$$\frac{P_{61000}}{P_{50000}} = \frac{\pi_{61000}}{\pi_{50000}}\left[\frac{61}{50}\right]^{117}\left[\frac{43}{54}\right]^{83} = 77\,\frac{\pi_{61000}}{\pi_{50000}},$$

which is greater than unity unless the *a priori* probability of a mixture of 50,000 known words exceeded the *a priori* probability of a mixture of 61,000 known words by the factor 77. Since neither of these figures, 50,000 and 61,000, is in any way special before the test is made there would seem no justifiable basis for considering one more probable than the other in any such ratio as that just found. The answer to our problem would then be that 61,000 is the most probable answer for the number of words in the dictionary, this conclusion being reached regardless of the character of π_x outside of the one restriction stated in (11). It is understood, of course, that the character of the term π_x in the neighborhood of $x = 61{,}000$ might shift the most probable value slightly, but π_x would surely be changing very slowly for values of x in this vicinity, and the shift would be therefore very small, and negligible for practical purposes. Although the question of whether equation (11) states a reasonable restriction upon the character of π_x is primarily one for educators to settle it would certainly seem sensible to assume that it does. We should surely agree that the previous results were not sufficiently well established that we could consider them, *a priori*, 0.5×10^{24} times as likely to be true as any other result. We must say "any other" result since our estimate of the *a priori* probabilities, it being independent of the result of the test and therefore for psychological reasons best formed before the test takes place, could attach no special importance to the figure 61,000—a number which is not known until after the test is performed. All we could say might be, for example, that we consider a result lying between, say, 20,000 and 30,000 one hundred times more likely than a result lying outside this band, and that we consider it certain that the actual mixture contains more than 5,000 and less than 90,000 words that the average student knows. Such an assumption, coupled with the fact that the total area under the curve $y = \pi_x$ must be unity gives

$$\left.\begin{aligned} \pi_x &= 0 && 0 \leq x < 5000 \\ &= 9.302 \times 10^{-7} && 5000 \leq x < 20000 \\ &= 9.302 \times 10^{-6} && 20000 \leq x < 30000 \\ &= 9.302 \times 10^{-7} && 30000 \leq x < 90000 \\ &= 0 && 90000 \leq x \leq 104000 \end{aligned}\right\} \tag{12}$$

Then we have P_x given by the full line on the graph.

The vertical scale is again obtained from the fact that the area must equal unity. The probability of the most probable mixture is found to be 9.946×10^{-5}, a very small probability as was earlier suggested would be the case. The values of P_x outside the range shown are too small to be indicated.

It is to be especially noted that this curve will approximately represent P_x *whatever the assumption concerning* π_x, only provided, say, that

$$\pi_{25000} < .5 \times 10^{20}\, \pi_{61000} \tag{13}$$

This condition is slightly more stringent than (11), and insures that P_{25000} shall

small variability. Even more important than this, however, is the fact that we cannot in all strictness compare our problem with the analogous urn problem in case the test is used on more than one person, as it of course was. For the content of the dictionary, from our point of view, actually changes with the observer, depending, as it does, upon how many words *he* knows. This is an added source of variability, over and beyond that which would occur due to the ordinary errors of sampling. The result of the above paragraph is still qualitatively applicable.

Our final conclusion is, then, that the experimental evidence of the test completely justifies us in abandoning the old result and accepting the new: and that, moreover, probabilities of mixtures in the immediate neighborhood of the most probable mixture themselves follow the normal Gaussian law as given by equation (16).

A GRAPHICAL AID IN THE STUDY OF FUNCTIONS OF A COMPLEX VARIABLE.

By NORMAN MILLER, Queen's University.

The impossibility in three dimensions of representing graphically a function of a complex variable makes it necessary for the student to call on his imagination in other ways in order to realize the properties of these functions. Two methods are common in the geometrical theory of functions. One is to represent in two different planes or in two Riemann surfaces the variables z and w and to study the correspondence between the points of the two planes or surfaces, which is determined by the relation $w = f(z)$. The second method, which does much to illuminate the subject for the beginner, is to represent in one plane both the independent and dependent variables and to interpret the transformation kinematically as a flow of the points in the plane.[1]

A complete graph of the function $w = f(z)$ or $u + iv = f(x + iy)$ consists of a 2-dimensional manifold in space of four dimensions. Nevertheless the student, in his effort to visualize the function, thinks instinctively of a surface spread out over the plane of z. Such a surface is actually determined by taking for a third coördinate the absolute value of $f(z)$. Calling the third coördinate ζ the equation of the surface is

$$\zeta = \sqrt{[u(x, y)]^2 + [v(x, y)]^2},$$

only the positive square root being taken. In this representation all points on a circle of center 0 in the w-plane yield the same ordinate ζ. It is, in fact, by making no distinction among the points of such a circle that we are able to pass from a two-way spread in four dimensions to an actual surface in three dimensions.

It is interesting to enquire what properties of the function $f(z)$ are exhibited

[1] See in this connection an article by Cole, *Annals of Mathematics*, vol. 5, June, 1890.

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in the surface $\zeta = |f(z)|$ and what properties are lost. The following discussion will apply only to functions which are in general analytic. At a zero of such a function the surface reaches down to the plane of z and in the neighborhood of a pole it extends upward indefinitely. It will easily be seen that the following properties are among those put into evidence in the surface.

(1) A function which is analytic over the entire plane and remains finite is a constant. (Liouville.)

If $f(z)$ is analytic in the plane without exception and the surface $\zeta = |f(z)|$ does not rise indefinitely then it can only be a plane parallel to the z-plane.

(2) The zeros of an analytic function which is not identically zero are isolated points.

The surface cannot touch the z-plane along a curve unless it is the z-plane itself. A zero may give rise to a conical point on the surface, it may belong to an edge, or it may be an ordinary point at which the z-plane is tangent.

(3) The poles of a function which is otherwise analytic are isolated points.

If from each pole a perpendicular be erected to the plane of z the surface rises indefinitely about each of these lines and nowhere else.

(4) Let $f(z)$ be any function other than a constant, analytic except for poles within and on the boundary of a region S. Then, excluding the zeros and poles, $|f(z)|$ can have neither a minimum nor a maximum at any other interior point of S. The same property may be stated more precisely as follows. In any subregion of S which includes no poles either within or on its boundary $|f(z)|$ takes on its maximum value on the boundary and in any subregion which includes no zeros $|f(z)|$ takes on its minimum value on the boundary.

If, then, a cylinder be erected on any closed curve C of S, with elements perpendicular to the z-plane it will cut from the surface a portion which will have its highest point on the boundary provided C includes no poles and which will have its lowest point on the boundary provided C includes no zeros.

(5) In the neighborhood of an isolated essentially singular point the function comes arbitrarily near to every assigned value (Weierstrass). It follows that $|f(z)|$ comes arbitrarily near to every positive real value. If a line be drawn upward from the singular point perpendicular to the plane of z then a point moving in the surface can approach as closely as we like every point on this line. The effort to visualize the surface in the neighborhood of an essentially singular point brings out clearly the nature of the singularity.

Models of the surfaces $\zeta = |f(z)|$ for some of the simpler functions would be of value in teaching the elementary theory of functions. Evidently $\zeta = |z|$ is one sheet of a circular cone with vertex 0 and axis of ζ for axis. $\zeta = |z^2|$ is a paraboloid of revolution with vertex touching the z-plane at the origin. The character of the surfaces for some other simple functions is shown in the figures 1–5. Only a portion of each surface is drawn.

(1) $\zeta = |z(z-1)|$ touches the z-plane at the points 0 and 1. Let us obtain the Cartesian equation of this surface.

$$z(z-1) = (x+iy)(x+iy-1)$$

one above the plane $\zeta = 1$ and to the right of the plane $x = 0$, the other below the plane $\zeta = 1$ and to the left of $x = 0$. · Each sheet of the surface approaches the plane $\zeta = 1$ asymptotically.[1] Every section of the surface by a plane parallel to the z-plane is a circle. The nature of the essential singularity at $z = 0$ is well illustrated.

(4) $\zeta = |\sin z|$ and (5) $\zeta = |\sin^2 z|$. The Cartesian equations of these surfaces are easily found to be $\zeta = \sqrt{\sin^2 x + \sinh^2 y}$ and $\zeta = \sin^2 x + \sinh^2 y$. The surfaces show the periodic character of the functions and the fact that $|\sin z|$ may take on any positive value in a period strip.

These examples are intended only to illustrate and not to exhaust the possibilities of the subject. The student should try to visualize the surface $\zeta = |\sin 1/z|$ in the neighborhood of its essential singularity. A few hours spent in constructing these surfaces will contribute towards overcoming the feeling of "a stranger in a strange land" which most students experience in entering upon the study of the theory of functions of a complex variable.

MODULAR GEOMETRY.

By ALBERT A. BENNETT, Baltimore, Md.

(Read before the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America, May 15, 1920.)

The study of relations in a single variable, in the case of finite number fields, is not, of course, a new idea, although it is awakening perhaps more interest now than in the last century. A study of several variables and in particular a characteristically geometrical treatment for finite fields may fairly be said to have been introduced within the past few years, and the only extended discussions of the whole subject to be found in any general treatise are those in Veblen and Young's *Projective Geometry*, and in a little book by G. Arnoux, *Essai de Géomêtrie Analytique Modulaire*, Paris, 1911. References are given in L. E. Dickson, *On Invariants and the Theory of Numbers* (Madison Colloquium Lectures), 1914, page 98.

§ 1. Plane Geometry, Modulo 3.

We shall for the sake of concreteness, consider the individual case of the field, modulo 3. This may be thought of as generated by the set of residues obtained by dividing integers by the number 3. Thus any integer, n, falls in one of three mutually exclusive classes according as n, $n - 1$, or $n - 2$ is an integral multiple of 3. We say that n is respectively congruent to 0, 1 or 2, expressed in symbols by $n \equiv 0 \ (mod\ 3)$, $n \equiv 1 \ (mod\ 3)$, $n \equiv 2 \ (mod\ 3)$, respectively. There are then only three numbers, 0, 1, 2, in the field. For these we have the following addition and multiplication tables.

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[1] Except in the $y\zeta$-plane. The plane $\zeta = 1$ intersects the surface in a line parallel to the y-axis.—Editor.

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[1] Except in the $y\zeta$-plane. The plane $\zeta = 1$ intersects the surface in a line parallel to the y-axis.—Editor.

Addition:

	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

Multiplication:

	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

Subtraction and division (except by 0) are always uniquely possible. Incidentally $-1 = \frac{1}{2} = 2$, in this field. The equation $x(x-1)(x-2) = 0$, has 0, 1, 2, as roots, and therefore $x^3 - x = 0$ is identically satisfied by every number of the field. Every linear equation with coefficients in the field has a unique root in the field, but quadratic equations exist with coefficients in the field whose roots are imaginary with respect to the field, that is, are not themselves in the field but may be introduced by a consistent extension of the system. Thus $x^2 + 1 = 0$, is not satisfied by 0, 1, or 2, and hence has no real roots in the field. Every polynomial with coefficients in the field may be successively reduced if of degree higher than 2 by means of the identity $x^3 = x$, so as to be represented as of degree not greater than 2. The following are therefore the totality of essentially distinct polynomials in the field:

$$\begin{array}{lllllllll} 0, & 1, & 2, & x^2, & x^2+1, & x^2+2, & 2x^2, & 2x^2+1, & 2x^2+2, \\ x, & x+1, & x+2, & x^2+x, & x^2+x+1, & x^2+x+2, & 2x^2+x, & 2x^2+x+1, & 2x^2+x+2, \\ 2x, & 2x+1, & 2x+2, & x^2+2x, & x^2+2x+1, & x^2+2x+2, & 2x^2+2x, & 2x^2+2x+1, & 2x^2+2x+2. \end{array}$$

One may proceed in this manner to study functions of one variable. But our present interest is geometry, taken for simplicity as of two dimensions, so that we shall not inquire further into properties of polynomials in one variable.

By points in the non-homogeneous modular plane (*mod* 3) will be meant any set of objects each of which is regarded as determining uniquely and being uniquely determined by an ordered pair of numbers of the modular field. The number of distinct points is equal to the number of distinct ordered pairs, (m, n), where each of these is in the field. The plane consists therefore of nine points. By a straight line in the plane is meant the set of points satisfying a single linear equation in two unknowns, the whole being in the field. Each line is found to contain three points. The set of distinct lines corresponds to the following, and are twelve in number.

$$\begin{array}{llll} x = 0, & y = 0, & x + y = 0, & x + 2y = 0, \\ x = 1, & y = 1, & x + y = 1, & x + 2y = 1, \\ x = 2, & y = 2, & x + y = 2, & x + 2y = 2. \end{array}$$

The equation $2x + y = 1$, for example, is identical in the field with $x + 2y = 2$, either equation being obtained from the other by multiplying by 2. Through any point there will be four distinct lines of the plane. Lines are either parallel or intersect in a point of the plane as in the usual system. For example,

$2x + y = 1$, and $x + y = 2$, intersect in a point whose coördinates may be obtained by the usual method of determinants, viz.,

$$\left(\frac{\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix}}{\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}}, \frac{\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}}{\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}} \right) = (2, 0).$$

Quadratic loci may be studied in the usual manner. In particular, there are circles. These are of three classes according as the square of the radius is 0, 1 or 2. A circle with imaginary radius, *e.g.*, $\sqrt{2} = \sqrt{-1}$, has in this geometry real points. For example, $x^2 + y^2 = 2$, contains the points (1, 2), (2, 1), (1, 1), (2, 2), which are as numerous as the points on the circle, $x^2 + y^2 = 1$, which has a real radius, namely the points, (1, 0), (2, 0), (0, 1), (0, 2). The circle of zero radius contains as usual but one point. Thus the three circles with any given common center, together contain all the points of the plane, or in other words a circle may be drawn with a given center and passing through a given point. It is interesting to notice that the circle of radius one is also of radius two, since in this system $2 = -1$.

The trigonometry in this plane is rather primitive since the only angles occurring are multiples of 45 degrees. The trigonometric functions yield the usual table reduced modulo 3.

	sin	cos	tan	cot	sec	csc
0°	0	1	0	–	1	–
45°	$2i$	$2i$	1	1	i	i
90°	1	0	–	0	–	1
135°	$2i$	i	2	2	$2i$	i
180°	0	2	0	–	2	–
225°	i	i	1	1	$2i$	$2i$
270°	2	0	–	0	–	2
315°	i	$2i$	2	2	i	$2i$

where however i is not a real number but may be introduced as an imaginary quantity satisfying the relation $i^2 = 2 = -1$.

Ellipses, parabolas, hyperbolas, all exist with their usual properties although many of these become trivial. Higher plane curves do not exist in the real plane.

As in the ordinary geometry, real points fall into three classes with respect to a non-degenerate circle: those points from which there are two real distinct tangents,[1] those with but one real tangent, those with no real tangents to the circle. Ordinarily the first is the class of exterior points, the second, the class of points on the circle, and the third the class of interior points of the circle.

[1] By tangent we mean the line whose equation is formed as in the ordinary analytical geometry. For example, the tangent to the circle $x^2 + y^2 = 1$ at a point x', y' will be the line whose equation is $xx' + yy' = 1$.

In the second place finite groups, in particular multiply transitive groups can regularly be interpreted and frequently with profit, in the geometrical language of a suitably chosen modular space. The abstract group becomes in many cases geometrically intuitive only by reference to an interpretation of this sort. In the third place the subject still presents novelty and has interest on its own account, the methods of investigation are extremely simple, and most of the results for a given modular space are obtained with ease and completeness.

QUESTIONS AND DISCUSSIONS.

EDITED BY W. A. HURWITZ, Cornell University, Ithaca, N. Y.

REPLIES.

15 [*1914*, 278; *1916*, 353; *1920*, 114]. In the *Proceedings of the Royal Society of Edinburgh*, Vol. 7, p. 144, in some mathematical notes by Professor P. G. Tait, it is stated:

"If $x^3 + y^3 = z^3$, then $(x^3 + z^3)^3y^3 + (x^3 - y^3)^3z^3 = (z^3 + y^3)^3x^3$.

"This furnishes an easy proof of the impossibility of finding two integers the sum of whose cubes is a cube."

How does this "easy proof" follow?

REMARKS BY THE EDITOR.

A number of communications relative to this question have been received. As regards content they fall into three classes. In the first (and largest) class may be placed mere demonstrations of the algebraic premiss of the question,—that if $x^3 + y^3 = z^3$, then

$$(x^3 + z^3)^3y^3 + (x^3 - y^3)^3z^3 = (z^3 + y^3)^3x^3.$$

This is indeed easy, and we are not to suppose that the question implied any difficulty in obtaining an identity within the reach of any high school student who has learned the fundamentals of algebraic notation. It seems necessary to emphasize, therefore, that the truth of the algebraic identity is granted, and that what is desired is an "easy" proof, based on this identity, that the equation $x^3 + y^3 = z^3$ has no solutions in positive integers.

In a second group fall the replies indicating that Tait may have been mistaken. One suggestion is that he may have noted that any of the relations $x = y$, $x = -z$, $y = -z$ will reduce the second equation to an identity, but not the first. This is not impossible; it amounts to supposing that Tait considered the two equations as equivalent,—that he regarded the second as being not only a necessary, but also a sufficient condition for the first. If so, he would not be the only mathematician of note who has confused necessary and sufficient conditions; but such an error would be so obvious in the present instance as to seem rather unlikely. Another correspondent suggests that Tait took for granted that an equation of the form $x^n + y^n = z^n$ cannot hold for an infinite number of sets of values of x, y, z when $n > 1$. Of course, however, this

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of definitions and theorems concerned only with continuity and forming a science which may be called the Analysis Situs of the given space; in the other case there is a corresponding science treating of algebraic relations among figures describable in geometrico-algebraic language.

By employing the term "algebra" in a wide sense to apply to commutative number fields (that is, number fields with commutative multiplication), one may study a "general algebraic geometry," which will include as sub-cases all those distinct instances that arise by further characterizing the commutative number field under consideration. The entire science may be regarded as the study, in geometrical language to be sure, of algebraic relations in a given number of variables defined over a general commutative number field. Theorems in this general algebraic geometry are clearly independent of Analysis Situs since, as has been shown, number fields may be selected in which continuity has no application.

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relation *does* hold for an infinite number of cases when $n = 2$; and we can scarcely ascribe to Tait such ignorance of the status of Fermat's theorem as this assumption would imply.

The third class of replies consists of those which seek to show that from a solution of $x^3 + y^3 = z^3$ can be deduced another whose elements are smaller, and hence to apply the method of infinite descent. This is the most reasonable direction in which to look for Tait's "easy" proof. Unfortunately all proofs of this type which have been sent in are fallacious.

30 [*1916*, 88, 354; *1920*, 114]. A certain Normal University wishes to offer thirty-five hours of college mathematics for the benefit of high-school teachers. What should these courses be in order that, primarily, they may be of the greatest value to high-school teachers of mathematics and, secondarily, that they may furnish stimulus for a more extended pursuit of the subject?

Reply by H. E. Webb, Central High School, Newark, N. J.

An answer to this question is conditioned upon the state of mind of the high-school teachers in question. If they have never taken college mathematics at all, thirty-five hours is as a teacupful of water to a thirsty horse. It should be borne in mind that high-school students often have capacity for reasoning far beyond that with which they are credited; but at the same time that they object to various canonical forms.

A teacher of high-school algebra should have a knowledge of the fundamental principles of number such as are found in the early pages of any good text in the elementary theory of functions of the real variable. He should have also some *clear* notions regarding series, equations generally, and the significance of a complex number system. He will then find that many of the difficulties which his students encounter are due to a dimly realized sense of the need of distinctions which he himself is able to draw clearly. He should also be familiar with the elements of the analytic geometry of the straight line and the conics, in order properly to weigh the value of various topics and examples for the future practical or theoretical development of mathematics. He is presumably familiar with plane trigonometry, as this is now accounted a high school subject.

A teacher of elementary geometry should above all else be familiar with the fundamentals of synthetic projective geometry, together with the logical foundations of metrical Euclidean geometry, in order to escape the embarrassment of finding that his pupils' logic is better than his own.

To bring these various topics into a compass of thirty-five hours calls for a high degree of skill on the part of the instructor. But it may be noted that no great amount of drill in technique is needed, and that many sub-topics which are often presented in college courses may be omitted from an intensive course of this sort.

The college course which seems least of all to bear directly upon secondary mathematics is the calculus, excepting in so far as the derivative may be used in analyzing higher equations. But this seems to many out of place in the high school.

37 [*1919*, 151; *1920*, 115]. Criticize the following as fundamental definitions of elementary geometry:

A *plane surface* is the limit approached by a finite portion of the surface of a sphere as the radius increases without limit.

A *straight line* is the limit approached by a finite portion of the circumference of a circle as the radius increases without limit.

I. Reply by A. A. Bennett, Baltimore, Md.

The above will be criticized from three aspects,—logical, postulational, and psychological.

From a logical standpoint, the above "definitions" are faulty in that they fail to define. While it is true that a sphere or circle may be viewed abstractly to the extent that the location of the center with respect to other objects need not be specified, caution must be observed when limits are to be used. The limit of a "finite portion" is not definite. As the radius is increased, the finite portion must change, but *how*? If the center remain fixed, a plane or line obtained in accordance with directions could contain no finitely accessible point. If the center move, why might not the subtended angle vary also? By steadily diminishing the angle, the resulting limit might be bounded or even reduce to a point. Unless care be exercised there will be no limit at all. Logical objections (which are numerous) can be met, however, by sufficient care in statement.

Geometry cannot afford to resemble a circle in having neither beginning nor end; nor ought its center of interest be removed "to infinity." Somewhere terms must be introduced for the first time which shall become fundamental for all further developments. While a geometry of inversion may properly start with spheres and circles, Euclidean geometry is usually found to be more tractable when starting with lines and planes. The above "definitions" imply a reversal of practice and leave one wondering as to what shall be done with spheres and circles. Any discussion involving isolated definitions implies a background of a system in which these are to find their place. No extended discussion of the proposed definitions is appropriate until at least a sketch of the development of Euclidean geometry from spheres and circles, the notion of radius, applications of limits, etc., all prior to the mention of lines and planes, is vouchsafed.

Psychologically, curved surfaces and lines appear more complicated than linear ones. Any use of the notion of limits is at least unfortunate in definitions, until the existential character of the object seems to demand it. In the definition of the length of an arc, area of a surface, etc., limits are usually necessary and appropriate. The "incommensurable case" depending upon the existence of irrational numbers, is a different but analogous instance in which limits are sometimes at least appropriate. Even where limits are freely employed few applications would be more objectionable in an elementary geometry on a non-projective basis than the discussion of what happens when a point moves off "to infinity," a region for which the Euclidean geometer has an abhorrence. The circle as the limit of polygons suggests merely an abstraction which can be visualized in a finite portion of space, and even lines were formerly avoided in favor of finite segments.

A careful introduction in a course supplementary to the usual elementary geometry, of the notion of points at infinity, coincident points, tangents as the limit of secants, general and degenerate cases, and incidentally of lines as special members in linear systems of circles—but *not* for the purpose of definition—would surely give the student valuable breadth of vision.

II. Reply by H. E. Webb, Central High School, Newark, N. J.

These definitions are open to many serious objections: In the second definition, the expression "the radius increases" can be interpreted only by employing the straight line concept itself. A beginner is not slow to notice this. The definition also violates the usual assumptions of order for points on a line in Euclidean geometry, and renders impossible many familiar demonstrations. A necessary consequence of the definition would be the equality of radii of two straight lines, since otherwise there would be no laws of congruence in the Euclidean sense. If, however, straight lines are assumed to have equal radii, then any two of them must intersect, and we have an elliptic geometry, which is interesting in its way, but is not Euclidean. From an elementary standpoint it is undesirable to define one configuration as the "limit" of another. Much confusion has in the past resulted from the misuse of the term "limit" in elementary texts. A "limit" should be regarded as a number, and a variable as any one of a class of numbers, all of which are determined according to some law. The idea of a Dedekind cut, explained in simple terms and not too nicely, is easily understood by students who have had two years of elementary mathematics. The numerical measure of the circumference of a circle can be explained in this way. But at the outset of the study of geometry the attempt to define a straight line as the limit of anything else would make a certain unavoidable confusion worse confounded.

Attempts in elementary geometry to define a *straight line* are usually worse than useless.

The first definition could presumably be so reshaped as to be correct, as far as phraseology is concerned, but would be open to the other objections mentioned above; it would also necessitate a restatement of most of the familiar definitions, and would serve no useful purpose as compared with the usually accepted criterion for points on a plane.

III. Remarks by the Editor.

It does not seem certain that the second definition necessitates equal "radii" of straight lines or leads to an elliptic geometry, as indicated in Mr. Webb's reply. In fact, the definition is so vague as to permit a variety of conjectures as to the interpretation which must be made in order to draw conclusions from it. Both the preceding replies exhibit beyond doubt the utterly unsuitable character of the proposed definitions for inclusion in a logical basis of elementary geometry. With respect to the last sentence of Professor Bennett's reply the Editor would express his belief

The tendency to stress technique in algebra and logic in geometry seems to grow out of three sources:

1. It is generally believed that technique is more easily acquired by the immature student than logic, and traditionally algebra comes first in the high-school course;
2. Geometric figures form a more concrete basis for logic than algebraic symbols;
3. Elementary geometry has been bequeathed to this modern world as a monument of rigorous logic which too many feel it would be profane to treat in any other way.

From experience, observation and experiment I am convinced that the above tendency, whatever its cause, violates sound principles of teaching. Algebra and geometry should be taught side by side in the high school, not one before the other, and technique and logic should receive about equal stress throughout the course. Technique without logic and logic without action are alike foreign to the adolescent. It must, however, be borne strictly in mind that insistence upon difficult technique causes as many failures as insistence upon rigorous logic; both must be kept within the ability of the high-school student. In a word, the development of technique and logic should receive equal attention; the material for the development of technique should be simplified and enriched by the introduction of geometrical drawing and construction work; the logic sought should be the logic of the high-school student, not that of the expert logician; technique should gradually increase in fineness and logic in rigor as the course advances.

This procedure will aid to develop the power to understand the quantitative relations of the everyday world that confront the student and the ability to put this understanding into successful execution. The procedure of question 38 has a tendency to develop memorists in logic and jugglers in technique.

NEW QUESTIONS.

40. How great emphasis is laid in freshman mathematics upon the elementary algebra of complex numbers? A recent paper by an eminent electrical engineer seems to indicate the need of a knowledge of this subject on the part of draftsmen and mechanics of very limited educational opportunities. The syllabus of the College Entrance Examination Board mentions the topic under the caption "Advanced Algebra," but the question papers call for only the slightest study of numbers of this type. Is Euler's theorem ($e^{i\theta} = \cos\theta + i\sin\theta$) usually presented in college algebra, or is it left to the calculus? Does the topic deserve greater emphasis than it usually gets, for the sake of applications in the field of periodic currents?

41. A reader asks for an elementary proof of the following two propositions in number theory, either of which can readily be obtained from the other:

Every positive integer of the form $8n + 3$ *is the sum of three odd squares.*

Every positive integer is the sum of not more than three triangular numbers.

NOTE. Bachmann[1] states that these theorems have been proved only by the use of the theory of ternary quadratic forms, and refers to the discovery of the theorems by Gauss, and a comparatively simple proof by Dirichlet,[2] by means of this theory.—EDITOR.

DISCUSSIONS.

Professor Noble, discussing the familiar fact that the addition of two rational algebraic fractions each in its lowest terms, by the usual method of reducing to the least common denominator, may lead to a result not in its lowest terms, shows in what way this situation presents itself, and verifies the process suggested by one text-book for securing the result in reduced form. That a similar contingency may arise in arithmetic is clear to anyone who has performed the addition of $\frac{1}{2}$ and $\frac{1}{6}$.

Professor Johnson calls attention to a very simple formula for an approximation to the smaller acute angle of a right triangle in terms of the sides, in which the error is surprisingly small. The formula was given, as the author states,

[1] *Niedere Zahlentheorie*, Leipsic and Berlin, 1910, Teil 2, p. 325.

[2] *Crelle's Journal*, Vol. 40, p. 228; *Liouville's Journal*, Series 2, Vol. 4, p. 233.

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By equating to zero the derivative of the expression

$$\frac{172 \sin B}{2 + \cos B} - \frac{180B}{\pi}$$

we find that the maximum positive deviation occurs for $B = 22° \; 20'$ approximately, and is equal to about 0.0122° or less than 45″. The maximum negative deviation, as indicated by the author's tables, is 0.0730° or between 4′ and 5′.

I. Note on the Solution of Fractional Equations.

By Charles A. Noble, University of California.

In Wilczynski and Slaught's *College Algebra* the following interesting statement occurs on page 242:

"If, however, we express every one of the rational fractions as a sum of simple partial fractions, then unite all of the partial fractions which have the same denominator into a single one, and finally add these partial fractions together, using as a common denominator the lowest common multiple of the denominators of the simple partial fractions, we may be sure that the sum obtained in this way is in its lowest terms."

The correctness of this rule was considered evident by the authors and the proof of it was omitted. The following is a formal proof:

Let F, G, P, and Q be polynomials in x. Assume first that G and Q have only one common linear factor, and that each has only one other linear factor, where each factor may be present in any order of multiplicity. Let F/G and P/Q, assumed to be in their lowest terms, be expressed in partial fractions:

$$\frac{F}{G} = \frac{A_1}{(x-\alpha)^k} + \frac{A_2}{(x-\alpha)^{k-1}} + \cdots + \frac{A_k}{x-\alpha} + \frac{B_1}{(x-\beta)^\lambda} + \frac{B_2}{(x-\beta)^{\lambda-1}} + \cdots + \frac{B_\lambda}{x-\beta},$$

$$\frac{P}{Q} = \frac{C_1}{(x-\beta)^\mu} + \frac{C_2}{(x-\beta)^{\mu-1}} + \cdots + \frac{C_\mu}{x-\beta} + \frac{D_1}{(x-\gamma)^\nu} + \frac{D_2}{(x-\gamma)^{\nu-1}} + \cdots + \frac{D_\nu}{x-\gamma}.$$

[1] *Nouvelle trigonométrie, où l'on trouve le moyen de calculer toutes sortes de triangles rectilignes sans les tables de sinus, . . .,* 1699.

[2] *De mathematica perfectione; Opera,* Paris, 1514.

[3] F. A. Protche, *Mémoires de la société académique de l'Aube,* Vol. 51 (1887), pp. 149–162. H. Brocard, *Mathesis,* 1889, p. 161; P. Mansion, ibid., p. 162; P. Mansion, ibid., p. 181. Further references on closely related formulas may be found in Th. Vahlen, *Geometrische Konstruktionen und Approximationen,* Leipsic, 1911, pp. 188–206.

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would be

$$\frac{F}{G}+\frac{P}{Q}=\frac{7x^3-35x^2+55x-27}{(x-1)^2(x-2)(x-3)}=0,$$

from which $7x^3-35x^2+55x-27=0$, or $x=1,\ 2\pm\frac{1}{7}\sqrt{7}$.

The root $x=1$ is extraneous. The sum of F/G and P/Q is not in its lowest terms. The reason is that $C_1=-B_1$. If the rule stated above is followed, one gets

$$\frac{F}{G}+\frac{P}{Q}=\frac{3}{x-1}+\frac{1}{x-2}+\frac{3}{x-3}=\frac{7x^2-28x+27}{(x-1)(x-2)(x-3)}=0,$$

which yields the true roots $x=2\pm\frac{1}{7}\sqrt{7}$.

II. Determination of an Angle of a Right Triangle, without Tables.

By Roger A. Johnson, Hamline University.

If a, b, c denote respectively the longer leg, the shorter leg, and the hypotenuse, then the value, in degrees, of the smaller acute angle is given approximately by

$$B=172\frac{b}{a+2c}.$$

This formula was given by Ozanam, 1699.

In practice it often happens that all three sides of a right triangle are known, and it is desired to find the angles quickly; the above formula will give them without the use of tables, with almost four-place accuracy for angles up to 35°, and better than three-place accuracy up to 45°. When the sides are given in integers, this method is simpler than the ordinary one. Again, it is useful as a check-formula in testing solutions.

The proof is easily effected by means of Taylor's Series. If we write

$$\frac{b}{a+2c}=\frac{\sin B}{2+\cos B}=f(B),$$

the expansion of $f(B)$ is easily found to be

$$f(B)=\frac{B}{3}\left(1-\frac{1}{180}B^4-\frac{1}{1512}B^6\cdots\right),$$

whence it is evident that for small values of B, $f(B)$ is nearly equal to $\frac{1}{3}B$ radians. That is, if we desire to have B in degrees,

$$B\text{ (degrees)}=\frac{3\times 180}{\pi}f(B)+e,$$

where e is a small correction. We may correct the error in part by replacing the coefficient $540/\pi$, or $171.89\cdots$, by the simpler number 172. The degree of

accuracy is best exhibited by working out the actual value given by the formula for various angles. The subjoined table shows that the angles given by the formula are too large, though the error scarcely exceeds .01 of a degree, up to about 33°; that thereafter the results are too small, and after the angle exceeds 45°, the discrepancy becomes rapidly larger. Since we can always use the formula to compute an angle less than 45°, this later divergence does not affect its usefulness.

True Value.	Value by Formula.	True Value.	Value by Formula.
5°	5.0033	30°	30.0067
10°	10.0065	35°	34.9945
15°	15.0094	40°	39.9703
20°	20.0115	45°	44.9270
25°	25.0112	50°	49.8562

RECENT PUBLICATIONS.

REVIEWS.

General Theory of Polyconic Projections. By OSCAR S. ADAMS, Geodetic Computer. Published by the Department of Commerce, U. S. Coast and Geodetic Survey, Serial No. 110, Special Publication No. 57, Washington, D. C., 1919. 174 pages. Price 25 cents.

To quote from the author's preface, "In this publication an attempt has been made to gather into one volume all of the investigations that apply to the system of polyconic projections." The author gives[1] Tissot's definition of a polyconic projection as "one in which the parallels of latitude are represented by arcs of a non-concentric system of circles with the centers of these various circles lying upon a straight line." Polyconic projections of the sphere and the ellipsoid of revolution only are considered, the whole purpose of the work being the construction of maps of the surface of the earth either as a whole or in part. The table of contents has thirty-one headings: Determination of ellipsoidal expressions, Development of general formulas for polyconic projections, Classification of polyconic projections, Rectangular polyconic projections, Stereographic meridian projection, Derivation of stereographic meridian projection by functions of a complex variable, Construction of stereographic meridian projection, Table for stereographic meridian projection, Stereographic horizon projection, Derivation of stereographic horizon projection by functions of a complex variab'e, Proof that circles project into circles in stereographic projections, Construction of stereographic horizon projection, Solution of problems in stereographic projections, Conformal polyconic projections, Determination of the conformal projection in which the meridians and parallels are represented by circular arcs, Special cases of the projection, General study of double circular projections,

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Conformal double circular projections, Cayley's principle, Discussion of the magnification on the conformal double circular projection, Equivalent, or equal area, polyconic projections, Conventional polyconic projections, non-rectangular double circular projections, Projection of Nicolosi or globular projection, Projection of P. Fournier, Ordinary or American polyconic projection, Tissot's indicatrix, Tables of elements of the ordinary, or American polyconic projection, Transverse polyconic projection, Projection for the international map on the scale of 1 : 1 000 000, Tables for the projection of the sheets of the international map of the world.

The book contains forty-eight figures and a folded frontispiece giving a transverse polyconic projection of the North Pacific Ocean.

The author states in the preface that in the preparation of the volume the following works were especially consulted: M. A. Tissot, *Mémoire sur la Représentation des Surfaces et les Projections des Cartes Géographiques*, Paris, 1881; A. Germain, *Traité des Projections des Cartes Géographiques*, Paris, 1866(?); N. Herz, *Lehrbuch der Landkartenprojectionen*, Leipzig, 1885; W. W. Hendrickson, U. S. N., *Notes on Stereographic Projection.*

The work may be described on the whole as a careful and detailed discussion of all such representations of the earth's surface on a plane as come under the definition of polyconic projection which have been found useful in practical map making. In general it may be said that the purpose of the discussion of each projection considered is first to determine and prove properties of the map as a whole, second to derive formulas for the coördinates of the point in the plane map representing a point of the earth's surface of given latitude and longitude, third to determine the scale or magnification of the map at any point in the directions of the circles of latitude and longitude. Both geometrical and analytical methods are extensively used. Not much mathematical knowledge is required of the reader, indeed we suppose the book to be intended rather for the practical maker of maps than for the mathematician. The author says, in the preface "it is hoped that the treatment may be found complete enough to serve all practical purposes." The figures are good, the book is well printed and remarkably free from misprints; we have noticed but two: on page 18, line 8, for M read N, on page 156, line 10 from the bottom, for "about" read "by."

Two paragraphs we find of particular interest, the analytical discussion of rectangular polyconic projection, pages 13 to 18, and the determination by the use of the complex variable of the conformal projection in which both meridians and parallels are represented by circular arcs, pages 80 to 86, presumably after Lagrange.

Considered as a mathematical treatise the book has some faults. There is an almost total lack of exact references, except to the author's earlier publication, *General Theory of Lambert Conformal Projection*, Special Publication No. 53, U. S. Coast and Geodetic Survey. It would add greatly to the interest of the work to give the names of the originators of the various projections with dates and exact references. A more serious fault is the nearly complete absence of any

By failing to consider this possibility one solution of the problem considered is lost, that is $s = \rho$, giving in the map circles of latitude passing through one point.

JAMES K. WHITTEMORE.

Lezioni di Calcolo Infinitesimale dettata nella R. Università di Bologna e redatte per uso degli studenti. S. PINCHERLE. Seconda edizione riveduta. Bologna, N. Zanichelli, 1920. 8vo. 8 + 785 pp. Price 40 lire.

Translation of an extract from the "avvertenza alla seconda edizione": "The lectures on the infinitesimal calculus which I gave to the press at the end of 1915, not without fear and trembling, have won favor, beyond all expectations, with the mathematical public, so that the call for a new edition came too soon to permit those modifications and additions that I had in mind to introduce.

"The second edition differs from the first only by slight changes, and only those parts have been retouched in which greater clarity or precision of statement seemed to me necessary."

Contents—Introduction, 1–50; Section first: Differential calculus, 51–302; Section second: Integral calculus, 303–484; Section third: Geometrical applications of infinitesimal calculus, 485–630; Section fourth: Differential equations, 631–764. There is a 14-page alphabetical index.

Leçons de Géométrie Supérieure. Par E. VESSIOT. Edition revue et augmentée. Avec une préface de M. G. Koenigs. Paris, J. Hermann, 1919. 10 + 376 pp. Price 30 francs.

Preface by E. Vessiot: "La première édition de ces leçons autographiée [1906], ayant été rapidement épuisée, j'ai accepté l'offre de réimpression que m'a faite M. Hermann. Les fautes d'impression avaient été corrigées par M. Anzemberger en vue de cette réédition. J'ai revu et amélioré la rédaction; et j'y ai fait des additions importantes. . . ."

Contents—I: Révision des points essentiels de la théorie des courbes gauches et des surfaces développables, 1–18; II: Surfaces, 19–34; III: Etude des éléments fondamentaux des courbes d'une surface, 35–60; IV: Les six invariants, La courbure totale, 61–81; V: Surfaces réglées, 82–120; VI: Congruences de droites, 121–160; VII: Congruences de normales, 161–189; VIII: Les congruences de droites et les correspondances entre deux surfaces, 190–237; IX: Les complexes de droites et les équations aux dérivées partielles du premier ordre, 238–268; X: Complexes linéaires, 269–292; XI: Transformations de contact, Transformations dualistiques, Transformations de Sophus Lie, changeant les droites en sphères, 293–315; XII: Systèmes triple orthogonaux, 316–325; XIII: Congruences de sphères et systèmes cycliques, 326–354; Exercices, 355–371.

Physical Bases of Ballistic Table Computations. Ordnance Textbook. [By Professor A. A. BENNETT.] (War Department, Document 972). Washington, Government Printing Office, 1920. 4to. 17 pp.

This monograph constitutes Part I of the Introduction of New Ballistic Tables being prepared by the Ordnance Department.

Quotation from 'Prefatory Remarks': "These new tables for exterior ballistics . . . were designed and supervised by the author of this introduction. The circumstances demanding their construction will not be reviewed in this part, nor will any account be given here of their form and content, nor of the technique and special devices used in their computation. It is only the physical facts and theories upon which these tables are founded that will here be treated, and even the history of these theories will be left with little more than mention.

"The matter treated here is qualitative rather than quantitative. The methods of numerical integration used in computing the trajectories, while in themselves only methods of approximation, are capable of giving results with any preassigned degree of accuracy, and as used yield vastly more precise figures than justified by the physical data available or the physical assumptions employed. This precision is obtained, however, at no extravagant sacrifice of labor, and secures results which are conveniently regular. Less accurate methods hitherto in vogue are now insufficient. The total number of physical factors in the problems of exterior ballistics is practically infinite. The elimination of all but a few is justified only by careful quantitative

experiments and computations which are not discussed here. To catalogue and describe, merely, the various influences affecting the projectile in flight, leaves the relative importance assigned to each apparently hap-hazard and arbitrary. Thus this introduction seeks to mention only those physical assumptions which enter explicitly into the computations of the accompanying tables, and attempts only such justification for them as is in keeping with the expository nature of this paper."

Solid Geometry with Problems and Applications. Revised edition. By H. E. SLAUGHT and N. J. LENNES. Boston, 1919. 12mo. 8 + 211 pp. Price $1.00.

Quotations from the Preface: "In re-writing the *Solid Geometry* the authors have consistently carried out the distinctive features described in the preface of the *Plane Geometry*. . . . Owing to the greater maturity of the pupils it has been possible to make the logical structure of the *Solid Geometry* more prominent than in the *Plane Geometry*. The axioms are stated and applied at the precise points where they are to be used. Theorems are no longer quoted in the proofs but are only referred to by paragraph numbers; while with increasing frequency the student is left to his own devices in supplying the reasons and even in filling in the logical steps of the argument. For convenience of reference the axioms and theorems of plane geometry which are used in the Solid Geometry are collected in the Introduction.

"In order to put the essential principles of solid geometry, together with a reasonable number of applications, within limited bounds (156 pages), certain topics have been placed in an Appendix. This was done in order to provide a minimum course in convenient form for class use and not because these topics, Similarity of Solids and Applications of Projection, are regarded as of minor importance. In fact, some of the examples under these topics are among the most interesting and concrete in the text. . . .

"The treatment of incommensurables throughout the body of this text, both Plane and Solid, is believed to be sane and sensible. In each case, a frank assumption is made as to the existence of the concept in question (length of a curve, area of a surface, volume of a solid) and of its realization for all practical purposes by the approximation process. Then, for theoretical completeness, rigorous proofs of these theorems are given in Appendix III, where the theory of limits is presented in far simpler terminology than is found in current text books and in such a way as to leave nothing to be unlearned or compromised in later mathematical work."

Plane Trigonometry for Secondary Schools. By C. DAVISON. Cambridge: at the University Press, 1919. 12mo. 4 + 334 pp. Price 6 shillings and 6 pence.

Contents—I: Trigonometrical ratios of an acute angle, 1–30; II: Circular measure, 31–40; III: Circular functions of an angle of any magnitude, 41–56; IV: Graphs of the circular functions, 57–65; V: Circular functions of compound angles, 66–84; VI: Circular functions of multiple angles, 85–98; VII: Transformation of trigonometrical expressions, 99–111; VIII: Solution of trigonometrical equations, 112–141; IX: Inverse functions, 142–147; X: Circular functions of sub-multiple angles, 148–158; XI: The geometry of the triangle, 159–174; XII: The solution of triangles, 175–187; XIII: Practical applications, 188–208; XIV: The principal circles associated with a triangle, 209–223; XV: The geometry of the quadrilateral, 224–232; XVI: Areas of regular polygons and circles, 233–238; XVII: Inequalities, 239–247; XVIII: Approximations and errors, 248–256; XIX: DeMoivre's theorem, 257–274; XX: Series, 275–287; Problem papers, 288–308; Answers, 309–334.

Elementary Applied Mathematics. A practical course for general students. By W. P. WEBBER. New York, Wiley, 1920. 12mo. 10 + 115 pp. Price $1.25.

Preface: "This course is a response to a demand in this university [of Pittsburg]. It is an effort to provide a course that is complete in itself and sufficiently general and practical to meet the needs of a large class of students who are not to specialize in mathematics but who do want some elementary mathematical training that they can use in everyday affairs.

"While the course has been developed simultaneously with Webber and Plant's *Introductory Mathematical Analysis* and there is some material common to the two courses, yet they are addressed to distinctly different groups of students.

"A knowledge of elementary algebra and geometry is presupposed. Students with little or no knowledge of formal geometry have succeeded with the course.

"The course has been given by lecture or in mimeograph form to classes during the past several years. . . ."

Contents—I: Review of elementary algebra, 1–9; II: Geometric theorems, 10–13; III: Methods of calculation, 14–27; IV: Graphic representation, 28–41; V: Ratio, proportion and variation, 42–47; VI: Geometric problems, 48–54; VII: Rectangular coördinates, graphs of equations, empirical formulas, 55–71; VIII: Applications of percentages, 76–79; IX: Analysis of food and receipts, 80–85; X: Individual and family accounts, 86–96; Tables 97–115.

ARTICLES IN CURRENT PERIODICALS.

American Machinist, New York, volume 52, March 25, 1920: "War on the decimal system" by F. Franz, 682–684; "Octaval notation and the measurement of binary inch fractions" by A. Watkins, 685–688.

Annals of Mathematics, second series, volume 21, no. 3, March, 1920: "A Green's theorem in terms of Lebesgue integrals" by H. E. Bray, 141–156; "Bilinear operations generating all operations rational in a domain Ω" by N. Wiener, 157–165; "On the enumeration of proper and improper representations in homogeneous forms" by E. T. Bell, 166–179; "A proof of Jordan's theorem about a simple closed curve" by J. W. Alexander, 180–184; "Linear order in three dimensional Euclidean and double elliptic spaces" by G. H. Hallett, Jr., 185–202; "Further properties of the general integral" by P. J. Daniell, 203–220; "Summability of double series" by L. L. Smail, 221–223; "The fundamental theorem of celestial mechanics" by J. L. Coolidge, 224.

Bulletin of the American Mathematical Society, volume 26, no. 7, April, 1920: "Parametric equations of the path of a projectile when the air resistance varies as the nth power of the velocity" by F. H. Safford, 289–293; "Infinite systems of functions" by W. E. Milne, 294–300; "On certain related functional equations" by W. H. Wilson, 300–312; "The equation $ds^2 = dx^2 + dy^2 + dz^2$" by E. T. Bell, 312–318; "A property of permutation groups analogous to multiple transitivity" by W. B. Carver and Mrs. Estella F. King, 319–322; "College algebras" by Elizabeth B. Cowley, 323–329 [review of E. B. Skinner's *College Algebra* (New York, Macmillan, 1918), of W. C. Brenke's *Advanced Algebra* (New York, Century, 1917) and of Helen A. Merrill and Clara E. Smith's *A First Course in Higher Algebra* (New York, Macmillan, 1917)]; "Notes," 329–333; "New publications," 334–336.

Canadian Chartered Accountant, Toronto, Ontario, volume 9, no. 4, April, 1920: "Bond schedules for the amortization of a premium or accumulation of a discount" by S. D. Killam, 231–236.

Educational Review, volume 59, no. 4, April, 1920: "Junior high school mathematics" by T. Lindquist, 296–303. [Quotation: "Students who have completed a satisfactory junior high school course in mathematics should be able during their senior high school to complete plane and solid geometry in one year, physics in one year, and algebra III together with trigonometry or college algebra the final year."]

Engineering and Contracting, Chicago, volume 53, February 4, 1920: "Instruction and tables for reducing labor in curve computation" by J. H. Lilly, 137–138.

Engineering News-Record, New York, volume 84, February 19, 1920: "Solution of compound curve problems" by L. R. Brown, 378–379 [Reprinted from *Electric Railway Journal,* Jan. 17, 1920]—March 4: Table for converting from scale 1 in. = 100 ft. in plotting curves," 486.

Industrial Management, volume 59, no. 4, April, 1920: "The mathematics of labor turnover" by C. G. Barth, 315–318.

International Marine Engineering, New York, volume 25, March, 1920: "Pointing off when using the slide rule" by B. W. Manier, 228–229.

Journal of Educational Research, University of Illinois, volume 1, no. 3, March, 1920: "A shorter method for computing the coefficient of correlation" by L. P. Ayres, 216–221; "A bibliography of standard tests for the high school" by W. S. Monroe, 240–242 [VIII. Mathematics].

"While the course has been developed simultaneously with Webber and Plant's *Introductory Mathematical Analysis* and there is some material common to the two courses, yet they are addressed to distinctly different groups of students.

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American Machinist, New York, volume 52, March 25, 1920: "War on the decimal system" by F. Franz, 682–684; "Octaval notation and the measurement of binary inch fractions" by A. Watkins, 685–688.

Annals of Mathematics, second series, volume 21, no. 3, March, 1920: "A Green's theorem in terms of Lebesgue integrals" by H. E. Bray, 141–156; "Bilinear operations generating all operations rational in a domain Ω" by N. Wiener, 157–165; "On the enumeration of proper and improper representations in homogeneous forms" by E. T. Bell, 166–179; "A proof of Jordan's theorem about a simple closed curve" by J. W. Alexander, 180–184; "Linear order in three dimensional Euclidean and double elliptic spaces" by G. H. Hallett, Jr., 185–202; "Further properties of the general integral" by P. J. Daniell, 203–220; "Summability of double series" by L. L. Smail, 221–223; "The fundamental theorem of celestial mechanics" by J. L. Coolidge, 224.

Bulletin of the American Mathematical Society, volume 26, no. 7, April, 1920: "Parametric equations of the path of a projectile when the air resistance varies as the nth power of the velocity" by F. H. Safford, 289–293; "Infinite systems of functions" by W. E. Milne, 294–300; "On certain related functional equations" by W. H. Wilson, 300–312; "The equation $ds^2 = dx^2 + dy^2 + dz^2$" by E. T. Bell, 312–318; "A property of permutation groups analogous to multiple transitivity" by W. B. Carver and Mrs. Estella F. King, 319–322; "College algebras" by Elizabeth B. Cowley, 323–329 [review of E. B. Skinner's *College Algebra* (New York, Macmillan, 1918), of W. C. Brenke's *Advanced Algebra* (New York, Century, 1917) and of Helen A. Merrill and Clara E. Smith's *A First Course in Higher Algebra* (New York, Macmillan, 1917)]; "Notes," 329–333; "New publications," 334–336.

Canadian Chartered Accountant, Toronto, Ontario, volume 9, no. 4, April, 1920: "Bond schedules for the amortization of a premium or accumulation of a discount" by S. D. Killam, 231–236.

Educational Review, volume 59, no. 4, April, 1920: "Junior high school mathematics" by T. Lindquist, 296–303. [Quotation: "Students who have completed a satisfactory junior high school course in mathematics should be able during their senior high school to complete plane and solid geometry in one year, physics in one year, and algebra III together with trigonometry or college algebra the final year."]

Engineering and Contracting, Chicago, volume 53, February 4, 1920: "Instruction and tables for reducing labor in curve computation" by J. H. Lilly, 137–138.

Engineering News-Record, New York, volume 84, February 19, 1920: "Solution of compound curve problems" by L. R. Brown, 378–379 [Reprinted from *Electric Railway Journal*, Jan. 17, 1920]—March 4: Table for converting from scale 1 in. = 100 ft. in plotting curves," 486.

Industrial Management, volume 59, no. 4, April, 1920: "The mathematics of labor turnover" by C. G. Barth, 315–318.

International Marine Engineering, New York, volume 25, March, 1920: "Pointing off when using the slide rule" by B. W. Manier, 228–229.

Journal of Educational Research, University of Illinois, volume 1, no. 3, March, 1920: "A shorter method for computing the coefficient of correlation" by L. P. Ayres, 216–221; "A bibliography of standard tests for the high school" by W. S. Monroe, 240–242 [VIII. Mathematics].

Machinery, New York, volume 26, January, 1920: "Development of time slide-rule" by J. B. Conway, 453–454; "To draw a circle tangent to three given circles" by C. N. Pickworth, 459–460—March: "Computation of diameters and circumferences of circles" by E. T. Meehan, 656.

Mathematics Teacher, volume 12, no. 3, March, 1920: "Association of Mathematics Teachers of New Jersey: Report of the Committee of First-Year High-School Mathematics," 89–100; "Proposed syllabus in algebra" by C. F. Wheelock, 101–114; "Mathematics for the physiologist and physician" by H. B. Williams, 115–123 [Paper read before the Mathematical Association of America, January 1, 1920]; "Love mathematical" by R. C. Gillies, 124–125; "New Books," 126–127.

Mind, new series, no. 114, April, 1920: Review by C. D. Broad of A. N. Whitehead's *The Principles of Natural Knowledge* (Cambridge, 1919), 216–231.

Monist, volume 30, no. 2, April, 1920: "Philip Bertrand Jourdain," 161–182 + frontispiece portrait [Nearly half of this appreciation is from "memories" written by Jourdain's youngest sister Milly; the article concludes with a list of thirty-one articles written for the *Monist,* 1908–1920]; "Elliptic orbits and the growth of the third law of Newton" by P. E. B. Jourdain, 183–198; "Newton's theorems on the attraction of spheres" by P. E. B. Jourdain, 199–202; "Leonardo da Vinci (Born 1452. Died 1519)" by M. Jourdain, 280–291.

Philosophical Magazine, sixth series, volume 39, April, 1920: "Applications of quaternions to the theory of relativity" by H. T. Flint, 439–449—May: "On the advance of the perihelion of a planet, and the path of a ray of light in the gravitation field of the sun" by A. Anderson, 626–628; Review of A. N. Whitehead's *An Enquiry concerning the Principles of Natural Knowledge* (Cambridge, 1919), 629–631 [Last paragraphs: "It will be seen then that this book is simply invaluable to anyone who wishes to bring himself into line with the new principle of relativity, whether his interest be scientific in the narrow sense or philosophical in the wide sense.

"John Stuart Mill tells us in his Autobiography that he was at times actually depressed by the thought that musical chords though practically infinite in the number of combinations they admitted were yet in reality finite and exhaustible. Our feeling as we close Professor Whitehead's book is one almost of elation at the thought of how little we know, and how uncertain is the little we think we know, when we form our concepts of the framework of infinite Nature."]

Revue de l'Enseignement des Sciences, volume 13, nos. 127–130, July–December, 1919: "Enveloppes des courbes et des surfaces à un paramètre" by F. Meyer, 177–186; "Sur les fonctions linéairement distinctes" by P. Montel, 186–190; "Sur la surface du triangle" by P. Montel, 190–197; "Sur l'équation en nombres entiers $a^2 + b^2 = c^2$" by A. Lévy, 197–211; "Limite de $(1 + 1/x)^x$ pour x infini" by R. Dontot, 211–214; "Questions de forme" by J. Juhel-Rénoy, 215–218; "Problèmes de mathématiques donnés au baccalauréat en Octobre 1918," 218–223, 229–234; "Examens et Concours de 1919: Agrégation des sciences mathématiques et Ecole Normale Supérieure," 241–251.

Revue de Mathématiques Spéciales, volume 13, no. 1, October, 1919: "Sur la somme des puissances semblables des n premiers nombres entiers" by R. Dontot, 1–4; Solutions of questions in algebra, analytic geometry, analysis, and mechanics, 4–10, 17–22; Questions in the written and oral examinations for the Ecole Polytechnique, 1919, 10–15; Ecole Centrale Concours de 1919, 23–24—No. 2, November: "Sur la somme des puissances semblables des n premiers nombres entiers" (suite) by R. Dontot, 25–28; Problems proposed and solved in examinations for the Ecole des Ponts et Chaussées, Ecole Polytechnique, Ecole Centrale, etc., 29–56—No. 3, December: "Sections planes et sphériques du tore" by A. Sainte-Laguë, 57–67; Problems and Solutions, 60–80—No. 4, January, 1920: "Sections planes et sphériques du tore" (fin) by A. Sainte-Laguë, 81–83; Problems and Solutions, 83–104—February, 1920: "Enveloppe des droites obtenus en faisant tourner d'un angle constant, α, les tangentes à une courbe plane autour de leurs points de contact" by Jean Mourret, 105–106; Problems and solutions, 107–128.

Revue du Mois, volume 15, January, 1920: "Radioactivité, probabilité et déterminisme" by E. Borel, 33–40— March: "La réforme de l'enseignement et les anciens combattants" by E. Borel, 225—227.

Revue Scientifique, April 10, 1920: "La vie et l'oeuvre de Lord Kelvin" by E. Picard, 193–207.

School and Society, volume 11, March 20, 1920: "The National Committee on Mathematical Requirements," 343–344—April 17, 1920: "The Bode theory of transfer applied to the teaching of mathematics" by E. B. Lytle, 457–462.

Science Progress, volume 14, April, 1920: Recent advances in pure mathematics, by Dorothy M. Wrinch, 536–543; Recent advances in applied mathematics, by S. Brodetsky, 543–550; "Octaval notation for inch fractions," 646–647 [Quotation: "A rule for engineer's precision measurements has just been issued which embodies a proposal for a logical notation for the inch binary fractions. Such notation is not new, but for the first time is put into a practical form for the workshop. . . . In place of the radix of 10 used for decimal notation, this rule . . . uses a radix of 8. Just as in decimals $0.234 = \frac{2}{10} + \frac{3}{10^2} + \frac{4}{10^3} \ldots$ so in octavals $0_0234 = \frac{2}{8} + \frac{3}{8^2} + \frac{4}{8^3}$;"] "Logic and mathematics" by the late P. E. B. Jourdain [review of J. B. Shaw's *Lectures on the Philosophy of Mathematics* (Chicago and London, 1918) and of B. Russell's *An Introduction to Mathematical Philosophy* (London and New York), 669–674; Reviews by Dorothy Wrinch of H. Lamb's *An Elementary Course of Infinitesimal Calculus* (3d edition, Cambridge, 1919), and of L. C. Karpinski, H. Y. Benedict, and J. W. Calhoun's *Unified Mathematics* (Boston and London, 1919), 678–679.

Scientia, volume 27, no. 3, March, 1920: Review by G. Loria of H. Bateman's *Differential Equations* (London, 1918), 228–229; Review by W. J. Greenstreet of L. Huxley's *Life and Letters of Sir Joseph Dalton Hooker* (London, 1918), 236–239.—No. 4, April: Review by G. Loria of E. Bortolotti's *Italiani scopritori e promotori di teorie algebriche* (Modena, 1919), 315–317; Review by R. Morcolongo of P. Appell and S. Dautheville's *Précis de mécanique rationnelle* (2e éd., Paris, 1918), 317–318.

Technology Review, Cambridge, Mass., volume 22, no. 2, April, 1920: "Richard Cockburn Maclaurin" by J. P. Munroe, 252–262; "Resolutions by the Alumni Council," 263–264 [regarding R. C. Maclaurin]; "The work of Professor Edward C. Pickering at the Massachusetts Institute of Technology, 1867–1877" by C. R. Cross, 277–288 [Professor Pickering was Thayer professor of physics at the Institute for the ten years preceding his appointment as professor of astronomy and director of the Harvard Observatory in 1876; he established the first physical laboratory in the United States]; "Joseph John Skinner, Died November 12, 1919" by W. T. Sedgwick, 289–290 + portrait [Professor Skinner, who was assistant professor in mathematics at M. I. T. 1896–1904, was born in Vermont in January, 1842. "From 1874 to 1881 he taught mathematics at the Sheffield Scientific School, and also some classes in physics and astronomy. One of his pupils in 1874 was the writer, whose recollections of 'Tutor Skinner' are of a man always genial and especially courteous to his students. It was in his classes that some students were for the first time in their lives treated as gentlemen and equals, and the reaction was encouraging and stimulating. On the other hand 'Tutor Skinner' did not escape chaffing in a familiar rhyme which ran:

"There was a bold tutor and he was a sinner.
His first name was Joseph, his last name was Skinner.
He wrestled the Freshmen and brought them to time,
With his tangent, co-tangent, co-secant, co-sine."]

Tôhoku Mathematical Journal, volume 17, nos. 1–2, February, 1920: "On the conservative field of force" by K. Ogura, 1–6; "A substitute for Duhamel's theorem" by G. James, 7–9; "Ueber Kriterien für Irreduzibilität ganzzahliger algebraischer Gleichungen" by M. Fujiwara, 10–17; "On the law of errors in the space of p dimensions" by A. Guldberg, 18–23; "Aeronautical photogrammetry with an appendix on reconnoitring photogrammetry" [in Japanese] by T. Iwat-uki, 24–63; "On a certain transcendental integral function in the theory of interpolation" by K. Ogura, 64–72; "Sur les polygones podaires d'un polygone plan donné" by V. Thébault, 73–83; "On the converse of the theorem of Scheeffer's" by Y. Uchida, 84; "Proof of the theorems due to Messrs. Kojima and Okada" [in Japanese] by M. Watanabe, 85–87 "Groups of order g containing $(g/2) - 1$ involutions" by G. A. Miller, 88–102; "On some methods of construction in elementary geometry of three dimensions" by K. Yanagihara, 103–108; "On the nature of the integrals of a certain linear differential equation" by K. Ôishi, 109–117; "Sur des points rémarquables du triangle" by V. Thébault, 118–122; "Remarks on the note 'On the Fourier constants'" by M. Plancherel and K. Ogura, 123–128; "On the theory of interpolation" by K. Ogura, 129–145.

University Bulletin, Louisiana State University, new series, volume 12, no. 3: March, 1920: "The mathematical process and some of its disciplines" by S. T. Sanders, 31 pp.

AMERICAN DOCTORAL DISSERTATIONS.

H. J. ETTLINGER, "Existence theorems for the general real self-adjoint linear system of the second order," *Transactions of the American Mathematical Society*, vol. 19, January, 1918, pages 79-86. (First part of dissertation, Harvard, 1919).

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL AND OTTO DUNKEL.

Send all communications about Problems and Solutions to B. F. FINKEL, Springfield, Mo.

PROBLEMS FOR SOLUTION.

[N.B. The editorial work of this department would be greatly facilitated if, on sending in problems, the proposers would also enclose their solutions—*when they have them.* If a problem proposed is not original the proposer is requested *invariably* to state the fact and to give an exact reference to the source.]

2850. Proposed by SARAH BEALL, U. S. Coast and Geodetic Survey.

An unknown star is observed at the altitudes h_1 and h_2 at the respective times t_1 and t_2, the latitude being known also. Obtain formulas for the right ascension and declination of the star: (1) when the time interval $t_2 - t_1$ is large: (2) when the time interval is so small that $(h_2 - h_1)/(t_2 - t_1)$ may be taken as the value of dh/dt corresponding to the mean altitude $(h_1 + h_2)/2$ and the mean time $(t_1 + t_2)/2$. This problem sometimes arises when a bright star is observed through the clouds.

2851. Proposed by HILLEL PORITSKY, Cornell University.

Does there exist an analytic function, satisfying the functional equation, $f(z + 1) = e^{f(z)}$?

2852. Proposed by D. H. RICHERT, Bethel College, Newton, Kan.

What is the radius of a cylinder inscribed in a right cone, radius of base $R = 5$ inches, and altitude $h = 18$ inches, the volume of the cylinder to be $(1/n = 3/4)$ that of the cone?

2853. Proposed by J. S. BROWN, Southwest Texas State Normal College, San Marcos, Texas.

Find the side and apothem of a regular pentagon inscribed in a circle, without the use of extreme and mean ratio.

2854. Proposed by C. N. MILLS, Heidelberg University.

Solve the simultaneous equations for x and y,

$$x^n + y^n = a_n, \qquad x^{n-1} + y^{n-1} = a_{n-1}.$$

2855. Proposed by J. L. RILEY, Stephenville, Texas.

Show that the circle of curvature at any point of the ellipse cannot pass through the centre unless the eccentricity be greater than $1/\sqrt{2}$.

2856. Proposed by O. S. ADAMS, U. S. Coast and Geodetic Survey.

Show that for the real domain defined by $+1 > x > -1$, s a positive integer,

$$\frac{1}{(1-x^s)^{1/s}}\int_0^x \frac{dx}{(1-x^s)^{(s-1)/s}} = x + \sum_{n=1}^{n=\infty} \frac{2(s+2)(2s+2)\cdots(ns-s+2)}{(s+1)(2s+1)\cdots(ns+1)} x^{ns+1}$$

and

$$\frac{1}{(1-x^s)^{(s-1)/s}}\int_0^x \frac{dx}{(1-x^s)^{1/s}} = x + \sum_{n=1}^{n=\infty} \frac{n!\, s^n}{(s+1)(2s+1)\cdots(ns+1)} x^{ns+1}.$$

2857. Proposed by the late L. G. WELD.

A savings bank offers to pay 3% interest on deposits, said interest to be continuously compounded, *i.e.*, compounded at infinitesimal intervals of time. What would be the amount of $1.00 for one year?

K_5 with center at F. Let G be one of the points of intersection[1] of K_4 and K_5. Draw the line $l_4 = GE$ intersecting[1] the line l_3 at the point H. With H as center and radius HA draw the circle K_6; let J be one of its intersections with the line l_4. With AJ as radius and P as center draw the circle K_7 intersecting[3] the circle K in the points B and B_1. Draw the chords $l_5 = BPB'$ and $l_6 = B_1PB_1'$. These are the required chords. Our construction has required the drawing of six lines l_1, l_2, l_3, l_4, l_5, l_6 and seven circles K_1, K_2, K_3, K_4, K_5, K_6, K_7, the locating of ten necessary points A, B, B_1, D', D, E, F, G, H, J. (If one assumes that the center C is not given, as one might from the statement of the problem, the construction is much more difficult.)

Also solved by George Agins, E. H. Clarke, H. N. Carleton, H. H. Downing, Emanuel Goldfarb, E. D. Grant, Laura Guggenbuhl, R. A. Johnson, Marcia L. Latham, E. W. Martin, F. V. Morley, A. G. Montgomery, H. L. Olson, W. B. Pierce, Arthur Pelletier, Joseph Rosenbaum, Elijah Swift, Charles Schuman, L. G. Weld, C. C. Yen, and the Proposer.

2754 [1919, 73]. Proposed by J. W. LASLEY, JR., University of North Carolina.

Given $\bar{x} = \tan^{-1} \dfrac{z}{\sqrt{x^2 + y^2}}$, $\bar{y} = \tan^{-1} \dfrac{y}{x}$, $\bar{z} = \log \sqrt{x^2 + y^2 + z^2}$, solve for x, y, and z in terms of $\bar{x}$, $\bar{y}$, and $\bar{z}$.

Solution by E. S. Smith, University of Cincinnati

From the given equations, we have $z/(\sqrt{x^2 + y^2}) = \tan \bar{x}$ (1), $y/x = \tan \bar{y}$ (2), and

$$x^2 + y^2 + z^2 = e^{2\bar{z}}. \tag{3}$$

Substituting the value of y from (2) in (1), gives

$$z = x \sec \bar{y} \tan \bar{x}. \tag{4}$$

Substituting the values of y and z from (2) and (4) in (3), we have

$$x = \pm\, e^{\bar{z}} \cos y \cos \bar{x}. \tag{5}$$

Hence, from (2) and (5),

$$y = \pm\, e^{\bar{z}} \sin \bar{y} \cos \bar{x}. \tag{6}$$

From (4) and (6), we have

$$z = \pm\, e^{\bar{z}} \sin \bar{x}. \tag{7}$$

Hence, the result is $x = \pm\, e^{\bar{z}} \cos \bar{x} \cos \bar{y}$, $y = \pm\, e^{\bar{z}} \sin y \cos \bar{x}$, and $z = \pm\, e^{\bar{z}} \sin \bar{x}$.

Also solved by Marcia L. Latham, E. W. Martin, H. L. Olson, George Paaswell, Arthur Pelletier, C. H. Richardson, and D. L. Stamy.

2755 [1919, 73]. Proposed by J. L. RILEY, Stephenville, Texas.

Every number whose square is the sum of the squares of two consecutive integers is equal to the sum of the squares of three integers of which two, at least, are consecutive.

Solution by Elijah Swift, University of Vermont.

It is well known that the solution of the equation $a^2 + b^2 = c^2$, where a, b, c are relatively prime integers, is given by the formulas, $a = m^2 - n^2$, $b = 2mn$, $c = m^2 + n^2$, where m and n

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points corresponding to G exist. The radius most easily specified for which the existence of G can be proved is the radius AF. It would not do, for example, to say "let us take any radius greater than a half of AF" because that does not specify an exact radius and it would need to be proved that any radius which was used was actually "greater than a half of AF". The simplest radius to use for the construction and a logical proof is the radius AF.

[3] Here we have assumed as in previous cases that there are points of intersection, but in this case there are none unless $AP \leqq AJ$. Since AJ is one-third of the required chord BB', when it exists, it follows that $AP \leqq \frac{1}{3}BB'$, which is impossible if AP is greater than one-third of a diameter of K. If $AP \leqq \frac{1}{3}$ of a diameter, there always exists a chord which is trisected at P.

are relatively prime. In the case above $a - b = \pm 1$, whence either (1) $(m - n)^2 - 2n^2 = 1$, or (2) $(m + n)^2 - 2m^2 = 1$. The equation (2) may be derived from (1) by interchanging m and n and changing the sign of n. This would not affect the value of c, nor the algebraic work below,[1] hence we need consider only (1).

Writing (1) in the form $(m - n)^2 - 1 = 2n^2$, and factoring we have

$$(3) \qquad [m - n - 1][m - n + 1] = 2n^2.$$

Since the two factors differ by 2 and their product is even, each is even, 2 is the H.C.F., and one factor is twice an odd square and the other the square of an even number. Accordingly we have

$$(4) \begin{matrix} m - n - 1 = 2\alpha^2 \\ m - n + 1 = 4\beta^2 \end{matrix} \qquad \text{or} \qquad (5) \begin{matrix} m - n - 1 = 4\beta^2 \\ m - n + 1 = 2\alpha^2, \end{matrix}$$

where α and β denote integers. From (3), (4) and (5) we find

$$(6) \qquad n = 2\alpha\beta, \qquad m = \alpha^2 + 2\beta^2 + 2\alpha\beta, \qquad \pm 1 = 2\beta^2 - \alpha^2,$$

the upper sign resulting from equations (4): the lower, from (5). Substituting these values in the expression for c, we find

$$c = m^2 + n^2 = (\alpha^2 + 2\beta^2 + 2\alpha\beta)^2 + (2\alpha\beta)^2 = (\alpha^2 + 2\alpha\beta)^2 + (2\alpha\beta + 2\beta^2)^2 + (2\alpha\beta)^2.$$

But $(\alpha^2 + 2\alpha\beta) - (2\alpha\beta + 2\beta^2) = \mp 1$ from (6), so that c is expressed in the desired form.

It is worthy of note that every solution of the equation $\lambda^2 - 2\mu^2 = \pm 1$ leads to a triangle having the given property, and such triangles can be obtained only from such solutions. All these can be found by developing $\sqrt{2}$ into a continued fraction and taking the numerator and denominator of any convergent as λ and μ respectively. In this way we get the following set of solutions:

λ	μ	m	n	a	b	c
1	1	2	1	3	4	$5 = 0^2 + 1^2 + 2^2$
		5	2	21	20	$29 = 2^2 + 3^2 + 4^2$
3	2	12	5	119	120	$169 = 3^2 + 4^2 + 12^2$
		29	12	697	696	$985 = 12^2 + 20^2 + 21^2$
7	5	70	29	4059	4060	$5741 = 20^2 + 21^2 + 70^2$.

3]. **Proposed by PAUL CAPRON, U. S. Naval Academy.**

Given a parallelogram with center O, vertices $PQP'Q'$, mid-points of sides $ABA'B'$ (cyclic order $PAQBP'A'Q'B'$). Let K be any point of OA. Draw KLH parallel to $Q'Q$ cutting AQ at L, and draw BLM, meeting OA produced at M. Draw MH, parallel to PP', to meet KLH at H, and draw $B'KE$ to meet BL at E. Repeat, changing A, B, P, Q to A', B', P', Q', respectively, and *vice versa*. Repeat each of the foregoing, changing P, P', B, B' to Q, Q', B', B, respectively, and *vice versa*.

What are the loci of E and H? Show that EH passes through A' when K is a point of OA, through A when K is a point of OA'. Consider the effect of interchanging the rôles of A and B.

(This construction, as commonly given, is specialized in these particulars: the parallelogram is rectangular, the divisions of OA are equal, and the locus of H is not found.)

I. Solution by Arthur Pelletier, Montreal, Can.

Let $OB = a$ and $OA' = b$ be the axes of coördinates and, taking $OK = -mb$, we find the following equations of the indicated straight lines:

(1)	(KLH)	$x/a + y/b = -m,$	(2)	(BLM)	$x/a - my/b = 1,$
(3)	(MH)	$x/a - y/b = 1/m,$	(4)	$(B'KE)$	$x/a + y/mb = -1.$

By eliminating m from (2) and (4) we find for the equation of the locus of E, $x^2/a^2 + y^2/b^2 = 1$. Hence when K passes from A to O the point E describes the part AB of an ellipse inscribed in the parallelogram, tangent to the sides at A, B, A', B'. The other parts of the ellipse will evidently be obtained by making the changes indicated in the problem. Similarly, we obtain from (1) and (3) the equation $x^2/a^2 - y^2/b^2 = -1$. Hence the locus of H is the part AHC of an hyperbola tangent to PQ at A etc.

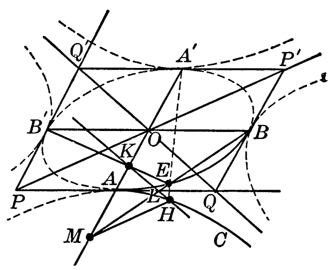
[1] The new n would be negative, consequently $\alpha\beta$ in (6) would be negative. Two of the three numbers, the sum of whose squares is c, would be $2\alpha\beta - \alpha^2$, $2\alpha\beta - 2\beta^2$ and the difference is again ± 1.

If we add to the two sides of (1) the corresponding sides of (3) multiplied by m, we obtain $(1 + m)(x/a) + (1 - m)y/b = 1 - m$, the equation of a straight line through H. The same process applied to (2) and (4) yields the *same* equation and, since the point $(0, b)$ satisfies the new equation, we see that H, E, and A' lie on this straight line.

Note.—A geometrical solution may be given. Starting with a square we easily find as loci, a circle and two conjugate equilateral hyperbolas. Then, by projection, we derive the loci found above. By the theorem of Meneläus we prove that the correlative points E, H, and A' are collinear, a property preserved in projection.

II. Solution by Otto Dunkel, Washington University.

The ranges of points K and L are projective and hence E is the intersection of two projective pencils with centers at B' and B, such that $B'B$ is not self-corresponding. It thus follows that the locus of E is a conic tangent at B' and B and passing through A and A' of the sides of the parallelogram. By using A and A' as centers it will be seen that it is tangent to the other two sides PQ and $P'Q'$. Similarly, the range of points M is projective with the ranges L and K, and hence H is the intersection of two pencils with centers at the points at infinity on OQ and OP. Thus the locus of H is a conic with OQ and OP as asymptotes, and the conic passes through A and A'. It may be shown by the theory of conics (analytic or projective theory) that PQ is a tangent. It also follows that the pencils A'



(E) and A' (H) are projective, that $A'A$, $A'B$, A' B' are self-corresponding rays of these two pencils and hence all the corresponding rays coincide. Therefore, A', E, H lie on a straight line, etc.

2757 [1919, 124]. Proposed by E. P. LANE, Rice Institute, Houston, Texas.

Integrate by quadrature the differential equation

$$\frac{d^2y}{dx^2} - 3y\frac{dy}{dx} + y^3 = 0.$$

Solution by Alexander Dillingham, U. S. Naval Academy.

Interchanging the variables and setting $q = dx/dy$, we have $d^2y/dx^2 = -(dq/dy)/q^3$. The given equation becomes, after making these substitutions,

$$\frac{dq}{dy} + 3yq^2 - y^3q^3 = 0.$$

By inspection we find a particular integral $q_1 = y^{-2}$ and hence we are led to put $q = q_1 + v$, where v is a function of y. The equation (1) now becomes a Bernoulli equation $dv/dy + 3v/y = v^3y^3$ which reduces, on putting $z = v^{-2}$, to the linear equation $dz/dy - 6z/y = -2y^3$ with the integrating factor y^{-6}. We then obtain $zy^{-6} = \int(-2y^{-3})dy = y^{-2} + c_1$ or $z = y^4 + c_1y^6 = v^{-2}$. Hence we have in turn

$$v = \pm\frac{1}{y^2\sqrt{1 + c_1y^2}}, \qquad q = \frac{1}{y^2} \pm \frac{1}{y^2\sqrt{1 + c_1y^2}} = \frac{dx}{dy},$$

and by integrating the last equation, we have finally

$$x = -\frac{1}{y} \mp \sqrt{y^{-2} + c_1} + c_2.$$

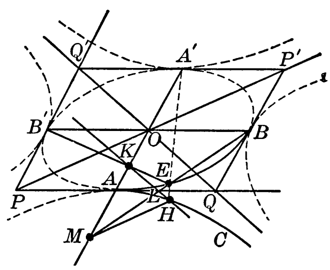
Also solved by R. D. Bohannan, P. J. da Cunha, E. B. Escott, M. Gutten, H. Halperin, H. L. Olson, and Elijah Swift.

If we add to the two sides of (1) the corresponding sides of (3) multiplied by m, we obtain $(1 + m)(x/a) + (1 - m)y/b = 1 - m$, the equation of a straight line through H. The same process applied to (2) and (4) yields the *same* equation and, since the point $(0, b)$ satisfies the new equation, we see that H, E, and A' lie on this straight line.

Note.—A geometrical solution may be given. Starting with a square we easily find as loci, a circle and two conjugate equilateral hyperbolas. Then, by projection, we derive the loci found above. By the theorem of Meneläus we prove that the correlative points E, H, and A' are collinear, a property preserved in projection.

II. Solution by Otto Dunkel, Washington University.

The ranges of points K and L are projective and hence E is the intersection of two projective pencils with centers at B' and B, such that $B'B$ is not self-corresponding. It thus follows that the locus of E is a conic tangent at B' and B and passing through A and A' of the sides of the parallelogram. By using A and A' as centers it will be seen that it is tangent to the other two sides PQ and $P'Q'$. Similarly, the range of points M is projective with the ranges L and K, and hence H is the intersection of two pencils with centers at the points at infinity on OQ and OP. Thus the locus of H is a conic with OQ and OP as asymptotes, and the conic passes through A and A'. It may be shown by the theory of conics (analytic or projective theory) that PQ is a tangent. It also follows that the pencils A' (E) and A' (H) are projective, that $A'A$, $A'B$, A' B' are self-corresponding rays of these two pencils and hence all the corresponding rays coincide. Therefore, A', E, H lie on a straight line, etc.



2757 [1919, 124]. Proposed by E. P. LANE, Rice Institute, Houston, Texas.

Integrate by quadrature the differential equation

$$\frac{d^2y}{dx^2} - 3y\frac{dy}{dx} + y^3 = 0.$$

Solution by Alexander Dillingham, U. S. Naval Academy.

Interchanging the variables and setting $q = dx/dy$, we have $d^2y/dx^2 = -(dq/dy)/q^3$. The given equation becomes, after making these substitutions,

$$\frac{dq}{dy} + 3yq^2 - y^3q^3 = 0.$$

By inspection we find a particular integral $q_1 = y^{-2}$ and hence we are led to put $q = q_1 + v$, where v is a function of y. The equation (1) now becomes a Bernoulli equation $dv/dy + 3v/y = v^3y^3$ which reduces, on putting $z = v^{-2}$, to the linear equation $dz/dy - 6z/y = -2y^3$ with the integrating factor y^{-6}. We then obtain $zy^{-6} = \int(-2y^{-3})dy = y^{-2} + c_1$ or $z = y^4 + c_1y^6 = v^{-2}$. Hence we have in turn

$$v = \pm\frac{1}{y^2\sqrt{1 + c_1y^2}}, \qquad q = \frac{1}{y^2} \pm \frac{1}{y^2\sqrt{1 + c_1y^2}} = \frac{dx}{dy},$$

and by integrating the last equation, we have finally

$$x = -\frac{1}{y} \mp \sqrt{y^{-2} + c_1} + c_2.$$

Also solved by R. D. Bohannan, P. J. da Cunha, E. B. Escott, M. Gutten, H. Halperin, H. L. Olson, and Elijah Swift.

NOTES AND NEWS.

EDITED BY E. J. MOULTON, Northwestern University, Evanston, Ill.

At the Carnegie Institute of Technology Assistant Professor J. R. EVERETT, of Baker University, and Professor G. W. HESS, of Bethany College, have been appointed instructors in mathematics.

Dr. F. ELIZABETH LE STOURGEON, of Carleton College, has accepted a position as assistant professor of mathematics at the University of Kentucky.

Professor EUGENE TAYLOR, of the University of Wisconsin, has been appointed head of the department of mathematics at the University of Idaho.

Dr. R. B. ROBBINS, of the University of Michigan, has been promoted to an assistant professorship. He has been granted leave of absence for the year, which he will spend in practical study in statistics and insurance offices.

Professor J. MCMAHON, of Cornell University, has a sabbatical leave of absence for the year, and will retire at its close.

Professor P. H. GRAHAM, of Agnes Scott College, has been appointed instructor in mathematics in Washington Square College, New York University.

Instructor R. M. MATHEWS, of the University of Minnesota, has been appointed assistant professor of mathematics at Wesleyan University.

Mrs. MAYME I. LOGSDON, instructor in mathematics at Northwestern University, but more recently fellow at the University of Chicago, has been appointed associate in mathematics at the University of Chicago.

At Wellesley College Doctors MARY F. CURTIS and LENNIE P. COPELAND have been promoted to assistant professorships in mathematics.

At Harvard University the following new instructors in mathematics have been appointed for the year 1920–21: Assistant Professor C. A. GARABEDIAN of New Hampshire College; Messrs. R. E. LANGER, E. L. MACKIE and HARRY LEVY of Harvard University; and Mr. A. J. COOK of University of Alberta.

At the University of Minnesota, Professor W. H. BUSSEY has been promoted to the rank of professor and made assistant dean in the College of Science, Literature and the Arts; assistant professors A. L. UNDERHILL, R. W. BRINK, and W. L. HART have been promoted to associate professorships; F. K. BAIER, JR., of Pennsylvania State College, and Dr. GLADYS GIBBENS have been appointed instructors in mathematics.

At the University of Wisconsin, Professor C. S. SLICHTER has been appointed dean of the Graduate School to succeed Professor G. C. COMSTOCK; Associate Professor E. B. SKINNER has been promoted to a full professorship; Assistant Professor H. W. MARCH has been promoted to an associate professorship; Dr. WARREN WEAVER, of the California Institute of Technology, has been appointed to an assistant professorship; and Messrs. HAROLD DAVIS, of Harvard, and M. L. MACQUEEN, of Southwestern Presbyterian University, have been appointed instructors in mathematics.

At the University of Paris, Dr. P. PAINLEVÉ, professor of rational mechanics and a former premier of France, has been appointed professor of celestial mechanics in place of Professor P. APPELL. Dr. E. J. CARTAN, professor of differential calculus has been appointed to Professor Painlevé's chair, and Professor E. P. J. VESSIOT, new sub-director of the Ecole Normale Supérieure [*1920*, 338], to Professor Cartan's. Professors PAINLEVÉ and E. BOREL have gone to China on an educational mission; the former was in Washington last May.

Mr. A. E. JOLLIFFE, tutor in mathematics at Corpus Christi College, Oxford, has been appointed to the University chair of mathematics tenable at the Royal Holloway College, London.

Miss MARY A. COLPITTS, instructor in mathematics at the University of Wisconsin [*1919*, 419] died at Madison on July 11, aged twenty-eight years.

Dr. J. N. STOCKWELL, of Cleveland, O., author of contributions to mathematical astronomy, died on May 18, 1920, aged eighty-eight years.

Popular Astronomy reports that Dr. A. BERBERICH, the noted German astronomer, who had done much valuable work in the computation of comet and asteroid orbits, died on April 27, 1920.

Nature reports the death on June 20, 1920, of Dr. F. A. TARLETON of the board of Trinity College, Dublin, aged seventy-nine years. He was professor of natural philosophy at Trinity College from 1890 to 1901. His books, *An Introduction to the Mathematical Theory of Attractions* (1899, vol. 2, 1913), and *An Elementary Treatise on Dynamics*, in collaboration with B. Williamson (1885, third edition, 1900), were widely used in Great Britain. "As professor of natural philosophy, Dr. Tarleton followed the traditions of his distinguished predecessors, Williamson, Townsend, and Jellett, in treating the subject from a strictly mathematical point of view. Although he had considerable practical acquaintance with experimental science, he flatly ignored the judicial aphorisms of Francis Bacon and, instead of treating mathematics as the handmaid of physics, he rather inverted the order, and almost succeeded in reducing hydrodynamics, elasticity, magnetism, and electricity to branches of pure mathematics." (R. A. P. Rogers).

At the last Yale commencement in conferring the degree of Master of Arts on Dean HAWKES, of Columbia University, President Hadley said: "Herbert Edwin Hawkes: B.A., Yale, 1896; Ph.D., 1900; like many of his classmates, Dr. Hawkes became a member of the Yale faculty, and taught mathematics for twelve years. In 1910 he was called to Columbia as professor; he was such a conspicuous success in administration that he was made dean of the college. He is the author of books in his chosen field, but his chief distinction is a worker of miracles—he has made hundreds of young men love mathematics. Perhaps they would not love mathematics so much if they did not love him even more. A living force in education."

At a meeting of the Academy of Sciences of the Institute of France on May 17, 1920, (1) a committee of seven (the president of the academy and three representatives each from the divisions of physical and mathematical sciences) was appointed to present a list of candidates for the chair of General History of Sciences in the Collège de France. (2) Professor L. E. DICKSON, of the University of Chicago, was elected (41 votes out of 45) "correspondant" of the Academy, in the section of geometry. Two votes were cast for G. CASTELNUOVO and two for E. I. FREDHOLM.

By a decree of 1909 the number of correspondants of the Academy of Sciences was increased to 116, 10 for each section except astronomy where there are 16, The other sections are: geometry, mechanics, physics, geography and navigation, chemistry, mineralogy, botany, rural economics, anatomy and zoology, medicine and surgery. Of the 50 correspondants in the section of geometry since 1816 only six have been English or American, namely: Ivory (elected in 1828), Hamilton (1844), Sylvester (1863), Spottiswoode (1876), Salmon (1884), and Dickson (1920).

By way of celebrating Professor Dickson's election colleagues and friends in the University of Chicago tendered him a complimentary luncheon. Professor A. A. Michelson, the only other "correspondant" of the Academy of Sciences at the University, presided.

Professor Dickson's productive activity during the twenty years of his connection with the mathematics department of the University of Chicago has been notable. In *Publications of the Members of the University 1902–1916* (Chicago, 1917), 6 of his books (one in collaboration with Miller and Blichfeldt) and 123 of his papers are listed. Of his 29 papers published in this MONTHLY 14 only are here mentioned; the others appeared before 1902. Since 1916 have appeared the first two volumes of his monumental *History of the Theory of Numbers* (cf. *1919*, 396–403). The second volume (25–803 pages) was published in August.

Professor Dickson was managing editor of this MONTHLY 1902–1906 and associate editor, 1906–1908. He was associate editor of the *Transactions of the American Mathematical Society*, 1902–1910 and joint editor 1911–1916. He has been president of the American Mathematical Society, is a member of the Division of Physical Sciences of the National Research Council, and is chairman of the American Section of the International Mathematical Union whose membership was listed in our last issue (*1920*, 340).

CONTENTS

EDITORIAL CORRESPONDENCE AND BOOKS FOR REVIEW should be addressed to the EDITOR-IN-CHIEF, R. C. ARCHIBALD, Brown University, Providence, R. I.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

Fifth Summer Meeting of the Association, Chicago, September 6, 1920;
Fifth Annual Meeting, Chicago, December 28–29, 1920

The following are dates of Section meetings of the Association in 1920:

IOWA, Univ. of Iowa, Iowa City, May 1
KANSAS, State Agricultural College, Manhattan, April 3; Topeka, November
KENTUCKY, Centre College, Danville, April 17
MARYLAND–DISTRICT OF COLUMBIA–VIRGINIA, Goucher College, Baltimore, Md., May 15; Annapolis, Md., December
MINNESOTA, St. Catherine's College, St. Paul, June 5
MISSOURI, Kansas City, November 12–13
OHIO, Ohio State Univ., Columbus, April 2
ROCKY MOUNTAIN, Colorado College, Colorado Springs, April 2

THE AMERICAN MATHEMATICAL MONTHLY

OFFICIAL JOURNAL OF THE

MATHEMATICAL ASSOCIATION OF AMERICA

(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

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THE FIFTH SUMMER MEETING OF THE ASSOCIATION.

The fifth summer meeting of the Mathematical Association of America was held at the University of Chicago on Monday, September 6, 1920, in conjunction with, and immediately preceding, the summer meeting and colloquium of the American Mathematical Society. 132 were present at the meeting, including the following 114 members of the Association:

O. W. Albert, Grinnell College.
Edna Allen, Chicago, Ill.
R. C. Archibald, Brown University.
G. N. Armstrong, Ohio Wesleyan University.

P. M. Batchelder, University of Texas.
I. A. Barnett, University of Saskatchewan.
Suzan R. Benedict, Smith College.
A. A. Bennett, University of Texas.
G. D. Birkhoff, Harvard University.
Vevia Blair, Horace Mann High School.
G. A. Bliss, University of Chicago.
R. L. Borger, Ohio University.
J. W. Bradshaw, University of Michigan.
W. H. Bussey, University of Minnesota.

W. D. Cairns, Oberlin College.
D. F. Campbell, Armour Institute of Technology.
J. W. Campbell, University of Alberta.
A. L. Candy, University of Nebraska.
J. A. Caparo, Notre Dame University.
F. E. Carr, Oberlin College.
W. E. Cederberg, University of Wisconsin.
E. W. Chittenden, University of Iowa.
L. M. Coffin, Coe College.
A. R. Crathorne, University of Illinois.
D. R. Curtiss, Northwestern University.

H. H. Dalaker, University of Minnesota.
Sister Mariola Dobbin, St. Clara College.
E. L. Dodd, University of Texas.
J. E. Dotterer, Manchester College.
L. W. Dowling, University of Wisconsin.
Otto Dunkel, Washington University.

M. D. Earle, Furman University.
G. C. Evans, Rice Institute.
H. S. Everett, Bucknell University.

Zoe Ferguson, Crane Junior College.
B. F. Finkel, Drury College.
L. R. Ford, Rice Institute.
W. B. Ford, University of Michigan.

M. G. Gaba, University of Nebraska.
C. D. Garlough, Wheaton College.
W. H. Garrett, Baker University.
D. C. Gillespie, Cornell University.
R. E. Gilman, Brown University.
Cornelius Gouwens, State College of Iowa.
C. F. Gummer, Queen's University.

W. L. Hart, University of Minnesota.
M. W. Haskell, University of California.
Olive C. Hazlett, Mount Holyoke College.
E. R. Hedrick, University of Missouri.
Albert Heinz, Tsing Hua College.
T. H. Hildebrandt, University of Michigan.
L. A. Hopkins, University of Michigan.
Jewell C. Hughes, University of Arkansas.
W. A. Hurwitz, Cornell University.

Dunham Jackson, University of Minnesota.
G. H. Jamison, Kirksville (Mo.) State Normal School.

Claribel Kendall, University of Colorado.
S. D. Killam, University of Alberta.
J. M. Kinney, Hyde Park (Chicago) High School.
H. W. Kuhn, Ohio State University.

Gillie A. Larew, Randolph-Macon Woman's College.
D. A. Lehman, Goshen College.
Flora E. Le Stourgeon, University of Kentucky.
Mayme I. Logsdon, University of Chicago.
A. C. Lunn, University of Chicago.
E. B. Lytle, University of Illinois.

S. H. Macdonald, Colorado Agricultural College.
L. E. McCarty, Michigan College of Mines.
R. B. McClenon, Grinnell College.
J. V. McKelvey, Iowa State College.
Helen A. Merrill, Wellesley College.
Bessie I. Miller, Rockford College.

W. L. MISER, Armour Institute.
C. N. MOORE, University of Cincinnati.
E. H. MOORE, University of Chicago.
E. J. MOULTON, Northwestern University.
F. R. MOULTON, University of Chicago.

A. L. NELSON, University of Michigan.

H. L. OLSON, University of Michigan.

ANNA H. PALMIÉ, Western Reserve University.
ANNA J. PELL, Bryn Mawr College.
T. A. PIERCE, University of Nebraska.
A. D. PITCHER, Western Reserve University.

S. E. RASOR, Ohio State University.
R. G. D. RICHARDSON, Brown University.
H. L. RIETZ, University of Iowa.
W. J. RISLEY, James Millikin University.
MARIA M. ROBERTS, Iowa State College.
PERCIVAL ROBERTSON, The Principia, St. Louis.
W. H. ROEVER, Washington University.

MINNA J. SCHICK, University of Minnesota.
OSCAR SCHMIEDEL, Nebraska Wesleyan University.
IDA M. SCHOTTENFELS, Chicago, Ill.
E. W. SHELDON, University of Alberta.
H. A. SIMMONS, University of Michigan.
W. G. SIMON, Western Reserve University.
E. B. SKINNER, University of Wisconsin.
H. E. SLAUGHT, University of Chicago.
E. B. STOUFFER, University of Kansas.
C. E. STROMQUIST, University of Wyoming.
K. D. SWARTZEL, Ohio State University.

E. J. TOWNSEND, University of Illinois.

A. L. UNDERHILL, University of Minnesota.
P. H. UNDERWOOD, Ball High School, Galveston.

OSWALD VEBLEN, Princeton University.

J. H. WEAVER, Ohio State University.
W. P. WEBBER, University of Pittsburgh.
F. M. WEIDA, University of Iowa.
MARY E. WELLS, Vassar University.
W. D. A. WESTFALL, University of Missouri.
MARION B. WHITE, Carleton College.
C. E. WILDER, Northwestern University.
F. B. WILEY, Denison University.
VERA L. WRIGHT, University of Minnesota.

C. H. YEATON, School of Engineering of Milwaukee.
J. W. YOUNG, Dartmouth College.
J. W. A. YOUNG, University of Chicago.

It is noteworthy that there were in attendance at the meeting from more distant parts of the country nine from Texas, six from Massachusetts, three each from Alberta, New York and Rhode Island, two each from Colorado, Pennsylvania and South Carolina, and one each from China, Ontario, California, Maryland, New Hampshire, New Jersey, Virginia and Wyoming.

Pleasant arrangements were made for those attending the meetings. Comfortable rooms were furnished in Beecher and Hitchcock Halls, while all had meals as well as social opportunities at the Quadrangle Club. The courtesies shown to the members were recognized in a resolution of thanks offered by Professor Veblen. The joint banquet of the two organizations was held on Tuesday evening where about 110 members and friends were present. At this joint dinner brief speeches were made by the toastmaster, Professor Slaught, and by Professor Birkhoff as representing the Society, Professor Merrill as representing the Association, Professors Veblen, Killam, Hedrick and Hurwitz. On Thursday evening the members of the two bodies and their friends were delightfully entertained at a reception at the home of Professor and Miss Slaught. Professor Lunn contributed greatly to the enjoyment by his piano solos.

Vice-President MERRILL presided at the morning session and Professor VEBLEN at the afternoon session. The following papers were read:

(1) "On certain fundamental principles in the mathematics of life insurance" by Professor D. F. CAMPBELL, Armour Institute of Technology.

(2) "Certain features of the application of Makeham's laws of mortality" by Professor H. L. RIETZ, University of Iowa.

(3) "The plan of pensions and insurance recommended by the Carnegie Foundation for the Advancement of Teaching" by Professor E. L. DODD, University of Texas.

(4) Report of progress of the National Committee on Mathematical Requirements, by Professor J. W. YOUNG, Dartmouth College.

(5) "The debt of mathematics to the experimental sciences" by Professor A. C. LUNN, University of Chicago.

(6) Discussion of Professor Lunn's paper with respect to its bearing upon research in pure mathematics, by Professor E. H. MOORE, University of Chicago.

(7) Discussion of Professor Lunn's paper with respect to its bearing on mathematical curricula, by Professor M. W. HASKELL, University of California.

(8) "Retrospect and prospect for mathematics in America,"—retiring presidential address by Professor H. E. SLAUGHT, University of Chicago.

Abstracts of the papers and discussions follow below, the numbers corresponding to the numbers in the list of titles:

1. Professor Campbell's paper dealt with some of the theorems of mathematics underlying the principles of life insurance. He explained various of the principles of interest touching upon nominal and effective interest, force of interest, and present values. Then he took up some problems in probability in connection with the mortality table and derived a few formulas in annuities and insurance, pointing out the special technique of avoiding extensive calculations. He finally treated the subject of reserves and a method of calculating these.

2. In the introduction to this paper, Professor Rietz outlined briefly the historical development of the ideas in Makeham's functions expressive of human mortality. He then gave an exposition of the properties of these functions that make them of fundamental importance in the problems of joint life and survivorship insurance and annuities. It was shown in this paper not only how a Makehamized table following Makeham's first modification of the Law of Gompertz leads to economy of time and energy, but it is shown also how a mortality table following the second modification of the Law of Gompertz may be applied to advantage in the problems of joint life and survivorship insurances and annuities.

3. The Carnegie Foundation recommends, as especially suited to the needs of college teachers, a combination of insurance, non-convertible term insurance or decreasing whole-life insurance, with a savings account—interest at 4 per cent. or $4\frac{1}{2}$ per cent.—to purchase a pension upon retirement or to pass to the estate upon death. These forms of protection—as, indeed, also the more common forms of insurance—can be purchased by teachers from The Teachers Insurance and

Annuity Association at cost. No part of a teacher's payment is used for the expenses of the company.

The more common forms of insurance are (1) non-convertible term, (2) term, convertible to other forms, (3) whole life, (4) limited payment life, and (5) endowment. With the exception of (1), these forms all involve an endowment or cash value, which can be used to purchase a pension. Indeed, with the exception of (1), the forms are all good; and they are well adapted to the needs of a man on moderate income who has some ability to save. For men on very small income, forms (2) and (3) are to be preferred; for men who find it difficult to save, form (5) or the plans suggested by The Carnegie Foundation. The more common forms of insurance are more flexible; but the Foundation plans bind the accumulations strictly for pension purposes. As pension plans, however, they exhibit very considerable flexibility, because of the various options permitted.

4. The report of Professor Young is embodied in the reports printed in the July–September issue of the MONTHLY and elsewhere (pages 441–442) in the present issue.

5. An important portion of the concepts of present mathematical science has emerged by abstraction and generalization from notions originally quite special and concrete occurring in experimental sciences. The successive steps in development have often been carried quite far under the impulse of suggestion from the experimental relations.

Professor Lunn's paper was devoted primarily to a commentary of illustration from the theories of mechanics, heat and electromagnetism. Historical sketches were given of examples leading to familiar general notions in the theories of quadratic forms, modern geometry, differential and integral equations. It is to be hoped that this paper may be available to those not present at the meetings.

6. Professor Moore in his discussion discriminated between the different aspects of research. (1) The process. In research in applied mathematics this does not differ essentially from that in pure mathematics, except that the latter does not need the elaborate equipment necessary in the former. The main speaker, it was pointed out, had brought out the fact that he who comes to pure research with a large background of experimental knowledge has a great aid in his work. Professor Moore instanced by reference to his own study of matrices the possibility of the discovery through generalization of *tools* of great usefulness. (2) Ideals of research, (*a*) of the individual, (*b*) of the group of individuals. Here distinct advantages come directly from the "right" of the experimental sciences to the "left" of the abstract fields; the "foundations" may well serve to make a contribution in the reverse direction. We have much in evidence at the present time in the way of scholarship and of research ability. We must develop the ideal of the group, as is beginning to be done in this country. (3) The form. One should choose the form in which he casts his research such that it shall be most clearly understood by those not conversant with his subject.

There must be a differentiation between various parts of research, yet if science is to advance, there must be a compensatory unifying principle. While mathematical principles have emerged, sometimes directly and logically, sometimes by way of analogy, from the various sciences as described by Dr. Lunn, in the higher reaches of pure mathematics there should be a working out of the principles embodied in physical research in a form free from mere analogy.

7. The mathematical curriculum has changed greatly in recent years. We cannot legitimately include a subject in the curriculum unless we can justify its inclusion to those taking the course. Professor Haskell regarded Professor Lunn's paper as a strong plea for the introduction of the element of interest in mathematics; we must be able to give the student reasons for studying it and must first of all know the reasons ourselves. We must furthermore give the students something in mathematics which shall relate itself to their own lives. We should, for example, show them the advantage of a mathematical formulation; trigonometry has here its great appeal. With less ease, yet in a feasible way, we may make college algebra an attractive subject instead of a bugbear.

Professor Jackson spoke of the lamentable lack of agreement existing between mathematicians and physicists. A single example, he said, would serve, viz., it would aid greatly if we would present existence theorems not so much as proofs of the existence of solutions (which usually are found by entirely different methods) as means of finding solutions where other methods fail.

8. The inspiring retiring address of President Slaught will appear in the December issue of this MONTHLY.

MEETING OF THE COUNCIL OF THE ASSOCIATION.

Eleven members of the Council were present at the meeting.

The following seventy persons and three institutions, on applications duly certified, were elected to membership (making 181 new memberships since January 1, 1920):

NINA M. ALDERTON, A.M. (Columbia). Asst. and grad. student, Univ. of California, Berkeley, Calif.

H. E. ANDERSON, A.M. (Augustana Coll.). Muhlenberg Coll., Allentown, Pa.

W. A. AUSTIN, A.M. (Indiana). Head of dept. of math., High School and Junior Coll., Fresno, Calif.

KATE C. BARBOUR, A.B. (Oklahoma). Teacher, High School, Norman, Okla.

J. F. BARNHILL, A.B. (Kansas). Supt., City Schools, Parsons, Kans.

P. M. BATCHELDER, Ph.D. (Harvard). Instr. in pure math., Univ. of Texas, Austin, Tex.

W. D. BATEN, A.M. (Texas). Head of dept. of math., Grubbs Vocat. Coll., Arlington, Tex.

SÉVÉRIN BAYS, Ph.D. Professeur agrégé, Univ. of Fribourg, Switzerland.

FLORENCE A. BIXBY, A.M. (Columbia). Head of dept. of math., Riverside High School, Milwaukee, Wis.

ETTORE BORTOLOTTI, Dottore in Mat. Prof. ord. di analisi algebrica, Univ. of Bologna, Italy.

M. LUCILE BROWN, A.M. (Ohio State). Instr., Western Coll., Oxford, Ohio.

GLADYS-MARY E. CAMPBELL, A.B. (California). Asst., Univ. of California, Berkeley, Calif.

MICHELE CIPOLLA, Dottore in Mat. (Palermo). Prof. ord. di analisi algebrica, Univ. of Catania, Italy.

R. F. CLARK, A.B. (Williams), Pd.B. (Albany State Normal). Chairman of dept. of math., De Witt Clinton High School, New York, N. Y.

L. H. CUTTING, B.S. (Chicago) Teacher, Westport High School, Kansas City, Mo.

J. A. ELY, C.E. (Princeton). Prof., St. John's Univ., Shanghai, China.

STEPHEN EMERY, A.M. (Boston). Head of dept. of math., Erasmus Hall High School, Brooklyn, N. Y.

P. H. EVANS. Chief actuary, Northwestern Mut. Life Ins. Co., Milwaukee, Wis.

H. L. FASSETT, A.M. (Bucknell). Head of dept. of math., South Side High School, Newark, N. J.

D. D. FELDMAN, B.S. (Nebraska). Prin., Curtis High School, Staten Island, N. Y.

L. R. FORD, Ph.D. (Harvard). Asst. prof., Rice Inst., Houston, Tex.

Z. G. DE GALDEANO, Dr. in Ciencias matemáticas. Prof., Univ. of Zaragoza, Spain.

FALKA M. D. GIBSON, A.B. (California). Teacher, High School, Orland, Calif.

MARIA D. GRAHAM, B.S. (Teachers Coll.). Head of dept. of math., Teachers Training School, Greenville, N. C.

May V. Haworth, Ph.B. (California). Vice-prin., High School, Alameda, Calif.

W. S. HIGGINS, M.E.E. (Harvard). Prof. of math. and engg., Southwestern Presbyt. Univ., Clarksville, Tenn.

L. A. HOPKINS, Ph.D. (Chicago). Asst. prof., Univ. of Michigan, Ann Arbor, Mich.

M. H. INGRAHAM, A.B. (Cornell). Instr., Univ. of Wisconsin, Madison, Wis.

GEORGE JACKSON, B.S. (Cincinnati). Asst. headmaster, Asheville School, Asheville, N. C.

C. L. JOHNSON, B.S. (Ore. Agric. Coll.). Head of dept. of math., Ore. Agric. Coll., Corvallis, Ore.

S. D. KILLAM, Ph.D. (Göttingen). Asso. prof., Univ. of Alberta, Edmonton South, Alb., Canada.

A. V. LEBEUF. Prof. of astr. and dir. of the observatory, Univ. of Besançon, France.

E. J. LEWIS, A.B. (Olivet). Head of dept. of math., Tech. High School, Scranton, Pa.

L. P. LOOMIS, B.S. (Miss. College). Clinton, Miss.

JANE H. MATHEWS, B.S. (Columbia). Teacher, Peabody High School, Pittsburgh, Pa.

L. E. McCarty, A.M. (Texas). Asst. prof., math. and physics, Michigan College of Mines, Houghton, Mich.

Dora McFarland, A.B. (Monmouth). Instr., Univ. of Oklahoma, Norman, Okla.

Elsie J. McFarland, Ph.D. (California). Berkeley, Calif.

Emma L. Noonan, A.M. (Columbia). Teacher, Girls High School, San Francisco, Calif.

J. W. Pancoast, B.S. (Swarthmore). Prof., Guilford College, N. C.

Honor K. Pettit, A.B. (Park Coll.). Grad. student, Univ. of California, Berkeley, Calif.

T. A. Pierce, Ph.D. (California). Asst. prof., Univ. of Nebraska, Lincoln, Neb.

J. F. Pobanz, A.B. (Michigan). Asst., Univ. of California, Berkeley, Calif.

Inez D. Powelson, A.M. (California). Teacher, Bakersfield, Calif.

V. V. Ramana-Sastrin, Ph.D. Vedaraniam, Tanjore Dt., South India.

O. H. Rechard, Jr., A.M. (Penna. Coll.). Instr., Univ. of Wisconsin, Madison, Wis.

Clair Reid, A.B. (Earlham). Instr., Purdue Univ., LaFayette, Ind.

Theresa M. Renner, B.S. (Blackburn). Instr., Blackburn Coll., Carlinville, Ill.

C. P. Rockwell. Asst. actuary, Texas Dept. of Ins. and Banking, Austin, Tex.

Sarah A. Ruby, A.B. (Iowa). Head of dept. of math., Jefferson High School, Portland, Ore.

G. O. Sagen, A.B. (California). Asst., Univ. of California, Berkeley, Calif.

Meyer Salkover, A.M. (Cincinnati). Instr., Univ. of Cincinnati, Cincinnati, Ohio.

Minna J. Schick, A.M. (Northwestern). Instr., Univ. of Minnesota, Minneapolis, Minn.

L. Silberstein, Ph.D. Research laboratory, Eastman Kodak Co., Rochester, N. Y.

E. B. Skinner, Ph.D. (Chicago). Prof., Univ. of Wisconsin, Madison, Wis.

I. W. Smith, A.M. (Illinois). Prof., North Dakota Agric. Coll., Fargo, N. D.

W. A. Stafford, A.M. (Stanford). Head of dept. of math., High School, Oakland, Calif.

G. C. Staley, A.M. (Chicago). Instr., Parker High School, Chicago, Ill.

Stella Stephens, A.B. (Georgetown Coll.). Teacher, High School, Paris, Ky.

Helen Thompson, A.B. (Vassar). Head of dept. of math., Kentucky Coll. for Women, Danville, Ky.

Luis Octavio de Toledo. Prof., Univ. of Madrid, Spain.

R. S. Underwood, A.M. (Minnesota). Instr., Purdue Univ., LaFayette, Ind.

Louis Van Hee. Jesuit Father; Prof of math., Liége, Belgium.

P. W. Waterman, Ph.B. (Vermont). Head of dept. of math., Montclair Acad., Montclair, N. J.

J. H. M. Wedderburn, D.Sc. (Edinburgh). Asst. prof., Princeton Univ., Princeton, N. J.

Helen F. Weeks, B.S. (California). Head of dept. of sc. and math., High School, Alhambra, Calif.

B. C. Wong, A.M. (California). Asst., Univ. of California, Berkeley, Calif.
Ruth G. Wood, Ph.D. (Yale). Prof., Smith Coll., Northampton, Mass.
Jessica M. Young, M.S. (California). Instr., math. and astr., Washington Univ., St. Louis, Mo.
Vivian Young, A.B. (Willamette). Head of dept. of math., High School, Salem, Ore.

To institutional membership.

St. Mary-of-the-Woods College, St. Mary-of-the-Woods, Ind.
Northern Normal and Industrial School, Aberdeen, S. Dak.
University of Wyoming, Laramie, Wyo.

A report was made for the committee on life membership. It was agreed that more study should be devoted to this subject and a committee of three actuarial members of the Association was appointed which should report on the actuarial features of the proposed life membership fee, including the feasibility of a fee graduated according to age.

It was voted to accept the invitation of Wellesley College for the summer meeting in 1921.

It was voted to accept the invitation of the staff of the University of Chicago for the next annual meeting in affiliation with the meeting of the American Association for the Advancement of Science; and through a later report of the committees of the Association and the Society it was agreed that there should be morning and afternoon sessions of the Association on Tuesday, December 28, simultaneous sessions of the two bodies on Wednesday morning, a joint session of these with Section A of the American Association on Wednesday afternoon, and further sessions of the Society on Thursday.

The most important action of the Council was in regard to the financial situation of the Association. Like other scientific periodicals, the cost of publication of the Monthly has increased about 50 per cent. during the past year; and provision needs now to be made for editorial and clerical assistance in the office of the editor-in-chief. Because of these additions, it appears, according to a report made to the Council by the Secretary-Treasurer, that the reserve in the Association treasury will be more than wiped out in the next year if no additional income is secured. The Council after careful and full consideration, both by correspondence during the past summer and by conference at the Chicago meeting, believes that the members are so strongly committed to the ideals and accomplishments of the Association and of its worth to themselves, that they will loyally approve increased membership dues as one means of offsetting the increased expenses. The Council therefore voted to increase the individual dues to four dollars, the institutional dues to seven dollars, and subscriptions for non-members to five dollars, beginning with January, 1921. An exception was made to this in the case of those who at that time will have been members for less than one year; it seems only fair to exempt these from the increase until January, 1922, thus giving them a full year on the basis of the dues prevailing when they joined.

Since twice as much additional revenue must be secured as will come from the increase in membership dues, if the income of the Association is to equal its expenditures, a committee of the Council has been appointed to invite subscriptions, aside from membership dues, which may be applied to the current budget or to permanent endowment. A third method by which members can readily come to the financial aid of the Association is for each member to invite and urge others to become members; if we believe in the value of the Association's work, we can rightfully bring this to the attention of all the members of the staff and to advanced students of mathematics at each college and university, and to other persons definitely interested in mathematics.

To the end that our organization may legally receive donations and bequests, the Council took steps to incorporate the Association. Both the Council and the Association unanimously adopted resolutions empowering Professors E. H. Moore, H. E. Slaught and W. D. Cairns to apply for a charter, and this action was taken by them at once. Since then the charter has been granted and the Mathematical Association of America Incorporated has been organized under the statutes of the State of Illinois. The articles of incorporation vest the legal control in the first instance in three trustees, Professors H. E. Slaught, E. R. Hedrick and W. D. Cairns, but they have exercised their legal prerogative by enlarging the Board of Trustees to nineteen, thus including the present officers and all members of the present Council and by adopting a set of by-laws which include the by-laws and constitution of the original Association together with such modifications and additions as were necessary to meet the legal requirements. The charter of the Mathematical Association of America Incorporated now supersedes the old constitution. The property and assets of the old Association have been transferred to the newly incorporated body in accordance with the authority granted by vote of the Council. The By-Laws of the Mathematical Association of America Incorporated follow, preceded by the official minutes of the organization meeting and the Certificate of Incorporation.

W. D. Cairns, *Secretary-Treasurer.*

CERTIFICATE NO. 3651.

State of Illinois, Office of Secretary of State.

To all to whom these Presents shall come, Greeting:

Whereas a certificate, duly signed and acknowledged has been filed in the office of the Secretary of state, on the 8th day of September, A.D. 1920, for the organization of

THE MATHEMATICAL ASSOCIATION OF AMERICA

(INCORPORATED)

under and in accordance with the provisions of "An Act Concerning Corporations" approved April 18, 1872, and in force July 1, 1872 and all acts amendatory thereof, a copy of which certificate is hereto attached;

Now, therefore, I, Louis L. Emmerson, Secretary of State of the State of Illinois, by virtue of the powers and duties vested in me by law, do hereby certify that the said

THE MATHEMATICAL ASSOCIATION OF AMERICA
(INCORPORATED)

is a legally organized corporation under the laws of this State.

IN TESTIMONY WHEREOF I hereunto set my hand and cause to be affixed the Great Seal of the State of Illinois done at City of Springfield, this 8th day of September A.D. 1920 and of the Independence of the United States the one hundred and forty-fifth.

LOUIS L. EMMERSON,
Secretary of State.

[GREAT SEAL
OF THE STATE OF
ILLINOIS]

[Filed for Record September 10, 1920 in the office of the Recorder of Cook County, Illinois as Document No. 6935590.]

MINUTES OF THE ORGANIZATION MEETING OF THE TRUSTEES OF THE MATHEMATICAL ASSOCIATION OF AMERICA (INCORPORATED), HELD AT CHICAGO, ILLINOIS, ON THE 10TH DAY OF SEPTEMBER, A. D. 1920.

There were present Messrs. Slaught, Hedrick and Cairns, being all of the Trustees acting by virtue of Application for Charter filed with the Secretary of State at Springfield, Illinois.

The meeting was called to order by Mr. Slaught, who presided. Mr. Cairns acted as Secretary of the meeting and recorded the proceedings.

Mr. Slaught reported that Application for Charter had been duly filed with the Secretary of State, that Certificate of Organization had thereon been issued by the Secretary of State and that such Certificate had been filed with, and on this 10th day of September, A. D. 1920, recorded in the office of, the Recorder of Deeds of Cook County, all in accordance with the laws of the State of Illinois; that the Association had, under the law, been fully organized and might now proceed to business.

The Chairman announced that at a meeting of the members of The Mathematical Association of America, the unincorporated organization to which it is proposed to make this Association successor, held at Chicago, Illinois, September 6, 1920, a resolution had been adopted, authorizing the Officers and the members of the Council of that organization to do all acts and things necessary to be done to complete the organization of this Association, to transfer to it the property and business of the unincorporated organization, subject to its liabilities, and, so far as possible, to provide for the continuation of the Association's business and affairs substantially along the lines prevailing in the unincorporated organization.

The Chairman announced that the first order of business was the election of a President and Secretary of the newly incorporated Association.

Thereupon, on motion duly made, seconded and unanimously carried, it was resolved that until, and subject to, the legal effectuation of the amendment to the Articles of Association for the increase of Trustees, as contemplated and hereinafter set forth, HERBERT E. SLAUGHT and WILLIAM D. CAIRNS be, and hereby are, elected, respectively, President and Secretary of this Association.

The Chairman announced that the next order of business was the adoption of By-Laws for the government of the Association and thereupon submitted a code of By-Laws which was carefully considered and, upon motion duly made, seconded and unanimously carried, adopted as the By-Laws of this Association. A copy of the By-Laws so submitted and adopted was ordered certified by the Chairman of the meeting and included in the records of this meeting.

Thereupon, on motion duly made, seconded and unanimously carried, it was resolved that Section 3 of the Articles of Association of THE MATHEMATICAL ASSOCIATION OF AMERICA (INCORPORATED) be, and hereby is, changed and amended to read as follows: "The management of the aforesaid Association shall be vested in a Board of Nineteen (19) Trustees."

Thereupon, on motion duly made, seconded and unanimously carried, it was resolved that the Officers of this Association be, and hereby are, authorized and directed to do all acts and things necessary, proper or convenient to be done in and about making legally effectual the amendment of the Articles of Association aforesaid and the consummation of the transfer to this Association of the assets, property, business and membership of the present unincorporated Association bearing the same name, including payment of all fees and expenses incident thereto out of the funds of the Association.

No further business being presented, the meeting was, on motion, duly adjourned.

WILLIAM D. CAIRNS,
Secretary of the Meeting.

Approved:
HERBERT E. SLAUGHT,
Chairman of the Meeting.

BY-LAWS OF THE MATHEMATICAL ASSOCIATION OF AMERICA (INCORPORATED).

ARTICLE I—NAME, PURPOSE AND CORPORATE SEAL.

1. This organization shall be known as

THE MATHEMATICAL ASSOCIATION OF AMERICA (INCORPORATED).

2. Its object shall be to assist in promoting the interests of mathematics in America, especially in the collegiate field, by holding meetings in any part of the United States or Canada for the presentation and discussion of mathematical papers, by the publication of mathematical papers, journals, books, monographs and reports, by conducting investigations for the purpose of improving the teaching of mathematics, by accumulating a mathematical library and by cooperating with other organizations whenever this may be desirable for attaining these or other similar objects.

3. The Corporate Seal of the Association shall have inscribed thereon the name of the Association and the words "Corporate Seal—Illinois."

ARTICLE II—MEMBERSHIP.

1. Any person who is interested in the field of collegiate mathematics shall be eligible for election to membership in the Association.

2. Any institution in which the Calculus is regularly taught shall be eligible for election to institutional membership in the Association. Such an institution shall have the privilege of sending a voting delegate to the meetings of the Association.

3. Election to membership shall be by vote of the Board upon written application from the individual or institution seeking admission.

4. Those who were admitted to membership in The Mathematical Association of America (unincorporated) prior to October 1, 1920, and are in good standing as such are hereby admitted to membership in this Association.

ARTICLE III—BOARD OF TRUSTEES AND OFFICERS.

1. The control and management of the affairs and funds of the Association shall be vested in a Board of Trustees, who shall be members of the Association. The Board of Trustees, in the first instance, shall consist of the three (3) persons named in the Certificate of Organization. It is contemplated that such Board of Trustees shall, immediately after completing the organization of the Association, amend the Articles of Association, in the manner hereinafter prescribed, to provide for a Board of Trustees numbering Nineteen (19). Such Board of Nineteen (19) shall consist of a President, two (2) Vice-Presidents, a Secretary-Treasurer, three (3) members of the Committee on Publications and twelve (12) additional members.

2. For the terms set against their respective names, the following shall, upon amendment of the Articles of Association above mentioned, be the Officers and Trustees of the Association:

Office.	Name.	Term ending with the Annual Meeting in
President	D. E. Smith	December, 1920 or January, 1921.
Vice-President	Helen A. Merrill	December, 1920 or January, 1921.
Vice-President	E. J. Wilczynski	December, 1920 or January, 1921.
Secretary-Treasurer	W. D. Cairns	December, 1920 or January, 1921.
Committee on Publication	W. A. Hurwitz, H. E. Slaught, *Manager,* R. C. Archibald, *Editor in Chief*	December, 1920 or January, 1921.
Trustees	Florian Cajori Elizabeth B. Cowley E. L. Dodd G. A. Miller	December, 1920 or January, 1921.
Trustees	L. P. Eisenhart B. F. Finkel E. V. Huntington E. H. Moore	December, 1921 or January, 1922.
Trustees	R. D. Carmichael E. R. Hedrick Oswald Veblen J. W. Young	December, 1922 or January, 1923.

3. The President and Vice-Presidents shall be elected by the Association's members annually for a term of one year, and four members of the Board shall be elected by the Association's members annually for a term of three years. They shall be eligible for reelection, but not for more than two (2) consecutive terms. The Secretary-Treasurer, and the Committee on Publications, consisting of the Manager, the Editor and one other member, shall be appointed by the Board. All Trustees and Officers shall hold over until their respective successors are elected and qualify.

4. The Board shall transact the official business of the Association and shall report its actions at the annual meeting of the Association and in the official journal. Any proposed action of the Board which makes or alters a question of policy shall be published in the official journal before final action has been taken, so that members of the Association may make known to the Board their individual views.

5. The Board shall have authority to fill vacancies *ad interim* in any office, including vacancies in the Board and in the Committee on Publications.

8. At all meetings of the Board of Trustees a quorum shall consist of not less than five (5) members and no business may be validly transacted at a meeting at which less than a quorum is present; *provided* that any meeting of the Board, whether or not a quorum be present, may be adjourned to a specified time and place by a majority of the members present without notice to the members at large other than announcement at such meeting. Informal action based on correspondence among the members of the Board, if ratified at a properly convened meeting of the Board, shall be as valid and effective as if originally authorized at such meeting.

7. Two months before the date of the annual meeting all members shall be given an opportunity to nominate by mail a candidate for each office to be elected by the members for the ensuing year. One month before the annual meeting the Board shall announce two candidates for each office to be filled by the members, one being the person who received the highest vote in the nominations and the other being selected by the Board from among the several nominees next in order. The election shall be by mail or in person and shall close on the day of the annual meeting.

8. The President shall be the executive Officer of the Association, shall preside at all meetings of the Board of Trustees and at the annual meeting of the Association. He shall have the usual duties pertaining to his office and such other duties as may from time to time be devolved upon him by the Board of Trustees.

9. The Vice-Presidents shall, in the absence of the President, have and exercise the powers of the President, their order being determined alphabetically. The Board of Trustees may devolve upon the Vice-Presidents such duties as may from time to time be determined.

10. The Secretary-Treasurer shall have the usual duties pertaining to the office of Secretary and of Treasurer, including the custody of the records of the Association and of its Corporate Seal, the keeping of minutes of the meetings of the Board of Trustees and of the annual meeting and special meetings of members, the giving of due notice of all regular and special meetings of the Association and of the Board of Trustees and the supervision and safe-keeping of the funds

3. The annual dues of each institutional member shall be Seven Dollars ($7), including two (2) subscriptions to the official journal.

4. All dues shall be payable on the first of January of each year. Should the annual dues of any member remain unpaid beyond a reasonable time, his name shall be dropped from the list, after due notice.

5. New members entering the Association after April 1 of any year shall have their dues pro rated for the balance of the year, except when they desire to receive the full current volume of the official journal.

Article VIII—Amendments to the Articles of Association and By-Laws.

1. The Articles of Association may be changed to provide for a Board of Nineteen (19) Trustees by vote of a majority of the Three (3) Trustees named in the original Articles. Subsequent changes, amendments or modifications, of the Articles of Association and any amendments to the By-Laws may be made at any annual meeting of the Association, or at any adjourned session thereof, or at any special meeting of the Association called for such purpose, by a two-thirds (2/3) vote of those present and entitled to vote; *provided* that such amendment shall have been printed in the official journal at least one (1) month before the date of such meeting.

2. No change in the Articles of Association shall have legal effect until a Certificate thereof, verified by oath of the President and under Seal of the Association, attested by the Secretary-Treasurer, shall be filed in the office of the Secretary of State of the State of Illinois and recorded in the office of the Recorder of Deeds for Cook County, Illinois.

Article IX—Interpretation.

1. "Board," wherever used in these By-Laws, shall be taken to mean a Board of Trustees consisting of the Officers, the Committee on Publications and the Trustees elected as such, as provided in Article III.

2. It being the intent that this Association continue as successor to the unincorporated The Mathematical Association of America, these By-Laws shall be construed liberally to that effect.

The Undersigned, Chairman of the Organization Meeting of the Board of Trustees of The Mathematical Association of America (Incorporated), Hereby Certifies that the above is a full, true and correct copy of the code duly adopted at such meeting, held September 10, 1920, as the By-Laws of said Association.

Witness My Hand and the Corporate Seal, this 11th day of September, A. D. 1920.

[corporate seal] Herbert E. Slaught,
Chairman as aforesaid.

THE "DANGER AREA" CURVE.

By A. S. MERRILL, University of Montana.

A moving object (ship) is a target for a projectile (torpedo) which travels horizontally under its own power. Certain limiting conditions under which the projectile would be fired, or launched, are known. It is desired to determine the boundary curve of the area of probable positions of the firing agent relative to the target, at the time of firing.

Let the target, of length AB but of negligible width, be considered first as fixed in position. For certain known conditions of probable error, the projectile has a range R if fired from a point in the perpendicular bisector of AB. Construct a circle of diameter R through A and B (Fig. 1). Suppose AB to be small in comparison with R. The middle point of AB is then approximately at M, the end point of the diameter ML perpendicular to AB,—sufficiently close to be

3. The annual dues of each institutional member shall be Seven Dollars ($7), including two (2) subscriptions to the official journal.

4. All dues shall be payable on the first of January of each year. Should the annual dues of any member remain unpaid beyond a reasonable time, his name shall be dropped from the list, after due notice.

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THE UNDERSIGNED, CHAIRMAN OF THE ORGANIZATION MEETING OF THE BOARD OF TRUSTEES OF THE MATHEMATICAL ASSOCIATION OF AMERICA (INCORPORATED), HEREBY CERTIFIES that the above is a full, true and correct copy of the code duly adopted at such meeting, held September 10, 1920, as the By-Laws of said Association.

WITNESS MY HAND AND THE CORPORATE SEAL, this 11th day of September, A. D. 1920.

[CORPORATE SEAL] HERBERT E. SLAUGHT,
Chairman as aforesaid.

THE "DANGER AREA" CURVE.

By A. S. MERRILL, University of Montana.

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Let the target, of length AB but of negligible width, be considered first as fixed in position. For certain known conditions of probable error, the projectile has a range R if fired from a point in the perpendicular bisector of AB. Construct a circle of diameter R through A and B (Fig. 1). Suppose AB to be small in comparison with R. The middle point of AB is then approximately at M, the end point of the diameter ML perpendicular to AB,—sufficiently close to be

so considered for our problem. If the target is viewed from any other point upon the circle, as C, it is evident that it subtends an angle of the same size as that which it subtends when viewed from L. The probability of a hit is then the same for a projectile launched from any point upon the circle. In other words, the circle is the locus of points from which the projectile may be fired with the same probability of success for the fixed limiting conditions of probable error, and is the boundary curve of the area from which the projectile may be fired within the limits of probable success.

If now the target is considered as moving, there arise very difficult complications in the matter of aiming the projectile. With these, however, we are not at present concerned. For our problem the difficulties are not great.

Let the target be moving in the direction from A to B at the rate r, while the speed of the projectile is s. While the projectile moves a certain distance, the target moves r/s times this distance from a former position to the position AB. For projectiles fired from equal distances, the target, of course, moves through the same distance. The following scheme then enables us to plot very accurately the desired curve.

Construct Fig. 2 on tracing paper as follows. On the circle of Fig. 1 construct the tangents MK and LQ at M and L respectively. With M as a center and

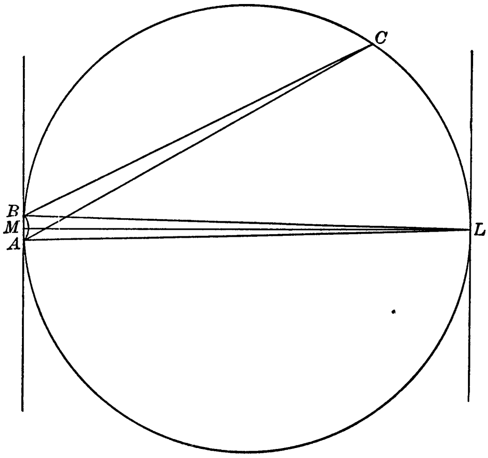


FIG. 1.

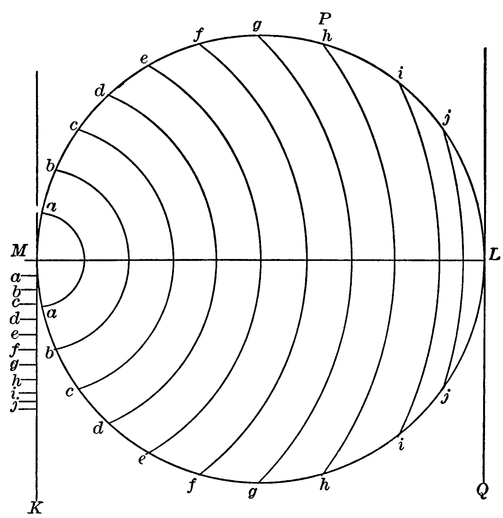


FIG. 2.

radii of lengths ranging at convenient intervals from 0 to R, strike arcs cutting the circle. From M toward K measure off distances equal to r/s times the lengths of these radii. Mark these points of division with the same symbols as the points of intersection of the corresponding arcs with the circle. On drawing paper draw two parallel lines at a distance R from each other, as OY and XV of Fig. 3. Place Fig. 2 over this drawing paper and, keeping MK and LQ coincident with OY and XV respectively, place the different division points of MK

successively over O. At each position, punch through the two papers with a pin point at the two points of the circle marked correspondingly. Draw a smooth curve (dotted in Fig. 3) through the points thus marked on the drawing paper. This is the required curve.

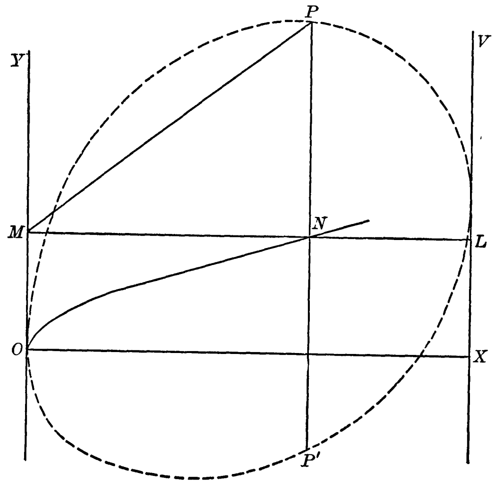


FIG. 3.

The area within the curve of Fig. 3 is referred to as the "danger area." It is evident that similar danger areas exist on both sides of AB, reflections of each other with respect to that line.

The above device is due to an English mathematician, and was used in naval strategical study during the late war. It has been interesting to work out the equations of a curve which has such an application. Results from two different simple methods are given below.

I. Let P be a point of the circle of Fig. 2 and the point determined from it in Fig. 3. Let $\angle\, LMP = \theta$. Then $MP = ML \cos\theta = R\cos\theta$, and

$$NP = MP \sin\theta = R\cos\theta \sin\theta.$$

Taking O as the origin, OX and OY as axes, we have

$$x = MN,$$

$$y = OM \pm NP \text{ (since } NP' = -\, NP).$$

But $MN = MP\cdot\cos\theta = R\cdot\cos^2\theta$, and $OM = r/s\cdot R\cdot\cos\theta$. Hence we have the parametric equations of the curve:

$$x = R\cos^2\theta,$$

$$y = R\cos\theta\cdot((r/s) + \sin\theta).$$

Whence we have

$$\cos\theta = \sqrt{x/R} = (1/R)\sqrt{Rx}, \qquad \sin\theta = \pm\,(1/R)\sqrt{R^2 - Rx};$$

and then

$$y = (r/s)\sqrt{Rx} \pm \sqrt{Rx - x^2}$$

is the equation of the curve.

II. The curve of Fig. 3 is the locus of points of intersection of corresponding curves of two families of circles. Of one of these families, the curves all have the same radius $R/2$, but their centers move along a line parallel to the line OY (Fig. 3) and midway between lines OY and XV. The radii, ρ, of the circles of

the second family vary from 0 to R, and the centers are along OY. The centers of corresponding curves are at the same distance above O, and the radius of a circle of the second family is equal to s/r times this distance. The equations of these families are then respectively:

$$(x - R/2)^2 + [y - (r/s)\cdot\rho]^2 = R^2/4, \tag{1}$$

$$x^2 + [y - (r/s)\cdot\rho]^2 = \rho^2. \tag{2}$$

Subtracting (1) from (2) we have

$$Rx - R^2/4 = \rho^2 - R^2/4,$$

or

$$\rho^2 = Rx.$$

Substituting in (2) we obtain

$$x^2 + [y - (r/s)\sqrt{Rx}]^2 = Rx,$$

$$y - (r/s)\sqrt{Rx} = \pm\sqrt{Rx - x^2},$$

or

$$y = (r/s)\sqrt{Rx} \pm \sqrt{Rx - x^2},$$

which is the equation already obtained in I.

The general shape of this curve is evident from an inspection of this equation. The locus of the equation

$$y = (r/s)\sqrt{Rx}$$

is evidently the upper branch of the parabola

$$y^2 = (r/s)^2\cdot Rx.$$

The quantity $\sqrt{Rx - x^2}$ is real for positive values of x less than R, and is zero for $x = 0$ or $x = R$. The danger area curve is thus seen to be a loop lying between the lines $x = 0$ and $x = R$. The points of this loop are at the distance $\sqrt{Rx - x^2}$ above and below the points (x, y) of the upper branch of the parabola $y^2 = (r/s)^2Rx$. If the objective had been considered as moving in the opposite direction, the danger area loop would have borne the same relation to the lower branch of this same curve. The danger area on the opposite side of the objective is similarly related to the parabola $y^2 = -(r/s)^2\cdot Rx$.

EXPRESSIONS FOR CERTAIN ACCELERATIONS OF A PARTICLE.

By GEORGE H. CRESSE, U. S. Naval Academy.

The object of this paper is to present a method of obtaining expressions for the magnitudes of certain accelerations of a particle in terms of its speed and from our knowledge of its path. Let the path be plane and considered only at non-singular points, and let its equation be $f(x, y) = 0$. By differentiation with respect to time, we see that the motion of the particle satisfies the equation,

$$f_x'x' + f_y'y' = 0 \tag{1}$$

where x', y' are time-derivatives. By differentiating (1) with respect to time and dividing by $\sqrt{f_x'^2 + f_y'^2}$, we have,

$$\frac{f_x''x'^2 + 2f_{xy}''x'y' + f_y''y'^2}{\sqrt{f_x'^2 + f_y'^2}} + \frac{f_x'x'' + f_y'y''}{\sqrt{f_x'^2 + f_y'^2}} = 0. \tag{2}$$

where x'', y'' are second time-derivatives. In the light of (1), we may put

$$\frac{f_x'x'' + f_y'y''}{\sqrt{f_x'^2 + f_y'^2}} = \frac{- y'x'' + x'y''}{\sqrt{x'^2 + y'^2}} = \alpha \cos \theta,$$

where α is the magnitude of the acceleration, and θ is the angle between the normal to the curve and the direction of acceleration. Hence for (2), we may write,

$$\frac{f_x''x'^2 + 2f_{xy}''x'y' + f_y''y'^2}{\sqrt{f_x'^2 + f_y'^2}} = - \alpha \cos \theta. \tag{3}$$

I. For example, let $f(x, y) = x^2 + y^2 - r^2$, and let the acceleration be directed toward the center of the circle. (3) now becomes the familiar formula:

$$\frac{x'^2 + y'^2}{\sqrt{x^2 + y^2}} = \alpha, \quad \text{or} \quad \alpha = \frac{v^2}{r}. \tag{4}$$

II. For another example, let $f(x, y) = b^2x^2 + a^2y^2 - a^2b^2$, and let the acceleration be directed toward one focus F of the ellipse. For this example, (3) is

$$\frac{b^2x'^2 + a^2y'^2}{\sqrt{b^4x^2 + a^4y^2}} = - \alpha \cos \theta. \tag{5}$$

If r and r' be the focal distances from F and F' respectively to the particle at P, and p and p' be perpendiculars drawn from the foci to the tangent which has

(Read at the meeting of the Maryland-Virginia-District of Columbia meeting of the Mathematical Association of America, May 15, 1920.)

P for its point of tangency, elementary properties of the ellipse are that,

$$\cos\theta = -\frac{p}{r} = -\frac{p'}{r'} = -\sqrt{\frac{pp'}{rr'}} = -\frac{b}{\sqrt{rr'}} = \frac{-b}{\sqrt{a^2 - e^2x^2}} = \frac{-ab^2}{\sqrt{b^4x^2 + a^4y^2}}.$$

And, since on the ellipse $b^2xx' + a^2yy' = 0$, the numerator in the first member of (5) has the value

$$\frac{b^4x'^2}{y^2} = \frac{a^4y'^2}{x^2}.$$

Hence (5) implies

$$(6_1)\ x'^2 = \frac{a\alpha y^2}{b^2}, \qquad (6_2)\ y'^2 = \frac{b^2\alpha x^2}{a^3};$$

whence

$$v^2 = \frac{\alpha(a^4y^2 + b^4x^2)}{a^3b^2} = \frac{\alpha rr'}{a},$$

or

$$\alpha = \frac{av^2}{rr'}, \tag{7}$$

which is a generalization of (4).

By differentiating (6_1) as to time, and setting in the value of y' from (6_2), we have,

$$x'' = \frac{1}{2a}\frac{d\alpha}{dx}(a^2 - x^2) - \frac{x\alpha}{a}.$$

But the x-component of acceleration is

$$x'' = \alpha\frac{ae - x}{a - ex}.$$

From the comparison of the last two equations, we have

$$\frac{d\alpha}{2\alpha} = \frac{e\,dx}{a - ex}.$$

Whence the Newtonian law:

$$\alpha = \frac{k^2}{(a - ex)^2} = \frac{k^2}{r^2}. \tag{8}$$

A comparison of (7) and (8) yields the following elegant form of the *vis viva* equation:

$$v^2 = \frac{k^2}{a}\frac{r'}{r}.$$

The above discussion of elliptical motion, with only the slightest modification is valid for hyperbolic and parabolic motion under a central force directed toward a focus. And the last equation is valid in all three cases without modification, r'/a having the limiting value 2 in the case of the parabola.

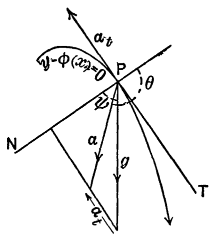
III. For the path of a projectile, we may take a parabola of higher order and let

$$f(x, y) = y - (a_0 + a_1x + a_2x^2 + \ldots + a_nx^n) = y - \varphi(x).$$

The range is assumed so short that the vertical direction may be regarded as constant. Now α is the resultant of the vertical gravitational acceleration $-g(y)$, and a tangential acceleration α_t. Obviously $\alpha \cos\theta = -g\cos\psi$, where ψ is the angle between the normal and the vertical. Hence (3) for this example is, by reference to (1),

$$-\frac{\varphi''(x)x'^2}{\sqrt{f_x'^2+f_y'^2}} = g\cos\psi = g\frac{dx}{\sqrt{dx^2+dy^2}} = g\frac{f_y'}{\sqrt{f_x'^2+f_y'^2}}.$$

or,



$$x' = \frac{\sqrt{g}}{[-\varphi''(x)]^{1/2}}.$$

Therefore,

(9) $$v = \frac{\sqrt{g}}{[-\varphi''(x)]^{1/2}}\sqrt{1+\left(\frac{dy}{dx}\right)^2};$$

and,

$$x'' = \frac{g}{2}\frac{\varphi'''(x)}{[\varphi''(x)]^2}.$$

But since the total horizontal acceleration is the horizontal component of α_t, it follows that

(10) $$\alpha_t = \frac{g\varphi'''(x)}{2[\varphi''(x)]^2}\cdot\sqrt{1+\left(\frac{dy}{dx}\right)^2};$$

or, by (9),

$$\alpha_t = v\frac{\sqrt{g}\varphi'''(x)}{2[-\varphi''(x)]^{3/2}}.$$

Now (9) and (10) can be written in the form $v = F_1(x)$ and $\alpha_t = F_2(x)$, where F_1 and F_2 are power series. The method of inversion of series gives

(11) $$F_2^{-1}(\alpha_t) = F_1^{-1}(v).$$

This entire discussion of the motion of a projectile is valid if the projectile is a finite sphere and also if the medium is not homogeneous. But the relation of α_t to v depends on the character and condition of the medium. So, (11) has the nature of a law for the particular projectile if, and only if, the medium in which the flight occurred was equally resistant at all points in the observed path.

QUESTIONS AND DISCUSSIONS.

EDITED BY W. A. HURWITZ, Cornell University, Ithaca, N. Y.

REPLIES.

34. [*1917*, 134, 341; *1920*, 114, 301.] Given the mixed integral and functional equation

$$\int_{x=0}^{x=h} f(x)dx = \frac{h}{6}\left[f(0) + 4f\left(\frac{h}{2}\right) + f(h)\right],$$

to determine the function $f(x)$. This equation is of rather fundamental practical value as it has to do with the most general solid whose volume is given by the prismatoid formula.

REPLY BY D. C. GILLESPIE, Cornell University.

The formula is a case of Simpson's Rule, where only two intervals are used. If $f(x)$ is a polynomial of degree not greater than three, this approximating formula overplays its rôle and evaluates the integral exactly. The function $f(x)$ being assumed continuous, one might surmise that the formula would not evaluate the integral exactly, for all values of h, unless $f(x)$ were a polynomial of degree three or less. The correctness of this surmise has not been demonstrated. In this note it is shown that *if $f(x)$ is continuous, $0 \leqq x \leqq b$, and has six continuous derivatives, $0 \leqq x \leqq H < b$, and the formula evaluates the integral for each value of h, $0 \leqq h \leqq b$, then $f(x)$ is a polynomial of degree not higher than three, $0 \leqq x \leqq b$.*

Suppose first that a function $g(x)$, continuous in $(0, b)$, satisfies the functional equation

$$\int_0^h x^n g(x)dx = \frac{h}{6}\left\{4\left(\frac{h}{2}\right)^n g\left(\frac{h}{2}\right) + h^n g(h)\right\} \tag{1}$$

for $0 \leqq h \leqq b$ and n a positive integer. The mean value theorem yields

$$\frac{h^{n+1}}{n+1} g(\theta h) = h^{n+1}\left\{\frac{1}{3}\cdot\frac{1}{2^{n-1}} g\left(\frac{h}{2}\right) + \frac{g(h)}{6}\right\}, \qquad 0 < \theta < 1. \tag{2}$$

The function $g(x)$ is continuous; hence, on dividing through by h^{n+1} and taking the limit of both sides as h approaches zero, we obtain

$$\frac{g(0)}{n+1} = \frac{1}{3}\cdot\frac{1}{2^{n-1}} g(0) + \frac{1}{6} g(0).$$

If $g(0) \neq 0$ we have

$$\frac{1}{n+1} = \frac{1}{3}\cdot\frac{1}{2^{n-1}} + \frac{1}{6}.$$

This equation is satisfied for n equal to 1 or 2 or 3 and for no other positive integral value of n. Hence, if n is to be greater than three, $g(0)$ must be zero.

We assume next $g(0) = 0$ and again find an upper limit for n. Let $\bar{h}$ be a value of x for which $|g(x)|$ assumes its upper limit in $(0, b)$. Equation (2) may now be written

$$\frac{\bar{h}^{n+1}}{n+1} g(\theta\bar{h}) = \bar{h}^{n+1}\left\{\frac{1}{3}\cdot\frac{1}{2^{n-1}} g\left(\frac{\bar{h}}{2}\right) + \frac{g(\bar{h})}{6}\right\}, \qquad 0 < \theta < 1. \tag{2'}$$

or

$$\frac{g(\theta\bar{h})}{n+1} = \frac{1}{3}\cdot\frac{1}{2^{n-1}} g\left(\frac{\bar{h}}{2}\right) + \frac{g(\bar{h})}{6}.$$

But

$$\left|g\left(\frac{\bar{h}}{2}\right)\right| \leqq |g(\bar{h})| \quad \text{and} \quad |g(\theta\bar{h})| \leqq |g(\bar{h})|$$

hence

$$\left|\frac{g(\bar{h})}{n+1}\right| \geqq \left|\frac{1}{3}\cdot\frac{1}{2^{n-1}} g\left(\frac{\bar{h}}{2}\right) + \frac{g(\bar{h})}{6}\right| \geqq \left(\frac{1}{6} - \frac{1}{3}\cdot\frac{1}{2^{n-1}}\right)|g(\bar{h})| \text{ for } n \geqq 2.$$

Now

$$\frac{1}{n+1} < \frac{1}{6} - \frac{1}{3} \cdot \frac{1}{2^{n-1}} \text{ for } n \geqq 6.$$

Equation (2′) then can not be satisfied for $n \geqq 6$, unless, of course, $g(x)$ is identically zero.

Suppose now that a function $f(x)$ which satisfies the given equation

$$(3) \qquad \int_0^h f(x)dx = \frac{h}{6}\left\{ f(0) + 4f\left(\frac{h}{2}\right) + f(h) \right\}, \qquad 0 \leqq h \leqq b$$

is continuous in $(0, b)$ and has six continuous derivatives in an interval $(0, H)$ in $(0, b)$. The Taylor expansion theorem applied for values of x in $(0, H)$ gives

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + f^{\text{iv}}(\theta x)\frac{x^4}{4!}, \qquad 0 < \theta < 1.$$

The quantity θ is, to be sure, not a constant but a function of x; nevertheless $f^{\text{iv}}(\theta x)$ is a continuous function of x in $(0, H)$. Susbtituting in equation (3) for $f(x)$ this expansion we obtain

$$\int_0^h \frac{x^4}{4!} f^{\text{iv}}(\theta x)dx = \frac{h}{6}\left\{ 4\left(\frac{h}{2}\right)^4 \frac{f(\theta h/2)}{4!} + \frac{h^4 f(\theta h)}{4!} \right\}.$$

This equation, as we have already seen, requires that $f^{\text{iv}}(0) = 0$; and one shows in the same way that $f^{\text{v}}(0) = 0$. Taking the sixth degree term in the expansion,

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + f^{\text{vi}}(\theta x)\frac{x^6}{6!}, \qquad 0 < \theta < 1.$$

If this expansion for $f(x)$ is substituted in (3) there results

$$\int_0^h \frac{x^6 f^{\text{vi}}(\theta x)}{6!}dx = \frac{h}{6}\left\{ 4\frac{h^6}{2^6}\frac{f^{\text{vi}}(\theta h/2)}{6!} + \frac{h^6 f^{\text{vi}}(\theta h)}{6!} \right\}.$$

We have shown that this equation can hold only when $f^{\text{vi}}(\theta x)$ is zero for all values of x in $(0, H)$. In the interval $(0, H)$ then $f(x)$ coincides with a cubic polynomial.

In QUESTIONS AND DISCUSSIONS *1920*, 302, it was pointed out by the EDITOR that two continuous functions satisfying equation (3) and coinciding in some interval from the origin to the right coincide throughout. Thus the result announced at the beginning is established.

The theorem, just quoted and used in the proof, is capable of a slight extension; *i.e.*, if a continuous function $f(x)$ satisfies equation (3) and is zero for $a \leqq x \leqq c$ where $c > 2a > 0$, then it is zero throughout $(0, b)$. For suppose $c \geqq h \geqq 2a$, then

$$\int_0^h f(x)dx = \frac{h}{6}f(0).$$

Since, however, $f(x) = 0$, $a \leqq x \leqq c$, as h increases from $2a$ to c the integral remains constant, hence $f(0) = 0$. Thus for $a \leqq h \leqq c$,

$$\int_0^h f(x)dx = \frac{2}{3}hf\left(\frac{h}{2}\right),$$

and as the integral maintains from $h = a$ to $h = c$ a constant value, which we have seen must be zero,

$$f\left(\frac{h}{2}\right) = 0, \qquad a \leqq h \leqq c,$$

or

$$f(x) = 0, \qquad \frac{a}{2} \leqq x \leqq \frac{c}{2}.$$

Continuing this process, we see that $f(x) = 0$, $0 \leqq x \leqq c$. Then by the previously quoted remark of the EDITOR, $f(x) = 0$ in the entire interval $(0, b)$.

DISCUSSIONS.

As the first discussion this month we present a paper which was read at the last annual meeting of the Mathematical Association of America, as part of a program devoted to the consideration of the sort of training in mathematics most useful for students specializing in fields in which mathematics finds frequent application. Professor Reed represents the point of view of the biometrist. His recommendations, briefly summarized, are: the usual courses in algebra, trigonometry, and analytic geometry; a short course in the calculus, emphasizing principles rather than technique; and a course in probability, with stress on statistical theory and the adjustment of curves to given data. It would seem, therefore, that the student of biometry will generally find ready at hand, in most of our colleges, courses agreeing reasonably well with the plan outlined by Professor Reed. Probably, too, the textbooks in elementary subjects are no longer so restricted in their treatment as Professor Reed implies. Few, if any, trigonometries of recent date, for example, fail to give applications to mechanics.

The brevity of the course in calculus in Professor Reed's scheme should be considered rather carefully. Is a student of statistics to accept Stirling's formula and the value of the probability integral on faith, or is he to receive demonstrations? The proofs, if given with logical precision, require more thorough treatment of the behavior of series and improper integrals than is usually found in even rather extensive first courses in calculus.

In the second discussion, Mr. Webb outlines a treatment of complex numbers, including the definition of exponential and trigonometric functions and their relationship. His scheme is similar to that found in some textbooks in trigonometry; seldom, if ever, is so full a discussion included in the ordinary course in algebra. Two items in his outline call for special comment. No. 6 prescribes "the customary development of e and of e^x by the binomial theorem as the limits of $(1 + 1/n)^n$ and $(1 + 1/n)^{nx}$ as $n \doteq \infty$." This customary scheme involves either a scandalously inaccurate treatment of the double limit process (as in most textbooks on calculus) or a degree of logical precision scarcely within the reach even of the average college graduate. The question involved is much more delicate than that of mere convergence. Either the properties of uniform convergence, or else some equivalent special process to avoid this concept must be used. It would be desirable, if possible, to find some less difficult approach to the exponential function. No. 8 implies that there must exist some k such that $\cos 1 + i \sin 1 = e^k$. Since the series for $\cos \theta$ and $\sin \theta$ are determined on the hypothesis that such a k exists, and then by use of the series the value $k = i$ is obtained, it is not easy to see how to avoid the hypothesis.

Professor Bell points out that the method of proof known as mathematical induction is valid only by virtue of a distinct assumption, which he formulates very clearly thus: *If a theorem is true for $n = 1$, and if its truth for $n - 1$ implies that it is true for n, then the theorem is true for all whole numbers.* This discussion should be helpful to our readers, inasmuch as the necessity of this explicit assumption is not always clearly recognized by teachers. However, the author's

and our readers may be interested in verifying the true result. If $\sqrt{PR/ES} = \alpha$, then α should be replaced by the next smaller integer n or the next greater integer $n + 1$, not according as α is less or greater than the arithmetic mean $n + \frac{1}{2}$, but according as α is less or greater than the geometric mean $\sqrt{n(n + 1)}$. For large values of n, the distinction between the two criteria is slight. Of course in any actual case it would be simple enough to substitute each of the two values in the expression for C in order to determine which was the better.

I. The Mathematics of Biometry.

By Lowell J. Reed, Johns Hopkins University.

(Read before the Mathematical Association of America, January 1, 1920.)

I think we are all agreed that the program for this meeting of the Mathematical Association of America is as important as any that the Association has ever discussed. Any science that fails to ally itself with the other sciences must of necessity have a narrow development, and this is perhaps more true of mathematics, due to its wide application, than of any of the other sciences. It is to be hoped that the Program Committee will carry out its own suggestion that at some later meeting we consider the converse of the present question, that is, the contribution of other sciences to the development of mathematics.

My own part in the present program is the discussion of the mathematics of biometry, and instead of presenting illustrations of the application of mathematics to this branch of biology I am going to outline what I consider to be the proper mathematical training for work in biometry. I wish first to call attention to the development of the present method of teaching mathematics. We have in the field of mathematics a number of branches that have been developed mainly as tools for the solution of practical problems. Thus, trigonometry has been developed mainly for the solution of problems in surveying; probability has been developed for games of chance. Now whenever this has been the case the teaching of the subject has been concentrated on this one particular application, to the exclusion of all others. In recent years the increasing use of mathematical methods in such sciences as chemistry, biology, etc., has led to an effort to broaden the teaching of mathematics by introducing new applications. The result has been a new group of textbooks under such titles as "Calculus for Chemical Students," "Mathematics for Agricultural Students," etc. All of these texts seem to me to miss the point in that they imply that calculus for chemistry students is distinct from calculus for other groups. This I do not believe to be the case. If we consider the mathematical needs of a student in any one of the sciences we find them about as follows: First, he needs a foundation in algebra, trigonometry, analytic geometry, and calculus; and secondly, he needs to be trained to take a problem in his particular field and translate it into mathematical language. The latter need is the greater of the two, and it is the one that it is the more difficult to satisfy.

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clearly brought out, mechanical illustrations being freely used. The correlation coefficient should be developed as a product moment with the mean of the variables taken as an origin. Following this the students should be taught the theory of curve fitting by the method of moments and this theory should be illustrated by examples in the derivation of empirical formulas. The method of fitting curves by least squares need not be presented as it is not so widely applicable as the method of moments. The student should also have impressed upon him the importance of the idea of probable error and should be trained to work out the probable error of all such constants as mean, standard deviation, coefficient of correlation, etc.

The course just outlined would furnish a student in biometry with the proper mathematical foundation for his work; his mathematical needs are similar to those of the student of any natural science. One of the most common quotations among biometricians is that of Sir Francis Galton: "Until the phenomena of any branch of knowledge have been submitted to measurement and number it cannot assume the status and dignity of a science," and as we see more and more of the leading scientists taking this point of view it impresses upon us the growing importance of mathematics. The methods of teaching mathematics should develop accordingly; and if this present meeting succeeds in stimulating the teaching of mathematics to live up to its increased importance, it will indeed have been a success.

II. Complex Numbers in Advanced Algebra.

By H. E. Webb, Central High School, Newark, N. J.

In many electrical laboratories it is desired that the workers have a certain familiarity with vectors, and in particular with the exponential notation $e^{i\theta}$ Examination of texts in algebra most commonly used in schools fails to bring to light an elementary explanation of this notation. Certain texts in trigonometry give such an explanation, but usually in a rather intricate form, and without a clear statement as to what it is all about. The following brief outline is suggested as an addition to a course in advanced algebra for the fourth year in high school or the freshman year in college; in this it is desired to escape from exhaustive analysis of various points which seemingly of necessity must be brought into any *rationale* of this notation.

1. The rules of combination of complex numbers to be presented as an arithmetic of number pairs, *without reference to Argand's diagram.*

2. The sine and cosine of *real positive and negative numbers* to be defined by reference to a unit circle, without mention of other trigonometric functions, leading only to the facts that in a circle of radius r the corresponding lines are $r \sin \theta$ and $r \cos \theta$, and that $\sin^2 \theta + \cos^2 \theta = 1$ for all values of θ.

3. The development of formulas for $\sin (x \pm y)$ and $\cos (x \pm y)$.

4. The polar notation of a complex number, shown as follows: If $a + ib$ is

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1. The rules of combination of complex numbers to be presented as an arithmetic of number pairs, *without reference to Argand's diagram.*

2. The sine and cosine of *real positive and negative numbers* to be defined by reference to a unit circle, without mention of other trigonometric functions, leading only to the facts that in a circle of radius r the corresponding lines are $r \sin \theta$ and $r \cos \theta$, and that $\sin^2 \theta + \cos^2 \theta = 1$ for all values of θ.

3. The development of formulas for $\sin (x \pm y)$ and $\cos (x \pm y)$.

4. The polar notation of a complex number, shown as follows: If $a + ib$ is

Dividing both members of the last equation by θ,

$$i\left(\frac{\sin\theta}{\theta}\right) = k + \frac{k^3\theta^2}{\underline{|3}} + \frac{k^5\theta^4}{\underline{|5}} \cdots.$$

As $\theta \doteq 0$, $(\sin\theta)/\theta \doteq 1$, and since k is constant,

$$k = i.$$

Substitution of i for k in the various formulas cited affords series development of $\cos\theta$ and $i\sin\theta$, in the first of which all terms are real, and in the second all imaginary; and also establishes the notation

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

9. If at this stage *and not sooner* the vector is defined the student will grasp perfectly the idea that such a directed line has two values, an algebraic value and an absolute value, which are different, in that its algebraic value depends upon its direction determined by the length of the arc of a unit circle described about the origin as a center.

It is the experience of the writer that in secondary school classes this development requires about as many days as he has topics outlined above, and that certain difficulties relating to the *value* of the ordinate are thereby avoided. A favorable opening is also provided for the teaching of hyperbolic functions, when this is desirable.

III. On Proofs by Mathematical Induction.

By E. T. Bell, University of Washington.

1. The best way to cure oneself of a crotchet is to confide it to some sympathetic listener. The crotchet in this note is one which has worried me since school days when I was induced to repeat the proof of the binomial theorem by mathematical induction. The same crotchet seems to trouble successive generations of freshmen, for occasionally one has obstinacy enough to balk at the magic formula "*and therefore the theorem is always true*," with which many authors conclude their proofs by recurrence. I hold that mathematical induction has no place in elementary teaching, particularly when such teaching strains at mathematical gnats, as in pseudo-rigorous presentations of the elementary theory of limits, the better to swallow logical camels such as some proofs of the binomial theorem or their equivalent quoted presently from Poincaré. This is the crotchet. In short, elementary teaching would be more convincing if it left rigor to that logistics which was Poincaré's bête noir.

2. It would be difficult to find a balder statement of the logical vice which characterizes many proofs by mathematical induction in the current text books, than the following extract from Poincaré's essay *On the Nature of Mathematical Reasoning*, in *Science and Hypothesis* (Halsted's translation, page 36, section IV).

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as a postulate, it would seem to be advisable to banish such phrases as "all whole numbers," and "the same operation can be repeated indefinitely," from elementary texts which make pretensions to rigor.

4. If reasoning by recurrence is, as Poincaré claims, mathematical reasoning *par excellence,* and if the objections put forth in § 3 are not groundless, it would seem to follow that mathematics is like any other science in that the conclusions which it legitimately draws are no more "general" or "universal" than those of other sciences. This contradicts what seems to be a current valuation of mathematical truth in the minds of laymen and some others who hold that mathematics has a timeless, eternal aspect, independent of all the empiricism which characterizes the conclusions of physical sciences.

There is one way of escape which is so obvious that it need only be pointed out. We can beat the mathematical devil round the logical bush by saying that (4) of § 3 is the rule, or law, of inference. But it would be a wise logician indeed who recognized (4) as one of his legitimate children. For where is either a proof of it or its explicit statement as a postulate of logic to be found?

IV. A Practical Printer's Problem in Maxima and Minima.

By Edgar E. DeCou, University of Oregon.

Dean Eric W. Allen, of the School of Journalism of the University of Oregon, presents a very interesting problem of frequent occurrence to the practical printer. The printer's only method of solution is by trial and error; and he states that on a large job of printing an added cost of \$100 or \$200 is often incurred by inability to solve the problem.

The conditions of the problem are as follows: 200,000 (P) prints are required; 1200 (S) prints per hour is the speed of the press; \$2.00 (R) per hour is the cost of running the press; 55 cents (E) each is the cost of the extra electrotypes, needed after the type is once set up. Required the number of electrotypes (x) that should be used to secure the minimum cost (C).

The problem is evidently one in determining the minimum value of C by the use of the differential calculus. The particular case takes the form,

$$C \text{ (in cents)} = \frac{200{,}000 \times 200}{1200(1+x)} + 55x = \frac{100{,}000}{3(x+1)} + 55x,$$

where x represents the number of electrotypes. Differentiating

$$\frac{dC}{dx} = -\frac{100{,}000}{3} \cdot \frac{1}{(x+1)^2} + 55 = 0,$$

for minimum value of C. Hence

$$x = \frac{100}{33} \cdot \sqrt{66} - 1 = 23.6 +.$$

In other words, the most economical number of electrotypes to use for this job is 24. Of course only the nearest integral value of x is used.

The general problem is stated thus:

$$C = \frac{PR}{S(1+x)} + E \cdot x,$$

$$\frac{dC}{dx} = -\frac{PR}{S(1+x)^2} + E = 0,$$

for a minimum.

From which

$$x = \sqrt{\frac{PR}{ES}} - 1.$$

This gives a formula involving only the arithmetical work of finding the square root to determine the number of electrotypes needed in any given case, and one of easy application by any practical printer.

RECENT PUBLICATIONS.

REVIEWS.

Differential Equations. By H. BATEMAN. London, Longmans, Green and Co., 1918. 8vo. 11 + 306 pp. Price 16 shillings.

The study of elementary methods of integrating differential equations is one which is taken up in many American colleges in a course following the integral calculus, or sometimes as a part of that course. When properly taught, it is a subject admirably adapted to developing in the student a skillful technique in using his calculus, a thing which he will find most helpful in his later work. Many students coming from calculus are woefully weak in many parts of the work which they have studied and "passed," so if such students are to go on to differential equations the beginning, at least, must be easy. They will then have some chance to develop and show their real ability. However, the manipulative side of the study must not be over-emphasized, for the extensive theoretical parts must be suitably developed. Moreover, there is far more opportunity for geometrical discussions than is generally given.

Since the time of Boole many text books on elementary differential equations have appeared in England and America. The general plan of all these books has, however, been much the same. Differential equations were classified into certain "standard forms," and, after having discussed the methods to be used in integrating these type forms, problems were given falling more or less closely under them. The number of real "clothed problems" was usually small. The book here under review is entirely different both in arrangement and content from

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the earlier books. This is especially true in the first four chapters. To quote from the preface: "The order in which the material has been arranged is slightly different from that which is usually adopted. Instead of beginning with the standard forms of equations which can be solved very easily, I have devoted the second chapter to integrating factors, and the third to the method of transformations."

When one comes to examine those changes in arrangement they appear to be considerable in the early chapters. The book has eleven chapters as follows: I. Differential equations and their solutions. II. Integrating factors. III. Transformations. IV. Geometrical applications. V. Differential equations, with particular solutions of a specified type. VI. Partial differential equations. VII. Total differential equations. VIII. Partial differential equations of the second order. IX. Integration in series. X. The solution of linear differential equations by means of definite integrals. XI. The mechanical integration of differential equations. At the end there are eight pages of miscellaneous examples, and an index.

With this list of chapter headings before him, let the reader ask himself where he would look for some topic with which he is familiar. Take, for instance, the general linear equation with constant coefficients. The index does not help, but a process of elimination leads one to look in the table of contents under Chapters II and III. Search there fails to give an exact reference, but linear second order equations are treated on p. 28 of Chapter II. Eight pages later we find the general linear equation with constant coefficients. However, the author does not treat it by means of an integrating factor, but by the symbolic method.

The question of the arrangement of the subject matter is of course extremely important. If a new arrangement is better than the old one, such comments as the foregoing would mean little. The author's plan in the first four chapters seems to be to classify his equations according to the methods used in discussing them, but he abandons this plan in the later chapters. The older plan classified differential equations according to their form, and integrated them by any available method. This older plan has one great advantage over the author's method, for it gives the student a classification of the equations themselves. When an ordinary differential equation is proposed for solution, the plan of attack is to discover which one of many methods is to be used. For this purpose some method of classification must be followed, and the most obvious classification is first according to order, and then by the form of the equation. If then the text book follows this same classification the student will be more ready to use it in attacking an unsolved equation. On the other hand the author's plan has the obvious advantage arising from the application of somewhat similar methods to equations of widely different forms. It leads to some curious results. Thus we find discontinuous solutions of linear equations with constant coefficients and a discussion of the general theory of a simultaneous system of n linear equations coming before such simple matters as the integrating factor of an equation of the

tions to important physical problems, including wave propagation, Maxwell's equations, theory of electrons, Laplace's equation and harmonic equations.

The last chapter describes various mechanical contrivances for solving certain differential equations of special forms. Several of these are due to E. Pascal, who has devised a number of instruments for this purpose.

Throughout the book there are many typographical errors. Most of them are easily corrected by the reader, but at least one of them has caused trouble to an unwary instructor.

The reviewer feels that the book will be difficult reading for a student beginning the study of differential equations, but that can be determined only by trying it out with a class. The book will surely prove to be a valuable addition to the library of the worker in mathematical physics.

CHARLES L. BOUTON.

HARVARD UNIVERSITY,
June, 1920.

The Theory of Determinants in the Historical Order of Development. By THOMAS MUIR. Volume 3, the period 1861 to 1880. London, Macmillan, 1920. 8vo. 26 + 503 pp. Price 35 shillings.

The first edition of the first volume of this work was reprinted in book form, in 1890, from the *Proceedings of the Royal Society of Edinburgh*; a second edition, with over 200 pages of additional material, appeared in 1906, and covered the history of general and special determinants up to 1841. The second volume (1911) made a similar survey for the period 1841 to 1860. The third volume under review covers an additional twenty year period. In June, 1918, the manuscript of a fourth volume bringing the record up to the end of the nineteenth century was nearly complete. Mathematicians must ever be grateful to Sir Thomas for his monumental work which is designed to contain a complete record of published results in connection with the theory of determinants.

The material is admirably arranged and indexed so that it is possible readily to trace the contributions to the theory of any individual, or the chronological development of any special type of determinants. For example chapter 14 in volume 2 and chapter 15 in volume 3 contain the history of circulants from the first paper of Catalan in 1846 to the last of Gegenbaur in 1880; skew determinants may be traced in a similar way in the fourteenth, ninth and tenth chapters of volumes 1, 2 and 3 respectively. Although a three page chapter is devoted to "cubic and n-dimensional determinants up to 1880" and various titles are listed, practically as in the article of 1900 by Professors Hedrick and Cairns,[1] the contents of the papers are not analyzed as in the other chapters because the work in question is a survey of determinants as ordinarily defined, and not of their generalizations.

The titles of the chapters are as follows—I: "Determinants in general, from 1860 to 1880," 1–82; II: "Determinants and linear equations, from 1861 to 1878," 83–93; III: "Axisymmetric

[1] "On three dimensional determinants," *Annals of Mathematics*, second series, vol. 1, pp. 49–67.

determinants, from 1846 to 1879," 94–122; IV: "Symmetric determinants that are not axisymmetric, from 1862 to 1879," 123–131; V: "Alternants from 1860 to 1879," 132–175; VI: "Compound determinants from 1862 to 1880," 176–207; VII: "Recurrents from 1858 to 1879," 208–247; VIII: "Wronskians from 1862 to 1874," 248-256; IX: "Jacobians from 1862 to 1877," 257–271; X: "Skew determinants and Pfaffians from 1862 to 1880," 272-283; XI: "Orthogonants from 1855 to 1879, 284–308; XII: "Persymmetric determinants from 1836 to 1879," 309-326; XIII: "Bigradients from 1859 to 1880," 327-362; XIV: "Hessians, from 1862 to 1879," 363-371; XV: "Circulants, from 1861 to 1880," 372-392; XVI: "Continuants from 1850 to 1880," 393–422; XVII: "Multilineants up to 1877," 423–428; XVIII: "Cubic and n-dimensional determinants up to 1880," 429–431; XIX: "Bordered determinants up to 1880," 432–446; XX: "Determinants whose elements are combinatory numbers up to 1880," 447–462; XXI: "Zero-axial determinants up to 1888," 463–468; XXII: "The less common special forms from 1839 to 1880," 469-496; "List of authors," 497–503.

R. C. ARCHIBALD.

College Teaching. Studies in Methods of Teaching in the College. Edited by PAUL KLAPPER with an Introduction by N. M. Butler. Yonkers-on-Hudson, New York, World Book Co., 1920. 16 + 583 pp.

This book contains twenty-eight chapters by as many different authors. Chapter VIII (pages 161–182) on "The Teaching of Mathematics" is by G. A. MILLER. The sub-headings of the chapter are as follows: Recent changes and some of their sources; Influence of researches in mathematics on methods of teaching; Range of subjects and preparation of students; Variety of college courses in mathematics; History of college mathematics; Relation of mathematics in secondary school and college; Aims of college mathematics: methods of teaching; Advanced work in college mathematics; Mathematics and technical education; Preparation of the college teacher of mathematics; The mathematical text-book.

The Teaching of Arithmetic. By JOHN C. STONE. Chicago, Benj. H. Sanborn & Company, 1918. 262 pp. Price $1.32.

This presents for teachers, supervisors, and those preparing to teach, "a discussion of the aims and purposes of a course in arithmetic and of the methods of presenting each topic that should find a place in our elementary schools." The final chapter, on "Measuring results," gives an account of the standard arithmetic tests developed during the last twenty years.

NOTES.

In T. ZIEHEN, *Lehrbuch der Logik auf positivistischer Grundlage mit Berücksichtigung der Geschichte der Logik*, (Bonn, A. Marucs & E. Webers Verlag, 1920), are discussed: "Die mathematische (symbolistische) Logik" pp. 227–236 including a bibliography of about 35 authors; "Mathematische Grundlegung der Logik," pp. 410–416.

A new list of the members of the Mathematical Association (England) was published in April, 1920. It contains the names of 9 "honorary members," of 747 "ordinary members", and of 83 "associates" of the London, Yorkshire, North Wales, and Sydney (New South Wales) Branches.

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"In speaking of *time* as a fourth dimension to space, have not our mathematicians been guilty of a carelessness in language which they would be the last to tolerate in symbols? They are always careful to associate (x, y, z) not with t but with ct, where c is the velocity of light. The fourth dimension is not *time* at all, but *distance travelled*, otherwise these same mathematicians would say its dimensions were wrong. Or if it is preferred to use time, then we must divide (x, y, z) by c, which comes to measuring distances in light-years before we can associate them with time.

"A further protest against the rather unnecessary mystification of the layman may not carry quite so much weight; but all the same let it be made. Is it vital to use imaginary time in our language and our thoughts? As a piece of mathematical mechanism it may be useful to treat a hyperbola $x^2 - y^2 = 1$, as a particular case of a circle $x^2 + y^2 = 1$; but it is surely straining our powers of conception needlessly to talk of the hyperbola as a particular case of the circle, except for the purpose of obtaining results quickly by the methods of projective geometry. The 'rigid body rotation' invoked for the transformation of $dx^2 + dy^2 + dz^2 - c^2dt^2$ calls up a false image in some of our minds: it is really a simple distortion with which we have to do, and could not this be mentioned in a cautionary way?"—From *The Observatory*, April, 1920, volume 43, page 171.

ARTICLES IN CURRENT PERIODICALS.

American Journal of Science, volume 49, May, 1920: "The mathematics of isostasy" I by I. C. Chamberlain, 311–318, II by W. D. MacMillan, 318–323.

Journal of the Indian Mathematical Society, volume 11, no. 6, December, 1919 (published May, 1920): "International Congress of Mathematicians," 201–202; "Computation of the ephemeris of a planet and a comet" by R. N. Apte, 203–218; "An extension of Feuerbach's theorem" by M. Bhimasena Rao, 219–227; "To find a cube the sum of whose divisors is a perfect square" by N. B. Mitra, 228–229; Note on a continued fraction by K. B. Madhava, 230–234; "Astronomical notes," 235–236; Problems and solutions, 237–240.

Journal of the Washington Academy of Sciences, volume 10, no. 2, Jan. 19, 1920: "A trigonometric computer" by F. E. Wright, 29-31.—No. 7, April 4: "A graphical method for plotting reciprocals" by F. E. Wright, 185–188.

Mathematical Gazette, volume 10, May, 1920: "The graphical treatment of differential equations" (continued) by S. Brodetsky, 49–60; "Ratio and proportion" by D. K. Picken, 60–62; "Gleanings from far and near," 62, 74; "Geocentric parallax" by A. A. Krishnaswami Aiyangar, 63–64; "A graphical treatment of simple harmonic motion" by W. G. Bickley, 64–65; "A trigonometrical lucubration" by R. F. Muirhead, 66–68; "Some incidental writings by De Morgan" (continued), 69–74; Review by J. M. Child of W. N. Rose's *Mathematics for Engineers* (London, 1920), 75–76; Review by A. Dakin of C. de la Vallée Poussin's *Leçons sur l'approximation des fonctions d'une variable réelle* (Paris, 1919), 76–77; Review by G. H. Hardy of Petrovitch's *Les spectres numériques* (Paris, 1919), and J. L. S. Hatton's *Theory of the Imaginary in Geometry,* 77–79; "Books received, contents of journals, etc.," i–iv.

Messenger of Mathematics, volume 49, no. 5, September, 1919: "On some series whose nth term involves the number of classes of binary quadratics of determinant $-n$" by L. J. Mordell, 65–72; "A note on a theorem of Riemann's" by Grace C. Young, 73–78; "The twelve elliptic functions related to sixteen doubly periodic functions of the second kind" by E. T. Bell, 78–80—October: "The twelve elliptic functions related to sixteen doubly periodic functions of the second kind" (concluded) by E. T. Bell, 82–84; "Notes on some points in the integral calculus (lii)" by G. H. Hardy, 85–90; "A new integral equation satisfied by the solution of a certain linear differential equation, which occurs in the theory of electrical oscillations and of the tides" by E. G. C. Poole, 91–96.

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Journal of the Indian Mathematical Society, volume 11, no. 6, December, 1919 (published May, 1920): "International Congress of Mathematicians," 201–202; "Computation of the ephemeris of a planet and a comet" by R. N. Apte, 203–218; "An extension of Feuerbach's theorem" by M. Bhimasena Rao, 219–227; "To find a cube the sum of whose divisors is a perfect square" by N. B. Mitra, 228–229; Note on a continued fraction by K. B. Madhava, 230–234; "Astronomical notes," 235–236; Problems and solutions, 237–240.

Journal of the Washington Academy of Sciences, volume 10, no. 2, Jan. 19, 1920: "A trigonometric computer" by F. E. Wright, 29-31.—No. 7, April 4: "A graphical method for plotting reciprocals" by F. E. Wright, 185–188.

Mathematical Gazette, volume 10, May, 1920: "The graphical treatment of differential equations" (continued) by S. Brodetsky, 49–60; "Ratio and proportion" by D. K. Picken, 60–62; "Gleanings from far and near," 62, 74; "Geocentric parallax" by A. A. Krishnaswami Aiyangar, 63–64; "A graphical treatment of simple harmonic motion" by W. G. Bickley, 64–65; "A trigonometrical lucubration" by R. F. Muirhead, 66–68; "Some incidental writings by De Morgan" (continued), 69–74; Review by J. M. Child of W. N. Rose's *Mathematics for Engineers* (London, 1920), 75–76; Review by A. Dakin of C. de la Vallée Poussin's *Leçons sur l'approximation des fonctions d'une variable réelle* (Paris, 1919), 76–77; Review by G. H. Hardy of Petrovitch's *Les spectres numériques* (Paris, 1919), and J. L. S. Hatton's *Theory of the Imaginary in Geometry*, 77–79; "Books received, contents of journals, etc.," i–iv.

Messenger of Mathematics, volume 49, no. 5, September, 1919: "On some series whose nth term involves the number of classes of binary quadratics of determinant $-n$" by L. J. Mordell, 65–72; "A note on a theorem of Riemann's" by Grace C. Young, 73–78; "The twelve elliptic functions related to sixteen doubly periodic functions of the second kind" by E. T. Bell, 78–80—October: "The twelve elliptic functions related to sixteen doubly periodic functions of the second kind" (concluded) by E. T. Bell, 82–84; "Notes on some points in the integral calculus (lii)" by G. H. Hardy, 85–90; "A new integral equation satisfied by the solution of a certain linear differential equation, which occurs in the theory of electrical oscillations and of the tides" by E. G. C. Poole, 91–96.

NATURE, volume 105, May 13, 1920: "Differential geometry" [review of R. H. Fowler's *The Elementary Differential Geometry of Plane Curves* (Cambridge, 1920)] by G. B. M[athews], 321–322—May 20: "Relativity and Geometry" [review of E. Freundlich's *The Foundations of Einstein's Theory of Gravitation,* translated by H. L. Brose (Cambridge, 1920)] by E. Cunningham, 350–351; "A new method for approximate evaluation of definite integrals between finite limits" by A. F. Dufton, 354–355 ["An approximate evaluation of $\int_0^1 F(x)dx$ is $\cdots \frac{1}{4}\{F(\frac{1}{10}) + F(\frac{4}{10}) + F(\frac{6}{10}) + F(\frac{9}{10})\}$"].—June 3: "A new method for approximate evaluation of definite integrals between finite limits" by C. F. Merchant, 422 ["The subject has a particular interest for naval architects, inasmuch as the majority of calculations relative to displacement, stability, strength, etc., of ships involve the finding of areas and volumes bounded by curved lines and surfaces.

The particular rule enunciated by Mr. A. F. Dufton in *Nature* of May 20 has been in use at this [Royal Naval] College for some years, and gives very accurate results in obtaining areas and volumes, and also, by a further application, the positions of their centers of gravity . . ..

An interesting paper dealing with this subject and giving a great variety of rules for approximate integration was read at the Institution of Naval Architects in 1908 (*Trans. I. N. A.*, Vol. 1) by Sir W. S. Abell entitled "Two notes on ship calculations."]

LA NATURE, volume 48, May 8, 1920: "Un nouvel appareil enregistreur pour l'étude des lois de la dynamique et la composition des mouvements vibratoires" by Paul Bud, 230–234—May 15: "Le centenaire de la machine à calculer industrielle" by L. Reverchon, 249–252; "Il y a quinze ans (27 août 1904) *La Nature* donnait un important et substantiel article de M. Maurice d'Ocagne sur les machines à calculer. L'auteur passait successivement en revue les additionneurs avec ou sans touches, les appareils à multiplier par additions successives et par application du principe des tables de Pythagore, les machines à différences spécialement employées à la construction des tables et les machines algébriques et analytiques dont quelques unes sont extraordinairement compliquées.

On pouvait lire dans cet article les lignes suivantes: 'C'est au financier alsacien Thomas, de Colmar, que revient sans conteste le très grand mérite d'avoir réalisé la première machine à multiplier et à diviser rapide, robuste et fonctionnant en toute sûreté. C'est en 1820 que Thomas créa son Arithmomètre dont depuis lors, le type n'a cessé de se perfectionner sous la direction du constructeur Payen. Très répandu dans les grands établissements financiers, il a fourni une carrière qui a depassé aujourd'hui trois quarts de siècle, attestant de hautes qualités pratiques.'" The article contains a portrait of Thomas, of Blaise Pascal, "inventeur de la premiere machine à calculer," and of Léon Ballée, "a 19 ans, manipulant une machine à calculer."]

NOUVELLES ANNALES DE MATHÉMATIQUES, volume 79, February, 1920: "Sur le maximum et le minimum des fonctions de deux variables" by G. Valiron, 41–50; "Equation angulaire d'un conoïde droit. Application au cylindroïde envisagé dans ses rapports avec la distribution des courbures autour d'un point d'une surface" by M. d'Ocagne, 51–55; "Sur les contacts des sphères tangentes à quatre plans" by V. Thébault, 55–59;" A propos de la transformation par tangentes orthogonales" by F. Balitrand, 59–60; "Certificats de calcul différentiel et intégral," 60–73; "Chronique," and "Questions," 76–80—March: "Exposé élémentaire d'une théorie rigoureuse des liaisons finies unilatérales" by E. Delassus, 81–93; "Sur les polygones harmoniques d'un nombre pair de cotés et sur certains cercles du triangle" by V. Thébault, 94–100; "Sur les tangentes aux trajectoires des sommets d'un triangle qui se déforme dans un plan," by R. Goormaghtigh, 100–102; "Chronique," 103–105; "Certificats de calcul différentiel et intégral" 105–117; "Questions," 117–120—April: "Sur l'application de la loi de Gauss à la position probable d'un point dans le plan ou dans l'espace" by J. Haag, 121–142; "Sur un théorème de Cornu relatif aux caustiques" by T. Lemoyne, 142–145; "Chronique," 145–147; "Bibliographie," "Certificat d'astronomie," "Certificat de physique mathématique," "Questions," 147–160.

PROCEEDINGS OF THE LONDON MATHEMATICAL SOCIETY, series 2, Vol. 19, part 1, June, 1920: "Groups involving three and only three operators which are square" by G. A. Miller, 51–56.

PROCEEDINGS OF THE NATIONAL ACADEMY OF SCIENCES, volume 6, no. 4, April, 1920: "The starting of a ship" by J. K. Whittemore, 182–185; "A thermodynamic study of electrolytic solutions" by F. L. Hitchcock, 186–197; "Functionals invariant under one-parameter continuous groups of transformations in the space of continuous functions" by I. A. Barnett, 200–204.

REVUE DE L'ENSEIGNEMENT DES SCIENCES, volume 14, January–February, 1920: "Elimination d'une inconnue entre plusieurs équations" by M. Stuyvaert, 1–6; "Apropos de la notion

41–45; "Die Perspektivität im geometrischen Unterricht der OII" by M. Enders, 46–51; "Die Kegelschnitte als Kreisprojektion" by J. Arneberg, 52–57; "Die Exponentialfunktion im Unterricht" by E. Götting, 58–62; "Kleine Mitteilungen," 62–65; "Aufgaben-repertorium," 65–71; "Bücher Besprechungen," 71–87 [not all of the books are mathematical.]

UNDERGRADUATE MATHEMATICS CLUBS.

EDITED BY U. G. MITCHELL, University of Kansas, Lawrence.

CLUB ACTIVITIES.

The attention of our readers is called to the fact that among the clubs reported this month are two which have not before been listed by the Department. One of these, "The Pascal Circle" of Trinity College, Washington, D. C., has been organized for four years. The other, "The Square" of Washington Square College, New York University, is, so far as the writer is aware, the most recently organized club having begun its activities in February of this year.

MATHEMATICS CLUB OF COLUMBIA UNIVERSITY, New York, N. Y.

[*1918*, 227–228; *1919*, 262.]

Officers 1919–1920: President, Joseph Feldt '21; vice-president, Barclay V. Heuill '21; secretary, Albert E. Meder, Jr., '22; faculty adviser, Professor Lewis P. Siceloff. These officers constituted the program committee.

The average attendance at meetings during the year was twenty-five. The programs were as follows:

October 31, 1919: "Geometric inversions" by Albert E. Meder '22.

November 21: "Cardinal numbers" by Joseph Feldt '21.

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January 9, 1920: "What is a point?" by Professor Cassius J. Keyser.

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March 12: "The Einstein theory" by Doctor Kenneth W. Lamson of Barnard College.

March 26: "Origins of the calculus" by Oscar Bodansky '22.

April 9: "Famous problems in the history of mathematics" by Professor David Eugene Smith, president of the Mathematical Association of America.

April 21: "The number π" by Frank Frink, Jr., '22.

May 7: "The geometry of the cubic equation" by Professor Henry B. Mitchell.

THE PASCAL CIRCLE, TRINITY COLLEGE, Washington, D. C.

An undergraduate Mathematics Club was organized at Trinity College, Washington, D. C., under the name of the "Pascal Circle," in 1916, to give

41–45; "Die Perspektivität im geometrischen Unterricht der OII" by M. Enders, 46–51; "Die Kegelschnitte als Kreisprojektion" by J. Arneberg, 52–57; "Die Exponentialfunktion im Unterricht" by E. Götting, 58–62; "Kleine Mitteilungen," 62–65; "Aufgaben-repertorium," 65–71; "Bücher Besprechungen," 71–87 [not all of the books are mathematical.]

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The Pascal Circle, Trinity College, Washington, D. C.

An undergraduate Mathematics Club was organized at Trinity College, Washington, D. C., under the name of the "Pascal Circle," in 1916, to give

students of mathematics an insight into the problems of current magazines and to study the principles underlying mathematical recreations. The Circle had twenty-two names on its roll that year and held fortnightly meetings. The club has improved each year in numbers, spirit and aims. During 1919–20 it had about thirty-five members and included on its program papers on the history of mathematics. The meetings have been informal in character and enlivened by discussions on present day topics or mathematical puzzles and recreations of interest.

The officers of the Circle for the year 1919–20 were: President, Mary C. Duncan '20; vice-president, Margaret M. Walsh '21; Secretary-treasurer, Julia Thomas '22; faculty adviser and honorary president, Professor Marie C. Mangold.

October, 1919: The initial meeting was devoted to the reception of new members and to a short interpretation of the constitution of the Circle by the president. At this meeting it was decided to have special pins for the society and all members were invited to submit designs.

November: "Blaise Pascal" by Mary C. Duncan '20. The speaker explained why the mathematical club of Trinity College chose the name of "Pascal Circle." The mathematical principle underlying "magic number cards" was discussed by the club.

December: "The history of arithmetic" by Margaret Walsh '21. Designs of pins were submitted and that of Margaret Sheehan '19 was accepted. The pin is of black enamel with gold edge and gold letters. It is in the form of a Pascal triangle bearing the Greek letters $\pi\tau\kappa$ representing the initial letters of "Pascal" Trinity College.

February, 1920: "The History of Geometry" by Professor Marie C. Mangold.

March: "A psychological test" by Margaret Sheehan '19: "A psychological problem" by Marguerite Hopper '22.

At a second meeting in this month a paper was read on "The history of algebra" by Julia Thomas '22 and the formation of magic squares was discussed and explained by various members of the club.

April: "Women who have played a part in the history of mathematics" by Margaret Crotty '21.

May: "The church and mathematics" by Catherine Manion '20.

The last meeting of the year was a social meeting.

THE SQUARE, WASHINGTON SQUARE COLLEGE, NEW YORK UNIVERSITY, N. Y.

The Mathematics Club of Washington Square College was organized February 18, 1920, and named the "Square." For the present it was decided not to limit the membership to advanced students and as a consequence a great deal of interest has been shown by freshmen.

The officers for 1920 were: President, Sophie Epstein '23; secretary-treasurer, Rose Beckenstein '23; faculty adviser, Assistant Professor Earnest J. Oglesby.

mathematicians and listening to delightful talks on Descartes, Newton and Euclid as seen in an interview.

March 18: Subjects for discussion were "Cranks and their follies" and "The eternal triangle." Part of the hour was spent in solving problems, a prize being awarded for the greatest number of correct solutions. After the business meeting refreshments were served.

April 27: The subject discussed was "The circle" including the nine-point circle and the problem of Apollonius, to construct a circle tangent to each of three given circles. A lighter supplement was added to the purely instructive part of the program by the reading of one of Stephen Leacock's essays entitled "An interview with our greatest scientist."

May 27: Spring picnic. The five-dollar prize for the greatest number of solutions to the problems presented during the year was awarded by Professor Cowley. Various members of the club had composed mathematical parodies of well-known songs and the entertainment feature of the meeting consisted in the singing of these parodies.

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL AND OTTO DUNKEL.

Send all communications about problems and solutions to B. F FINKEL, Springfield, Mo.

PROBLEMS FOR SOLUTION.

[N.B. The editorial work of this department would be greatly facilitated if, on sending in problems, the proposers would also enclose their solutions—*when they have them.* If a problem proposed is not original the proposer is requested *invariably* to state the fact and to give an exact reference to the source.]

2858. Proposed by C. P. SOUSLEY, Pennsylvania State College.

A boy can split wood as fast as his father can saw, and the father can split twice as fast as the son can saw. How should the money received for their labor be divided?

2859. Proposed by L. S. DEDERICK, U. S. Naval Academy.

Derive an expression for the limit of error in evaluating a definite integral by Simpson's Rule.

2860. Proposed by E. O. BROWN, Chicago, Ill.

A frustrum of a right circular cone has a volume v. The lateral area added to the lesser base is a sum which is a minimum. Determine the dimensions of the frustrum in terms of v.

2861. Proposed by B. F. FINKEL, Drury College.

Obtain by plane geometry, *i.e.*, without use of calculus, a construction for finding points on the envelope of a system of circles whose diameters are chords of a fixed circle passing through a given point on it. Also determine geometrically the nature of the locus.

2862. Proposed by J. L. RILEY, Stephensville, Texas.

Show that the whole area commanded by a gun on a hillside is an ellipse whose focus is at the gun, whose eccentricity is the sine of the inclination of the hill to the horizon, and whose semi-latus rectum is twice the greatest height to which the gun could send a ball.

SOLUTIONS.

129 (Average and probabilities) [**1902, 148; 1903, 81**]. **Proposed by J. K. ELLWOOD, Pittsburg, Pa.**

A and B play with two dice, A throwing. If he throws 7 or 11, he wins; if he throws 3, or two aces, or two sixes, B wins. But if he throws 4, 5, 6, 8, 9, or 10, he continues throwing to duplicate this throw, in which event he wins; if in throwing, however, he throws 7, B wins. What is the expectancy of each? [This is the regulation "crap" game, B being the banker.]

155 (Average and probabilities) [**1905, 76**]. **Proposed by E. B. WILSON, Yale University.**

The game of craps is played with two dice. If the player throws 7 or 11 on the first throw he wins. If he throws 12, 2, or 3 he loses. If the player throws any other number, that is to say, 4, 5, 6, 8, 9, 10, he is obliged to continue throwing until he throws that number again or until he throws 7. If he succeeds in throwing his first throw before he does 7, he wins—otherwise he loses. Required the odds against him. (Note that he can continue throwing indefinitely without getting either his original throw or the 7).

NOTE BY R. C. ARCHIBALD, Brown University.

The answers to these problems may be found in the article by Mr. Bancroft H. Brown ("Probabilities in the game of 'shooting craps'") in this MONTHLY, *1919*, 351–352; see also *1920*, 166–167.

2752 [1919, 72]. Proposed by the late R. E. MOORE.

Test for convergence, the series $\sum_{n=1}^{\infty} a_n$, in which

$$a_n = \left[\frac{1\cdot 3\cdot 5 \cdots (2n-1)}{2\cdot 4\cdot 6 \cdots 2n}\right]^2.$$

I. SOLUTION BY P. J. DA CUNHA, University of Lisbon.

On sait que lorsque le rapport a_n/a_{n+1} est développable suivant les puissances entières de $1/n$, il est très facile de décider, dans tous les cas, s'il y a ou non convergence. En effet, si l' on pose, en s'arrêtant aux termes du second ordre,

$$\frac{a_n}{a_{n+1}} = \alpha + \frac{\beta}{n} + \frac{\theta_n}{n^2},$$

θ_n restant fini pour $n = \infty$, il y a:

Divergence	si $\alpha < 1$ ou $\alpha = 1$ et $\beta \leqq 1$;
Convergence	si $\alpha > 1$ ou $\alpha = 1$ et $\beta > 1$

(Jordan, *Cours d'Analyse*, troisième édition, tome 1, page 313.)

Cela posé, en appliquant la règle à la série donnée, nous avons

$$\frac{a_n}{a_{n+1}} = \left(\frac{2n+2}{2n+1}\right)^2 = \frac{4n^2+8n+4}{4n^2+4n+1} = \frac{1+\dfrac{2}{n}+\dfrac{1}{n^2}}{1+\dfrac{1}{n}+\dfrac{1}{4n^2}}$$

ou, finalement

$$\frac{a_n}{a_{n+1}} = 1 + \frac{1}{n} + \frac{\theta_n}{n^2}.$$

Comme nous trouvons $\alpha = 1$, $\beta = 1$, la serie considerée est divergente.

II. SOLUTION BY OTTO DUNKEL, Washington University.

A proof of the same nature as that of Professor Cunha but requiring only elementary facts is as follows.

Omit the first factor $(\frac{1}{2})^2$ of each term and consider the series whose general term is

$$b_n = 4a_n = \left(\frac{3}{4}\right)^2\left(\frac{5}{6}\right)^2\cdots\left(\frac{2n-3}{2n-2}\right)^2\left(\frac{2n-1}{2n}\right)^2.$$

Since

$$\left(\frac{2n-1}{2n}\right)^2 = \left(1-\frac{1}{2n}\right)^2 = 1 - \frac{1}{n} + \frac{1}{4n^2} > \frac{n-1}{n},$$

we have, by multiplying all such inequalities from $n = 2$ to $n = n$,

$$b_n > \frac{1}{n}$$

and therefore the b series is divergent and hence also the a series.

Also solved by E. H. CLARKE, R. A. JOHNSON, H. L. OLSON, ARTHUR PELLETIER, and the Proposer.

2758 [1919, 124]. Proposed by LEONARD RICHARDSON, University of British Columbia.

Prove that, if r be a positive integer,

$$\int_0^{\pi/2} \frac{\sin (2r+1)\psi}{\sin \psi} d\psi = \frac{\pi}{2}.$$

and

$$\int_0^{\pi/2} \frac{\sin 2r\psi}{\sin \psi} d\psi = 2\left\{ 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + \frac{(-1)^{r-1}}{2r-1} \right\}.$$

I. SOLUTION BY ELIJAH SWIFT, University of Vermont.

We have the identity

$$\sin n\psi - \sin (n-2)\psi = 2 \sin \psi \cos (n-1)\psi,$$

or

$$\sin n\psi = 2 \sin \psi \cos (n-1)\psi + \sin (n-2)\psi.$$

This yields us an easy reduction formula for the two integrals. For the second,

$$\int_0^{\pi/2} \frac{\sin 2r\psi}{\sin \psi} d\psi = 2\int_0^{\pi/2} \cos (2r-1)\psi \cdot d\psi + \int_0^{\pi/2} \frac{\sin (2r-2)\psi}{\sin \psi} d\psi$$
$$= 2\frac{(-1)^{r-1}}{2r-1} + \int_0^{\pi/2} \frac{\sin (2r-2)\psi}{\sin \psi} d\psi.$$

Applying this r times, we reach the indicated result.

For the first integral,

$$\int_0^{\pi/2} \frac{\sin (2r+1)\psi}{\sin \psi} d\psi = 2\int^{\pi/2} \cos 2r\psi \cdot d\psi + \int_0^{\pi/2} \frac{\sin (2r-1)\psi}{\sin \psi} d\psi.$$

The first of these is 0. Repeating this process r times, the required integral

$$= \int_0^{\pi/2} \frac{\sin \psi}{\sin \psi} \cdot d\psi = \frac{\pi}{2}.$$

Solved similarly by A. M. HARDING and C. C. YEN.

II. SOLUTION BY R. D. BOHANNAN, Ohio State University.

Let $\cos \psi + i \sin \psi = z$ and $\cos \psi - i \sin \psi = 1/z$. Then, for the first integral, we have

$$\frac{\sin (2r+1)\psi}{\sin \psi} d\psi = \frac{z^{2r+1} - \dfrac{1}{z^{2r+1}}}{z - \dfrac{1}{z}} \frac{dz}{iz},$$
$$= \left\{ z^{2r} + z^{2r-2} + \cdots + z^2 + 1 + \frac{1}{z^2} + \cdots + \frac{1}{z^{2r}} \right\} \frac{dz}{iz}.$$

The first and last terms, after integration, give

$$\frac{1}{2ri}\left(z^{2r} - \frac{1}{z^{2r}} \right) \quad \text{or,} \quad \frac{1}{r} \sin 2r\psi \Big]_0^{\pi/2},$$

which is zero; likewise, all other pairs, equidistant from the ends. The middle term gives $(\log z)/i$ or ψ, since $z = e^{i\psi}$, and with the given limits, this reduces to $\pi/2$.

Similarly, the second integral comes from

$$\left\{ z^{2r-1} + z^{2r-3} + \cdots z + \frac{1}{z} + \frac{1}{z^3} + \cdots + \frac{1}{z^{2r-1}} \right\} \frac{dz}{iz}.$$

After integration, the first and last terms, give

$$\frac{1}{i(2r-1)} \left\{ z^{2r-1} - \frac{1}{z^{2r-1}} \right\} \quad \text{or} \quad \frac{2}{2r-1} \sin (2r-1)\psi \Big]_0^{\pi/2} = \pm \frac{2}{2r-1}.$$

The result given is obtained by starting with the central terms,

$$\left(z + \frac{1}{z} \right) \frac{dz}{iz}.$$

Also solved similarly by P. J. da Cunha, William Herberg, and H. L. Olson.

2759 [1919, 124]. Proposed by J. L. Riley, Stephenville, Texas.

Solve the simultaneous functional equations

$$\phi(x+y) = \phi(x) + \frac{\phi(y)\cdot\psi(x)}{1-\phi(x)\phi(y)},$$

$$\psi(x+y) = \frac{\psi(x)\cdot\psi(y)}{1-\phi(x)\phi(y)}.$$

I. Solution by C. F. Gummer, Kingston, Ont.

When $y = 0$ the equations show that either $\psi(x) = 0$, or $\phi(0) = 0$ and $\psi(0) = 1$. In the former case, $\phi(x)$ is constant. In the latter, the first equation gives

$$\frac{\phi(x+y) - \phi(x)}{y} = \frac{\phi(y) - \phi(0)}{y} \cdot \frac{\psi(x)}{1-\phi(x)\phi(y)};$$

so that, if we assume that $\phi'(0)$ exists and equals a, we deduce that

$$\phi'(x) = a\psi(x).$$

If further $\psi'(0)$ exists and equals b, the second equation gives

$$\psi'(x) = \{a\phi(x) + b\}\psi(x).$$

Eliminating $\psi(x)$ between the last two equations, and writing $X(x) = a\phi(x) + b$, we get

$$X''(x) = X(x)X'(x);$$

and on integration, since $X(0) = b$ and $X'(0) = a^2$,

$$X(x) = 2A \tan (Ax + B),$$

where $\sqrt{2a^2 - b^2} = 2A$ and $b = 2A \tan B$.

Hence

$$\phi(x) = \sqrt{2} \cdot \frac{\sin Ax}{\cos (Ax+B)}, \qquad \psi(x) = \frac{\cos^2 B}{\cos^2 (Ax+B)}.$$

This solution, however, fails to satisfy the functional equations unless $A = 0$. Hence the only solutions admitting derivatives for $x = 0$ are

$$\phi(x) = c, \quad \psi(x) = 0 \quad \text{and} \quad \phi(x) = 0, \quad \psi(x) = 1.$$

It is apparent that $\phi(x) = 0$, $\psi(x) = a^x$ is a solution.

Possibly the proposer intended to square the denominator on the right of the second equation. With this change and similar work we get, assuming $\phi'(0)$ and $\psi'(0)$ to exist, *either*

$$\phi(x) = c, \qquad \psi(x) = 0$$

Similarly, the second integral comes from

$$\left\{ z^{2r-1} + z^{2r-3} + \cdots z + \frac{1}{z} + \frac{1}{z^3} + \cdots + \frac{1}{z^{2r-1}} \right\} \frac{dz}{iz}.$$

After integration, the first and last terms, give

$$\frac{1}{i(2r-1)} \left\{ z^{2r-1} - \frac{1}{z^{2r-1}} \right\} \quad \text{or} \quad \frac{2}{2r-1} \sin (2r-1)\psi \Big]_0^{\pi/2} = \pm \frac{2}{2r-1}.$$

The result given is obtained by starting with the central terms,

$$\left(z + \frac{1}{z} \right) \frac{dz}{iz}.$$

Also solved similarly by P. J. DA CUNHA, WILLIAM HERBERG, and H. L. OLSON.

2759 [1919, 124]. Proposed by J. L. RILEY, Stephenville, Texas.

Solve the simultaneous functional equations

$$\phi(x+y) = \phi(x) + \frac{\phi(y)\cdot\psi(x)}{1-\phi(x)\phi(y)},$$

$$\psi(x+y) = \frac{\psi(x)\cdot\psi(y)}{1-\phi(x)\phi(y)}.$$

I. SOLUTION BY C. F. GUMMER, Kingston, Ont.

When $y = 0$ the equations show that either $\psi(x) = 0$, or $\phi(0) = 0$ and $\psi(0) = 1$. In the former case, $\phi(x)$ is constant. In the latter, the first equation gives

$$\frac{\phi(x+y) - \phi(x)}{y} = \frac{\phi(y) - \phi(0)}{y} \cdot \frac{\psi(x)}{1-\phi(x)\phi(y)};$$

so that, if we assume that $\phi'(0)$ exists and equals a, we deduce that

$$\phi'(x) = a\psi(x).$$

If further $\psi'(0)$ exists and equals b, the second equation gives

$$\psi'(x) = \{a\phi(x) + b\}\psi(x).$$

Eliminating $\psi(x)$ between the last two equations, and writing $X(x) = a\phi(x) + b$, we get

$$X''(x) = X(x)X'(x);$$

and on integration, since $X(0) = b$ and $X'(0) = a^2$,

$$X(x) = 2A \tan (Ax + B),$$

where $\sqrt{2a^2 - b^2} = 2A$ and $b = 2A \tan B$.

Hence

$$\phi(x) = \sqrt{2} \cdot \frac{\sin Ax}{\cos (Ax+B)}, \qquad \psi(x) = \frac{\cos^2 B}{\cos^2 (Ax+B)}.$$

This solution, however, fails to satisfy the functional equations unless $A = 0$. Hence the only solutions admitting derivatives for $x = 0$ are

$$\phi(x) = c, \quad \psi(x) = 0 \quad \text{and} \quad \phi(x) = 0, \quad \psi(x) = 1.$$

It is apparent that $\phi(x) = 0$, $\psi(x) = a^x$ is a solution.

Possibly the proposer intended to square the denominator on the right of the second equation. With this change and similar work we get, assuming $\phi'(0)$ and $\psi'(0)$ to exist, *either*

$$\phi(x) = c, \qquad \psi(x) = 0$$

when both (1) and (2) are satisfied, one has

$$\phi(x) = \tan ax, \qquad \psi(x) = \sec^2 ax,$$

which is the most general continuous solution, [also $\psi(x) \equiv 0$ and $\phi(x) = \text{const.}$—EDS.].

Also solved by ELIJAH SWIFT; partial solutions by E. B. ESCOTT, and H.L. OLSON.

2763 [1919, 170] Proposed by C. N. SCHMALL, New York City.

Show that the equation $ky - 2k^{1/3}a^{2/3}x + x^2 = 0$, where k is a variable parameter, represents a family of parabolas passing through a fixed point, and all having the same areas comprised between the curve and the x-axis.

Show, also, that the envelope of the family is the rectangular hyperbola whose equation is $xy = 2^5a^2/3^3$.

SOLUTION BY P. J. DA CUNHA, University of Lisbon.

Il est évident que l'équation donnée représente une famille de paraboles, et que toutes ces courbes jouissent de la propriété de passer par l'origine. L'abscisse du second point d'intersection de chacune des paraboles et de l'axe des x est $2k^{1/3}a^{2/3}$, de sorte que l'aire comprise entre la courbe et le même axe est

$$A = \frac{1}{k}\int_0^{2k^{1/3}a^{2/3}} (2k^{1/3}a^{2/3}x - x^2)dx = \frac{1}{k}\left[k^{1/3}a^{2/3}x^2 - \frac{x^3}{3} \right]_0^{2k^{1/3}a^{2/3}} = \frac{4}{3}a^2.$$

Finalement, l'équation de l'enveloppe s'obtenant par la règle connue, c'est à dire, en éliminant k entre l'équation donnée et celle-ci: $y - 2k^{-2/3}a^{2/3}x/3 = 0$, nous tombons sur l'équation

$$xy = \frac{2^5a^2}{3^3},$$

c. q. f. d.

After giving this same solution H. H. Downing, University of Kentucky, notes that the locus of the vertices of the parabolas is the equilateral hyperbola, $xy = a^2$.

The problem was also solved by C. A. BARNHART, A. M. HARDING, WILLIAM HERBERG, WILLIAM HOOVER, ARTHUR PELLETIER, and J. B. REYNOLDS.

2766 [1919, 171]. Proposed by N. P. PANDYA, Amreli, India.

Is it possible to find a harmonic series whose terms are positive integers such that the product of the first, second, seventh, and eighth terms is equal to the product of the third, fourth, fifth, and sixth terms?

SOLUTION BY A. M. HARDING, University of Arkansas.

Any harmonic series of eight terms may be written in the form

$$a, \quad \frac{a}{1+ak}, \quad \frac{a}{1+2ak}, \quad \cdots, \quad \frac{a}{1+7ak},$$

where a and k are constants, $a \neq 0$. If it is possible to find an harmonic series satisfying the conditions of the problem, then

$$a \cdot \frac{a}{1+ak} \cdot \frac{a}{1+6ak} \cdot \frac{a}{1+7ak} = \frac{a}{1+2ak} \cdot \frac{a}{1+3ak} \cdot \frac{a}{1+4ak} \cdot \frac{a}{1+5ak}.$$

That is, since $a \neq 0$,

$$(1+ak)(1+6ak)(1+7ak) = (1+2ak)(1+3ak)(1+4ak)(1+5ak),$$

or

$$16a^2k^2 + 112a^3k^3 + 120a^4k^4 = 0.$$

Supposing $k \neq 0$ and dividing by $8a^2k^2$, we have

$$2 + 14ak + 15a^2k^2 = 0;$$

whence

$$ak = \frac{-7 \pm \sqrt{19}}{15}.$$

Then

$$1 + ak = \frac{8 \pm \sqrt{19}}{15}, \quad \text{and} \quad \frac{a}{1 + ak} = \frac{15a}{8 \pm \sqrt{19}}.$$

That is, if a is a positive integer, the second term of the series is irrational. Hence, the only series which satisfies the condition of the problem is one for which $k = 0$; that is, a series in which the terms are all equal.

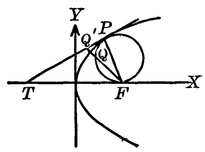
Also solved by H. L. Olson and Arthur Pelletier.

2779 [1919, 268]. Proposed by J. L. RILEY, Junior Agricultural and Mechanics College, Stephensville, Texas.

A parabola is placed with its axis horizontal; find the straight line of shortest descent from the curve to the focus.

I. Solution by A. M. Harding, University of Arkansas.

Construct a circle tangent to the axis of the parabola at F and tangent to the parabola at P[1]. The time of descent down all chords of this circle which pass through F is the same, so that the time down PF is the same as the time down any other chord QF, and is, therefore, less than the time down the focal radius $Q'F$.



II. Solution by H. S. Uhler, Yale University.

Let $p \equiv$ distance from vertex to focus, $g \equiv$ acceleration due to gravity, $s \equiv$ distance from any point (x, y) on the parabola and in the first quadrant to the focus $(p, 0)$, $t \equiv$ time of descent, and $\gamma \equiv$ angle which the straight line makes with the positive direction of the axis of x. From kinematics, $s = \frac{1}{2}at^2$ and $a = g \sin \gamma$; hence,

$$t^2 = \frac{2s}{g \sin \gamma}. \tag{1}$$

The polar equation of the parabola, with the pole at the focus, is

$$s = \frac{2p}{1 - \cos \gamma} = \frac{2p(1 + \cos \gamma)}{\sin^2 \gamma}. \tag{2}$$

Hence, from (1) and (2),

$$\frac{g}{4p} t^2 = \frac{1 + \cos \gamma}{\sin^3 \gamma}.$$

Differentiating,

$$\frac{gt}{8p} \cdot \frac{dt}{d\gamma} = \frac{-\sin^2 \gamma - 3(1 + \cos \gamma) \cos \gamma}{\sin^4 \gamma} = 0;$$

therefore,

$$2 \cos^2 \gamma + 3 \cos \gamma + 1 = 0$$

so that

$$\cos \gamma = -1 \quad \text{or} \quad \cos \gamma = -\tfrac{1}{2}.$$

[1] If this construction is possible and if the tangent at P meets the axis in T, $TP = TF = x + p$, the equation of the parabola being $y^2 = 4px$. Hence to get the coördinates of P we have $(x + p)^2 = 4x^2 + 4px$ or $x = \frac{1}{3}p$. It follows that the triangle TPF is an equilateral triangle, each side equal to $\frac{4}{3}p$. The coördinates of the center of the circle will be p, $4p/3\sqrt{3}$, and its equation $(x - p)^2 + y^2 - 8py/3\sqrt{3} = 0$. If we eliminate x or y from this equation and the equation of the parabola we shallfind for the resulting equation two equal roots corresponding to p, and two complex roots, showing that the circle must lie entirely within the parabola—which is an important consideration in connection with Professor Harding's discussion of the problem.—Editors.

An examination of the factored form of the derivative shows that $\gamma = 120°$ gives a minimum.

Incidentally,

$$t = 4\sqrt{\frac{p}{3g\sqrt{3}}}, \qquad s = \tfrac{4}{3}p.$$

Also solved by T. M. Blakslee, R. A. Johnson, H. L. Olson, Arthur Pelletier, S. W. Reaves, J. B. Reynolds, and Elijah Swift.

2794 [1919 458]. Proposed by B. J. BROWN, Kansas City, Mo.

Find the value of $x^{e^x} \div x^{x^x}$ when $x \doteq 0$ and when $x \doteq \infty$. I. C. S. 1902.

Solution by Paul Capron, U. S. Naval Academy.

A^{b^c} is taken to mean $A^{(b^c)}$.

(i) When $x \doteq 0$, it is well known that $e^x \doteq 1$, $x^x \doteq 1$;

$$x(\log x)^n \doteq \frac{(\log x)^n}{1/x} \doteq -n\frac{(\log x)^{n-1}}{1/x} \doteq \cdots \doteq \pm n!\, x \doteq 0. \qquad y = x^{e^x} \div x^{x^x} = x^{e^x - x^x}.$$

$$\log y = (e^x - x^x)\log x = \frac{e^x - x^x}{1/\log x} \doteq \frac{0}{0} \doteq \frac{e^x - x^x(1+\log x)}{-\dfrac{1}{x(\log x)^2}}$$

$$\doteq -xe^x(\log x)^2 + x^x(x(\log x)^2 + x(\log x)^3 \doteq 1.0 + 1(0+0) = 0.$$

Hence, as $x \doteq 0$, $y \doteq 1$.

(ii) When $x \doteq \infty$,

$$x^x \div e^x = \left(\frac{x}{e}\right)^x \doteq \infty;$$

hence,

$$e^x - x^x = e^x\left(1 - \left(\frac{x}{e}\right)^x\right) \doteq -\infty;$$

i.e., with the notation of (i), $\log y = (e^x - x^x)\log x \doteq (-\infty)(+\infty) \doteq -\infty$, or $y \doteq 0$.

2795 [1919, 458]. Proposed by C. N. SCHMALL, New York City.

A square is described touching the ellipse, $x^2/a^2 + y^2/b^2 = 1$, at the ends of its minor axis; a second ellipse is drawn circumscribing the square and tangent to the given ellipse at the ends of the major axis. The new ellipse is treated as the first and the process is continued until there are n new ellipses. Show that the last ellipse is a circle if the eccentricity of the original ellipse is $\sqrt{n/(n+1)}$.

Solution by Gertrude I. McCain, Oxford, Ohio.

If x', y' be the coördinates of a corner of the first square, then each equals b; and, lying on the second ellipse,

$$\frac{x'^2}{a^2} + \frac{y'^2}{b_1^2} = 1,$$

where b_1 is the minor axis of the first circumscribed ellipse. Substituting b for x' and y' and solving for b_1^2,

$$b_1^2 = \frac{a^2b^2}{a^2 - b^2}.$$

Similarly,

$$b_2^2 = \frac{a^2b_1^2}{a^2 - b_1^2} = \frac{a^2b^2}{a^2 - 2b^2},$$

and

$$b_n^2 = \frac{a^2b^2}{a^2 - nb^2}.$$

An examination of the factored form of the derivative shows that $\gamma = 120°$ gives a minimum. Incidentally,

$$t = 4\sqrt{\frac{p}{3g\sqrt{3}}}, \qquad s = \tfrac{4}{3}p.$$

Also solved by T. M. Blakslee, R. A. Johnson, H. L. Olson, Arthur Pelletier, S. W. Reaves, J. B. Reynolds, and Elijah Swift.

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$$\log y = (e^x - x^x)\log x = \frac{e^x - x^x}{1/\log x} \doteq \frac{0}{0} \doteq \frac{e^x - x^x(1+\log x)}{-\dfrac{1}{x(\log x)^2}}$$

$$\doteq -xe^x(\log x)^2 + x^x(x(\log x)^2 + x(\log x)^3 \doteq 1.0 + 1(0+0) = 0.$$

Hence, as $x \doteq 0$, $y \doteq 1$.

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$$x^x \div e^x = \left(\frac{x}{e}\right)^x \doteq \infty;$$

hence,

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i.e., with the notation of (i), $\log y = (e^x - x^x)\log x \doteq (-\infty)(+\infty) \doteq -\infty$, or $y \doteq 0$.

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Similarly,

$$b_2^2 = \frac{a^2b_1^2}{a^2 - b_1^2} = \frac{a^2b^2}{a^2 - 2b^2},$$

and

$$b_n^2 = \frac{a^2b^2}{a^2 - nb^2}.$$

If $\frac{a^2 - b^2}{a^2} = \frac{n}{n+1}$, then $n = \frac{a^2 - b^2}{b^2}$. Substituting this value of n, $b_n^2 = a^2$. Hence, the last ellipse is a circle.

Also solved by NORMAN ANNING, GRACE M. BAREIS, PAUL CAPRON, H. O. HANSON, OLIVE C. HAZLETT, E. J. OGLESBY, H. L. OLSON, ARTHUR PELLETIER, J. L. RILEY, C. P. SOUSLEY, ELIJAH SWIFT, and H. S. UHLER.

NOTES AND NEWS.

IT IS HOPED THAT READERS OF THE MONTHLY WILL COOPERATE IN CONTRIBUTING TO THE GENERAL INTEREST OF THIS DEPARTMENT BY SENDING ITEMS TO THE EDITOR-IN-CHIEF.

Mr. L. M. GRAVES of Chicago, has been appointed instructor in mathematics at Washington University.

Miss LOUISA M. WEBSTER, associate professor of mathematics at Hunter College, has been appointed principal of the Hunter College High School.

Mr. H. L. OLSON, of the University of Wisconsin, has been appointed instructor in mathematics at the University of Michigan.

At the University of Nebraska, Dr. T. A. PIERCE has been promoted to an assistant-professorship of mathematics; Mr. W. M. BOND and Mr. C. R. SHERER have been appointed instructors, and Miss C. RUMMONS assistant instructor.

Mr. OSCAR SCHMIEDEL has been appointed professor of mathematics at Nebraska Wesleyan University.

H. S. MYERS, for four years professor of mathematics at Huron (S. D.) College, has taken a similar position at Southwestern College, Winfield, Kansas.

Professor SOLOMON LEFSCHETZ of the University of Kansas will be absent on leave during this academic year most of which he will spend in Europe.

Professor HARRIET GLAZIER of Western College for Women has been given an extended leave of absence for a second year and will spend the year at Southern Branch of the University of California, filling the vacancy caused by the absence for the year of Professor MYRTIE COLLIER.

At the University of Texas Miss ANNA MULLIGAN and Miss HELMA HOLMES have been appointed instructors, and Mr. CLAUDE BAILEY, tutor, in mathematics.

Mr. O. C. COLLINS, of Oxford University, England, has been appointed instructor in mathematics at the University of Nebraska.

Mr. C. E. HARRINGTON, M. E. (Cornell), has been appointed as an instructor in mathematics at the University of Buffalo.

Professor E. H. THOMAS of Tabor College has been appointed professor of physics at Southeast Missouri State Teachers College.

Mr. J. B. ROSENBACH, last year at the University of Illinois, has been appointed to an instructorship in mathematics at the Carnegie Institute of Technology. During the summer vacation Mr. Rosenbach was employed in the valuation department of the Atchison, Topeka and Santa Fe Railway.

Miss CLARIBEL KENDALL of the University of Colorado will spend her year's leave of absence at the University of Chicago as Fellow in mathematics, her place being supplied by W. H. HILL of Greeley (Colo.) High School.

Dr. E. G. BILL has been promoted to a full professorship of mathematics at Dartmouth College.

Dr. O. J. RAMLER has been promoted to be associate professor of mathematics at the Catholic University of America.

Mr. F. V. MORLEY, of Johns Hopkins University, has left for study at Oxford University where he enters as a Rhodes Scholar.

Dr. DANIEL BUCHANAN, professor of astronomy and mathematics at Queen's University, has been appointed professor of mathematics and head of the department at the University of British Columbia.

In the department of mathematics at the University of Minnesota Associate Professor R. W. BRINK [*1920*, 382], who has been lecturer at the University of Edinburgh during 1919–20, has resumed his work. Mr. C. M. JENSEN, of the University of Minnesota, and Miss RUTH ASKELAND, of Hamline College, have been appointed teaching fellows; and Miss ELIZABETH CARLSON, of the University of Minnesota, a teaching assistant.

At the University of Wyoming Professor C. B. RIDGAWAY is retiring after twenty-four years' service; Professor C. E. STROMQUIST has been made head of the department; A. R. FEHN of Kansas Agricultural College has been made associate professor of mathematics, and LUCY A. FEDDERSEN has been appointed instructor in mathematics in the University High School.

At the University of Pennsylvania Assistant Professor F. H. SAFFORD has been promoted to a full professorship of mathematics; Dr. R. A. ARMS has resigned his instructorship to accept a professorship at Pennsylvania College; and Mr. H. M. GEHMAN and R. W. HARTLEY have been appointed instructors in mathematics.

At the University of Alberta, Lecturer GEORGE ROBINSON has been appointed assistant to Professor E. T. WHITTAKER in his mathematical laboratory in Edinburgh; Mr. T. H. MILNE, of the department of physics at the University of Toronto has been appointed lecturer in mathematics; Mrs. E. T. MITCHELL, of the University of Alberta, has been appointed instructor.

At the University of California, Dr. A. R. WILLIAMS, for three years instructor in the Shipping Board at Portland, Oregon, and Dr. C. D. SHANE have been appointed instructors; Mrs. A. D. B. SHANE, Ph.D., Miss NINA ALDERTON, Miss G. M. CAMPBELL, and Messrs. P. H. DAUS, B. C. WONG, VICTOR STEED and J. F. POBANZ have been appointed assistants.

At the University of Saskatchewan, Professor L. L. DINES [*1919*, 84] is acting head of the department of mathematics while Professor G. H. LING is spending a year in Europe. Dr. I. A. BARNETT has been appointed assistant professor of mathematics, having been released from his acceptance of an instructorship at the University of Illinois [*1920*, 190].

In the department of mathematics at the University of Illinois, Associate

Professor R. D. CARMICHAEL has been promoted to be a professor and Drs. L. L. STEIMLEY and C. F. GREEN have been advanced from assistantships to instructorships; Drs. B. MARGARET TURNER and MARGARET HASEMAN, of Bryn Mawr, have been appointed instructors; Associate Professor E. R. SMITH, of Pennsylvania State College, has leave of absence for a year to serve as an associate; Messrs. J. W. WAGNER, of Baldwin Wallace College, H. A. BENDER, instructor in Muskingum College, L. H. McFARLANE, of the University of Missouri, R. F. GRAESSER and J. W. HEARST, of the University of Illinois, and Mrs. FENNER STICKNEY, of the University of California, have been appointed assistants.

Dr. EINAR HILLE, of the University of Stockholm, is studying mathematics at Harvard University as a fellow of the Swedish-American Foundation.

ERIC DOOLITTLE, professor of astronomy in the University of Pennsylvania and director of the Flower Observatory since 1912, died on September 21, 1920, aged sixty-one years. He was instructor in astronomy at the University of Pennsylvania 1896–1904, and assistant professor 1904–1912. We have already referred [*1919*, 178] to the death of his father, Professor C. L. DOOLITTLE, his predecessor as director of the Flower Observatory.

T. R. DAVIES, assistant professor of mathematics at McGill University since 1909, died on August 19, 1920, aged fifty-six years. He was born in England and graduated at the University of Cambridge. For several years he was head-master of Abingdon school, Montreal. In 1907 he was appointed lecturer in mathematics at McGill.

K. H. STRUVE, director of the Berlin-Babelsberg Observatory, and professor of astronomy in the University of Berlin since 1904, died August 12, 1920, aged sixty-six years. Professor Struve was a Russian by birth. His father O. W. STRUVE (1819–1905) and grandfather F. G. W. STRUVE (1793–1864) were also astronomers—working for many years in Russian observatories.

Sir J. N. LOCKYER died August 16, 1920, aged eighty-four years. He was the founder of *Nature*, director of the Hill Observatory, Salcombe Regis, Sidmouth, England, and late director of the solar physics observatory, South Kensington, 1885–1913. For sketches of his life and work see *Nature*, August 19, 26, and September 2, 1920.

Dr. GIOVANNI CELORIA, director of the observatory of Brera, Milan, since 1900, and author of numerous books and papers on astronomical subjects, died August 17, 1920, aged 78 years.

Dr. P. G. H. BACHMANN, professor at the University of Münster from 1875 till made professor emeritus in 1900, died on March 31, 1920, aged eighty-three years. He wrote many books and papers in the field of theory of numbers, and was the author of the sections on "Analytische Zahlentheorie," 1900, and "Niedere Zahlentheorie," 1900 (French editions 1906–1910), in the *Encyklopädie der mathematischen Wissenschaften*. His most recent book was *Das Fermatproblem in seiner bisherigen Entwicklung*, 1919. Among his earlier works were the following: *Die Lehre von der Kreistheilung und ihre Beziehung zur Zahlentheorie*,

1872 (translated into Italian); *Die Elemente der Zahlentheorie*, 1892; *Die analytische Zahlentheorie*, 1894; *Die Arithmetik der quadratischen formen*, 1898; *Niedere Zahlentheorie*, 1902–1910; *Allgemeine Arithmetik der Zahlenkorper*, 1905; and *Grundlehren der neueren Zahlentheorie*, 1907.

JOHN PERRY, emeritus professor of mechanics and mathematics, Royal College of Science, South Kensington, London, died August 4, 1920, aged seventy years. He was born in Ulster and educated at Queens College, Belfast. From 1875 to 1879 he was professor of engineering in Japan, and from 1881–96 professor of engineering and mathematics at the City and Guilds of London Technical College, Finsbury. He was the author of many books and scores of papers; among the former are: *Practical Mechanics*, 1883; *Spinning Tops*, 1890 (2nd German ed. 1913); *Calculus for Engineers*, 1897 (German ed. 1902, 2nd ed. 1910; Russian ed. 1904); *Applied mechanics*, 1897 (French ed. 1913; German ed. 1908); *Practical Mathematics*, 1899, 1907, 1910 (Russian ed. 1909); and *Elementary Practical Mathematics*, 1913. There were many other translations of his books into foreign languages. Perry's *England's Neglect of Science*, 1900, and *Discussion on the Teaching of Mathematics* (which he edited), 1901, contain arguments against the culture value of mathematics—the basis of the "Perry movement" (Cf. *Report of the National Committee of Fifteen on Geometry Syllabus*, Washington, 1912, pp. 27–28). For a sketch of Perry's career see *Nature*, August 12, 1920.

The twenty-seventh meeting and ninth colloquium of the American Mathematical Society were held at the University of Chicago, September 7–11. One hundred and one members of the Society were present at the meeting and thirty four papers were read. The authors of thirty-two of these papers are also members of the Association; for details see the *Bulletin of the American Mathematical Society*, November, 1920. The attendance of eighty-eight persons at the colloquium was considerably in excess of that at any previous colloquium. Two courses of five lectures each were given "Dynamical systems" by Professor G. D. Birkhoff, and "Topics from the theory of functions of infinitely many variables" by Professor F. R. Moulton.

At the meeting of the American Academy of Arts and Sciences on October 13 the following communications were delivered by members of the Association: "Einstein's first theory of relativity" by C. L. E. Moore; "Einstein's generalized relativity or theory of gravitation" by G. D. Birkhoff.

The so-called International Congress of Mathematicians [*1920*, 191] was held at Strasbourg, September 22–30, 1920. The first and fifth days, and the last two days, were reserved for excursions. There were four general lectures, one by L. E. DICKSON, University of Chicago, on "Relations between the theory of numbers and other branches of mathematics"; the others by Larmor, De la Vallée-Poussin, Volterra, and Nörlund. The papers were read in each of four sections; among the speakers were eight American mathematicians. In Section I (Arithmetic, Algebra, Analysis) there were 33 papers of which the following were by American mathematicians: "Homogeneous polynomials

with a multiplication theorem" by L. E. Dickson; "On Stieltjes integrals and Volterra compositions" by P. J. Daniell, Rice Institute; "On certain iterative properties of bilinear operation" and "On the theory of sets of points in terms of continuous transformations" by N. Wiener, Massachusetts Institute of Technology; "Quelques remarques sur la multiplication complexe" by S. Lefschetz, University of Kansas; "On the location of the roots of a polynomial" by J. L. Walsh, Harvard University. In Section II (Geometry) the 13 papers presented included: "La géométrie des variables complexes" by J. S. Taylor, Massachusetts Institute of Technology; "Transformation des systèmes conjugués R" and "Transformation des surfaces applicables sur une quadrique" by L. P. Eisenhart, Princeton University; "Method of classifying all polygons having a given set of vertices" by F. H. Murray, Harvard University. In Section III (Mechanics, mathematical physics, applied mathematics) there were 23 papers; in Section IV (Philosophical, historical and pedagogical questions), 10 papers.

Only three Englishmen took part in the program: Larmor, Young, and Greenhill; Italians, Spaniards, and Scandinavians, as well as Germans and Austrians, were conspicuously absent. The total attendance was above 200. The International Mathematical Union voted to hold the International Mathematical Congress of 1924 in the United States.

The following 16 doctorates with mathematics as major subject were conferred by American universities in the academic year 1919–1920; the title of the dissertation is added in each case: E. M. Berry, Iowa, "Diffuse reflection"; J. D. Bond, Michigan, "Plane trigonometry in Richard Wallingford's Quadri partium de sinibus demonstratis"; J. Douglas, Columbia, "On certain two-point properties of general families of curves"; T. C. Fry, Wisconsin, "The use of divergent integrals in the solution of differential equations"; Gladys Gibbens, Chicago, "Comparison of different line-geometric representations for functions of a complex variable"; C. F. Green, Illinois, "On the summability and regions of summability of a general class of series of the form $\Sigma c_n g(x + n)$"; J. W. Lasley, Chicago, "Some special cases of the flecnode transformation of ruled surfaces"; Elsie J. McFarland, California, "On a special quartic curve"; J. J. Nassau, Syracuse, "Some theorems in alternates"; C. A. Nelson, Chicago, "Conjugate systems with conjugate axis curves"; E. L. Post, Columbia, "Introduction to a general theory of elementary propositions"; Susan M. Rambo, Michigan, "The point at infinity as a regular point of certain difference equations of the second order"; L. L. Steimley, Illinois, "On a general class of series of the form $Y(x) = C_0 + \Sigma C_n g(nx)$"; J. L. Walsh, Harvard, "On the location of the roots of the jacobian of two binary forms"; R. Woods, Illinois, "The elliptic modular functions associated with the elliptic norm curve E^7"; T. Yang, Syracuse, "A problem in differential geometry."

In this Monthly, *1920*, 341, a general statement was made concerning the inferiority of the doctorat de l'université, in contrast to the state doctorate, as awarded in France to mathematicians. Professor Harris Hancock has kindly

sent us a copy of a letter, written in July, 1900 by Professor Gaston Darboux, dean of the faculty of sciences, from which it appears that at the University of Paris an exception should be made to the general statement. The "doctorat de l'université de Paris" is considered by that faculty of sciences as on a par with the state doctorate; Darboux wrote: "les épreuves sont les mêmes, la sévérité est la même pour les deux doctorats."

There are several vacancies for the position of computer at the Ballistic Station of the U. S. Ordnance Department, situated in the Coca Cola Bldg., Baltimore, Md. It is hoped to fill these with recent college graduates, either young men or young women who have completed a course in calculus and would like experience in practical computational work for a year at least. Owing to the present urgency and the difficulties experienced, the initial salary offered has been usually $1400. Further particulars may be obtained by addressing the Ballistic Station.

The Work of the National Committee on Mathematical Requirements. September 15, 1920 [*1920*, 341–342].

The National Committee on Mathematical Requirements held a meeting at Lake Delavan, Wisconsin, on September 2, 3, and 4 at which a number of reports were discussed and adopted. A report on "The revision of college entrance requirements" received the greatest amount of discussion. It is hoped that this report may be released for publication late in October. It includes a general discussion of the present problems connected with college entrance requirements in mathematics. It is a report of an investigation, recently made by the National Committee, concerning the values of the various topics in elementary algebra as preparation for the elementary college courses in other subjects. It contains also a suggested revision of the definitions of entrance units in elementary algebra and plane geometry. In connection with the suggested requirements in plane geometry a list of fundamental propositions and constructions is attached. This list includes the propositions which may be assumed without proof or given informal treatment, a list of the fundamental theorems and constructions from which it is intended that questions on entrance examination papers other than originals be chosen, and a list of subsidiary theorems. It is proposed to prepare a mimeographed edition of this list of propositions and constructions at the earliest possible moment for the benefit of such teachers as may desire to make use of it in connection with their classes during the coming year. A copy will be sent to any person interested upon application to the Chairman of the Committee (J. W. Young, Hanover, New Hampshire).

A preliminary draft of a report on "Mathematics in experimental schools" was discussed at this meeting. Mr. Raleigh Schorling of the Committee has spent over a year collecting material for this report. It is hoped that it will be ready for publication early next spring. The report will be an extensive one and will describe in detail the work actually done in mathematics in experimental schools throughout the country.

Miss Vevia Blair of the Committee presented her report on the "Present status of disciplinary values in education." It is expected that this report also will be released for publication in October. It gives a critical review of the complete literature concerning the experimental work on the transfer of training as well as an evaluation of this literature terminating in the formulation of certain propositions concerning disciplinary values which appear justified by the experimental work. A particularly valuable feature of the report would seem to lie in the fact that a large majority of the most prominent psychologists in the country appear to be ready to subscribe to the propositions formulated.

Professor E. R. Hedrick presented a report which he prepared at the request of the National Committee on "The function concept in secondary school mathematics." This report also will be published in the near future and is intended ultimately to form a part of the final report of the Committee on the "Reorganization of the first courses in secondary school mathematics." (A preliminary report on this subject was published for the Committee by the U. S. Bureau of Education last February as Secondary School Circular No. 5.)

A preliminary report on "Junior high school mathematics" is in the press of the U. S. Bureau of Education and should be ready for distribution early in October. The National Committee desires the assistance of its cooperating organizations, which now number over 70, in the revision of this preliminary report. Comments, suggestions and criticisms should be sent to the Chairman of the Committee not later than January 1, 1921, in view of the fact that the Committee expects to take up the formulation of its final report on this subject immediately after this date.

A subcommittee under the chairmanship of Professor C. N. Moore is preparing a report on "Elective courses in mathematics in secondary schools." A committee under the chairmanship of Professor David Eugene Smith is preparing a report on "The standardization of terminology and symbolism" and Professor R. C. Archibald is preparing one on "The training of teachers." It is expected that two of these reports will be presented for the consideration of the National Committee in October.

The work of the National Committee and its recommendations were discussed in teachers classes at the summer sessions of colleges, universities and normal schools throughout the country. Addresses on the work of the Committee were given as follows: by Mr. Raleigh Schorling at Harvard University, by Professor E. R. Hedrick at the Universities of Texas and of Oklahoma, and by Mr. J. A. Foberg at the Universities of Iowa and Minnesota.

Present indications point to the fact that the work of the National Committee will have a prominent place on the programs of most teachers organizations throughout the country during the coming year. The National Committee stands ready as before to help in every possible way in the preparation of such programs and will be glad to furnish material for discussion.

It will also be pleased to furnish speakers for such meetings to the extent of its ability.

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EDITORIAL CORRESPONDENCE AND BOOKS FOR REVIEW should be addressed to the EDITOR-IN-CHIEF, R. C. ARCHIBALD, Brown University, Providence, R. I.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

Fifth Summer Meeting of the Association, Chicago, September 6, 1920;
Fifth Annual Meeting, Chicago, December 28–29, 1920

The following are dates of Section meetings of the Association in 1920, unless otherwise specified:

IOWA, Univ. of Iowa, Iowa City, May 1

KANSAS, State Agricultural College, Manhattan, April 3; Topeka, January 22, 1921.

KENTUCKY, Centre College, Danville, April 17

MARYLAND–DISTRICT OF COLUMBIA–VIRGINIA, Goucher College, Baltimore, Md., May 15; Annapolis, Md., December

MINNESOTA, St. Catherine's College, St. Paul, June 5

MISSOURI, Kansas City, November 12–13

OHIO, Ohio State Univ., Columbus, April 2

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ILLINOIS, Chicago, December 29

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P. 41, l. 22, *for* "Simson" *read* "Simpson."
P. 41, l. 29, *for* "DANIEL" *read* "DANIELL."
P. 45, ll. 13–14, *for* "Journal of the Mathematical Association of Japan for Secondary Schools" *read* "*Journal of the Mathematical Association of Japan for Secondary Education.*"
P. 46, l. 3, *for* "*Whitaker*" *read* "*Whittaker.*"
P. 72, ll. 25–26, *for* "H. PIOGGIO" *read* "H. T. H. PIAGGIO."
P. 80, l. 5, *for* "**2806**" *read* "**2806**A."
P. 92, l. 4 from bottom, *for* "THORNDYKE" *read* "THORNDIKE."
P. 192, l. 12, *for* "June 1" *read* "June 21."
P. 199, l. 7, *for* "meets" *read* "meet."
P. 221, l. 25, *for* "J. C. Camplain" *read* "J. C. Kamplain."
P. 221, l. 28, *for* "E. R. Brestich" *read* "E. R. Breslich."
P. 240, l. 24, *for* "E. T." *read* "E. J."
P. 241, l. 9 from bottom, *for* "Columbus" *read* "Calculus."
P. 281, ll. 7–8 from bottom, *for* "America, and one member each by the National" *read* "America and its representative on the National Research Council, and one member each selected by the National."
P. 286, l. 3, *for* "the catenary" *read* "the equation of the catenary."
P. 286, l. 5 from bottom, *for* "then" *read* "and."
P. 288, ll. 26–27, *for* "experimenters" *read* "experiments."
P. 321, l. 6 from bottom, *for* "Burman" *read* "Burnam."
P. 336, l. 10, *for* "associate professor of mathematics" *read* "associate in mathematics."
P. 336, l. 13, *for* "J. P. MUSSELMAN" *read* "J. R. MUSSELMAN."
P. 338, l. 18, *for* "Stackel" *read* "Stäckel."
P. 339, *for* ll. 6–7 *read* "l'Avancement des Sciences. He wrote: *La Mathématique. Philosophie. Enseignement* (Paris, 1898: second edition revised, 1907) and *Initiation Mathématique. Ouvrage étranger à tout programme, dedié aux amis de l'enfance* (Paris, 1906) which was translated into."
P. 339, l. 21, *for* "variants" *read* "invariants."
P. 340, l. 5 from bottom, *for* "1920" *read* "**1920**."
P. 341, l. 11, *for* "1903" *read* "1901."
P. 346, ll. 6–7, *for* "Julia Calpitts" *read* "Julia T. Colpitts."
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P. 385, l. 6 from bottom, *for* "S. H. MACDONALD" *read* "S. L. MACDONALD."
P. 389, l. 5 from bottom, *for* "PAIGE" *read* "PAGE."
P. 428, l. 10 from bottom, *for* "frustrum" *read* "frustum."
P. 481, l. 13, *for* "Carolyn Bonds" *read* "Carolyn A. Boudo."

"Thou hast made me known to friends whom I knew not. Thou hast given me seats in homes not my own. Thou hast brought the distant near and made me a brother of the stranger."

THE AMERICAN MATHEMATICAL MONTHLY

OFFICIAL JOURNAL OF
THE MATHEMATICAL ASSOCIATION OF AMERICA
(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

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BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES
IN THE MIDDLE WEST

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RETROSPECT AND PROSPECT FOR MATHEMATICS IN AMERICA.

RETIRING PRESIDENTIAL ADDRESS DELIVERED BEFORE THE MATHEMATICAL ASSOCIATION OF AMERICA, SEPTEMBER 66, 1920.

By H. E. SLAUGHT, University of Chicago.

On several occasions since January, 1913, when twelve men, representing as many universities and colleges of the Middle West, took over THE AMERICAN MATHEMATICAL MONTHLY and dedicated it to the interests of collegiate mathematics, editorial comment in this journal has naturally taken the form of gratitude for past successes and optimism for future achievement. One of those occasions[1] was at the end of the first year of this critical undertaking when the subscription list had been doubled and it was becoming apparent that the self-imposed duties and responsibilities of that first editorial board, in their effort to render genuine service to the teachers of collegiate mathematics, were eliciting widespread approbation. Another such occasion[2] was at the end of three years, when the subscription list had again doubled, the subvention being no longer needed had been released, and over one thousand charter members had joined the newly organized MATHEMATICAL ASSOCIATION OF AMERICA with the MONTHLY as its official journal.

It seems highly appropriate now, near the end of the eighth year, that my remarks on the present occasion should again fall into the form of "Retrospect and Prospect," not alone directly with respect to the Association and the MONTHLY, but more especially with respect to recent developments in mathematical activity as a whole in America and the rôle which the Association and the MONTHLY may reasonably be expected to play in future developments.

First, let us recall and emphasize the fundamental things for which the Association and the MONTHLY stand. By the terms of its constitution the object of the Association is "to assist in promoting the interests of mathematics in America, especially in the collegiate field." This is, indeed, a broad charter, but it is subject to one obvious limitation inherent in the very circumstances under which the Association was conceived and organized, namely, that there should be no conflict in ideals between the Association and the American Mathematical Society, whose object, as stated by its constitution, is "to encourage and maintain an active interest in mathematical science." Just as the Society, by the resolution of its Council[3] in April 1915, decided that it was unwise to enter into the activities then represented by THE AMERICAN MATHEMATICAL MONTHLY, but expressed its realization of the importance of work in that field and its

[1] "Retrospect and prospect" by H. E. Slaught, AMERICAN MATHEMATICAL MONTHLY, *1913*, 1–3.

[2] "A tentative platform of the Association" by E. R. Hedrick, AMERICAN MATHEMATICAL MONTHLY, *1916*, 31–33.

[3] *Bulletin of the American Mathematical Society*, Vol. 21, page 482.

value to mathematical science, and pledged its hearty good will and encouragement to a new organization, if such should be formed to deal specifically with this work; so the Association, on its part, concerns itself more largely with what someone has called the humanitarian aspects of mathematics and leaves to the Society the field of pure research in the technical sense of that term.

However, the Association in two or three important particulars fulfills its purpose "to assist in promoting the interests of mathematics," even in connection with research, and, indeed, thereby actually pledges its good will to the Society. Namely, (1) it has during the past three years contributed to the support of one of the American research journals at a time when such support was of vital importance, and has agreed to continue such support at least for a fourth year; (2) it has been helpful in reawakening to activity in research some who had lost interest or become discouraged through isolation or lack of sympathetic touch with others, or lack of a medium of communication whose standards were not beyond their reach; (3) it has, through the MONTHLY, consistently and persistently fostered the *beginnings of research* by helping to supply some of the lower rungs which had been lacking in the research ladder. To quote from one of the MONTHLY editorials[1]: "It is by no means true that we are without interest in the higher, technical, mathematical field. On the contrary, we have an interest that is far more vital than the mere supplying of technical papers which can be read by specialists only. We believe that large numbers who would become active and effective in higher mathematical research are now lost to the cause simply by reason of the fact that there are no intermediate steps by which they can climb to these heights. We believe that the MONTHLY has a mission to perform in holding the interest of such persons by providing mathematical literature of a stimulating character that is within their range of comprehension, and by offering an appropriate medium for the publication of worthy papers which the more ambitious among them may produce." Notable examples of the far reaching effects of this sort of helpfulness can be cited.

One other fundamental position of the Association needs to be re-emphasized, namely, its relation to the *teaching* of mathematics. Many friends of the Association at the time of its organization feared that it might be in danger of degenerating into a pedagogical debating society, whose discussions might all evaporate into futility. The best guarantee at that time against such an outcome was the record already established by the MONTHLY in the preceding three years, a record which has been maintained by both the Association and the MONTHLY in the succeeding five years. The Association is, indeed, most vitally concerned with the teaching of mathematics, especially in the collegiate field, but it has very little confidence in the efficacy of mere pedagogical devices, however clever these may be, to produce and maintain high standards of teaching.

As for pedagogy in a scientific sense, its discussions belong to the realm of psychology and the research departments of education. To quote from the

[1] AMERICAN MATHEMATICAL MONTHLY, *1914*, 2.

foreword to the MONTHLY[1] in 1913 and the tentative platform[2] of the Association in 1916: "It is not the province of the MONTHLY to enter the field of general pedagogy, nor will the MONTHLY entertain discussions that are concerned with research in general pedagogy, or that deal with new theories of pedagogy, however important these contributions may seem. . . . What we desire is to inspire, not a discussion along these lines, but rather a discussion of definite mathematical problems. . . . Just as research will be held to be within the province of the Association only if the word is given a broad interpretation, it may be said also that the discussions which this Association will foster may be termed pedagogical only if the word is used in a much broader sense than is common."

To sum up the matter now, after five years of experience, the activities of the Association bearing upon the teaching of mathematics include at least the following: (1) the presentation, discussion, and publication of papers calculated to stimulate general mathematical interest and professional esprit de corps, or of papers calculated to stimulate thoughtful consideration of possible new courses or reconsideration of the content and form of old courses; (2) the investigations and reports of committees such as the Committee on Libraries which recommended minimum lists of books for reading in the various college years, the National Committee on Mathematical Requirements, which is now engaged in a complete reconsideration of the curriculum content from the elementary school to the college, and the Committee on Mathematical Dictionary which desires to see in the hands of every teacher and student of mathematics a source of information giving in clear, compact, and scientific form an explanatory definition of every mathematical term which is likely to be found in their reading or study.

It should be added in this connection that the foregoing remarks concerning technical pedagogical matters refer to the formal meetings of the Association and formal papers in the MONTHLY. The numerous sections of the Association provide sufficiently small units to give full opportunity for informal discussions of any character and to any extent that may be deemed desirable by the members concerned; and the department of Questions and Discussions in the MONTHLY provides an open forum for free discussion in brief form of any questions deemed by the editors to be of interest and value.

Turning now to the broader retrospect of American Mathematical activities, we find the field well classified by Professor Fiske in his presidential address in 1904, namely, including the colonial period, up to the founding of Johns Hopkins University in 1876, the period from 1876 to the nationalizing of the New York Mathematical Society in 1891, and the period from 1891 to 1904. Our retrospect need not extend back of this latter date, except that it will be convenient to consider here the entire twenty year period from 1900 to 1920. The mathematical development during this period may be gauged pretty accurately on the scientific side by the activities of the American Mathematical Society and by the rapid expansion of the American colleges and universities; and on the teaching side by the activities of the secondary associations and more recently of the Mathe-

[1] Vol. 20, *1913*, 4. [2] Vol. 21, *1916*, 33.

matical Association of America, and by the rapid expansion of the American high schools, both within their own original boundaries and also downward to include the junior high schools and upward to include the junior colleges.

During this twenty-year period the Society has held six of its eight colloquia[1] at which series of from four to six lectures each were delivered by Professors Oskar Bolza, E. W. Brown, E. B. Van Vleck, H. S. White, F. S. Woods, E. H. Moore, Max Mason, E. J. Wilczynski, G. A. Bliss, Edward Kasner, L. E. Dickson, W. F. Osgood, Oswald Veblen, and G. C. Evans. Several of these colloquia lectures were subsequently published in book form. The attendance upon these lectures has averaged 48, the highest being 69 at Cambridge in 1916.[2]

The Society has held during this period twenty summer meetings, with an average attendance of 48 and a maximum of 80 at Ann Arbor in 1919, at which 606 scientific papers were read; twenty February meetings and twenty October meetings in New York, with an average attendance of 34 at which a total of 471 papers were read; eighteen December meetings in the east, chiefly in New York, with an average attendance of 68 and a maximum of 131; and twenty December meetings in the middle West, chiefly in Chicago, with an average attendance of 43 and a maximum of 84, at which 514 and 407 papers respectively were read; nineteen April meetings in New York with an average attendance of 49 and a maximum of 82; twenty April meetings in Chicago with an average attendance of 41 and a maximum of 69, at which 422 and 442 papers respectively were read; thirty-five meetings of the San Francisco Section with an average attendance of 14 at which 299 papers were read; and twelve meetings of the Southwestern Section with an average attendance of 18, at which 212 papers were read. The total number of meetings held by the Society and its sections during this period is thus 184 and the number of scientific papers read is 3373. Of these papers 2155 had been published when the annual list for 1919 was made up.

While this conspectus of meetings of the Society manifestly gives only a quantitative measure of some of the scientific activity of American mathematicians, it nevertheless provides a fairly definite indication of progress in research work. The lists of papers read before the Society and subsequently published show an average annual output for the twenty years of 108 papers. These papers and numerous others read before other scientific bodies such as local and state academies of science, the National Academy of Science, national and international congresses, together with monographs and treatises published by educational foundations, by universities, or by commercial publishers, together constitute a scientific output in mathematics which not only makes a favorable showing for the past but also augurs well for the future.

Just as the attendance upon meetings and the output of scientific papers provide a quantitative measure of the contribution to mathematical progress made through the American Mathematical Society, so the number of doctorates

[1] Not including the Evanston Colloquium in 1893 at which Professor Felix Klein delivered a series of lectures.

[2] The ninth colloquium was since held at Chicago with an attendance of 88.

in mathematics, with their rate of increase and distribution, furnishes a certain quantitative measure of the contribution made by the universities. From 1898 to 1908 the average yearly number of doctorates in mathematics was about 12, while from 1908 to 1918 the average more than doubled. In 1900 only five universities granted the doctor's degree in mathematics, and to a total of eleven candidates. In 1916, thirteen universities granted the degree to a total of 35 candidates, the largest number in any year up to that time. While it is true that more than three-fourths of all the doctorates in mathematics from 1900 to 1920 have been given by eight universities, namely, Chicago 93, Johns Hopkins 39, Yale 38, Harvard 37, Cornell 31, Columbia 30, Pennsylvania 21, and Clark 16, a total of 305, nevertheless, twenty-nine different universities have contributed to the grand total of 406 mathematical doctorates during this period. It was approximately in 1910 that several other institutions began seriously to make their contributions to such doctorates. These are notably, Illinois with a total to date of 17, California with 15, Princeton with 15, and Michigan with 10. Other universities with more than one doctorate in this period are: Wisconsin 7, Syracuse 6, Indiana 5, Bryn Mawr 5, Virginia 4, Catholic University 3, Boston, Missouri, and Kansas, each 2.

We thus have had eight universities steadily and consistently throughout the twenty years contributing highly trained men and women to the mathematical faculties of our institutions, and four other universities doing likewise during the past ten years, while still others are showing signs of similar activity. The significance of these figures is not so much in the actual output, for this is small indeed compared to the needs, as in the portent for the future. Not one of the twelve universities just mentioned has reached anything like its limit of capacity for turning out trained men, while at least another dozen may be added to the productive list in the next decade, stimulated, as they are bound to be, by the cumulative influence of the steadily increasing body of trained mathematicians. What has been said with respect to mathematics is equally true of practically all other sciences, and in some cases with greater emphasis, so that the combined influence of the rapidly growing body of trained scientific men is certain to elevate at least a score of American universities to high rank in the production of research scholarship in the next twenty years.

Having deliberately laid primary emphasis upon the development of research scholarship in this country during the past twenty years, let us now briefly review the activities of this period with respect to the improvement of mathematical teaching. It is not too much to say that the presidential address of E. H. Moore in December, 1902, started a train of thought and action whose far-reaching influence has extended through all these years to the present time. Its first immediate effect was the expansion of the Central Association of Physics Teachers, then just organized in Chicago, into the Central Association of Science and Mathematics Teachers, an organization which has always been and is now at the very front in the promulgation of progressive ideas on the teaching of secondary mathematics. Similar organizations at once followed in 1903 in New

England and in the Middle States and Maryland, and a rapid succession of state and local organizations ensued, especially in the Middle West, and reaching to the Rocky Mountains and the Pacific Coast. There seemed to be a spontaneous awakening and a universal desire to reconsider the very fundamentals of procedure in the teaching of mathematics. There followed at varying periods committee investigations[1] and reports on geometry, on algebra, on combined mathematics, on mathematics for students of engineering, and on all phases of facts concerning the teaching of mathematics in this country gathered and summarized by the fourteen American committees under the International Commission on the Teaching of Mathematics.

All of these activities are in a sense now centered in the National Committee on Mathematical Requirements which was organized by the Mathematical Association of America as one of its earliest official actions, and which later became still more truly representative of national interests when it was enlarged to include members from the three great secondary associations of mathematics teachers and from the southwestern and Pacific coast sections. This Committee is unique in several respects. It is adequately financed by the General Education Board which, after careful investigation, considered the work of the Committee of sufficient national importance to warrant a liberal appropriation of funds for its use last year and a largely increased appropriation for a second year. The committee is able, therefore, to hold periodic meetings with all members present; it can command the services of a chairman and vice-chairman who give their whole time and attention to the weighty questions with which it is concerned; it can keep in touch with all the organizations of mathematics teachers in the country, not merely by correspondence, but especially by personal contact in their meetings, thus gathering at first hand the results of cooperative activity from all sources. The scope of this committee's activity is unlimited. While it is at present engaged in consideration of the secondary field, it is bound eventually to reach down into the junior high school and elementary school curricula, and to reach upward into the junior college domain. Finally, this committee enjoys the confidence of the Bureau of Education at Washington, so that its reports are published as public documents.

Never before in connection with mathematics in America has there been a committee with such an organization and such an opportunity for effective action. Possibly never before has there been greater need of just such an organized body of friends of mathematics as this committee represents. The whole high-school curriculum has been called in question by would-be reformers who would seek to justify the retention of any subject in the curriculum only in so far as its direct usefuness in the practical affairs of life can be established. Mathematics has come in for its full share of criticism and by many has been weighed in the balance to their own satisfaction and found wanting. It is maintained by them that on the score of practical usefulness in the ordinary occupations of

[1] See an Editorial on "Incentives to mathematical activity," AMERICAN MATHEMATICAL MONTHLY, *1913*, 169–173.

life, no general brief for algebra and geometry can be defended; that on the score of training special faculties, such as memory and reasoning power, the acquirements in mathematics do not carry over in general to other fields; that on the score of general culture the claims of mathematics which have held it as a required course in the curriculum have been grossly exaggerated. Hence, they conclude that, except for those looking forward to technical courses or to teaching mathematics, algebra and geometry should be removed from the curriculum as required subjects.

It is not my purpose to discuss here the validity of any of these claims against mathematics—there has been much discussion pro and con in the public press. Suffice it to say that, while in some school systems action has been taken toward eliminating or greatly reducing the required work in mathematics, nevertheless, the public in general is not convinced of the wisdom of such action, the psychologists are not agreed among themselves as to their claims, and the conclusions drawn from systems of mental tests are far from convincing. It is doubtless true that the personal experiences of a large multitude of individuals belie the charges against mathematics and hence the thinking public stands unconvinced.

However, no one would claim that the content and method of presentation of our mathematical courses, whether in elementary school, high school, or college, are beyond improvement; in fact, probably the most merciless critics of content and method are the mathematicians themselves. In no other branch of the curriculum has there been so much thoughtful agitation for improvement. It is well that the plans for improvement are in the hands of experts in the mathematical field. This is the service which the Mathematical Association of America has rendered to the cause in establishing the National Committee on Mathematical Requirements. We have faith to believe that the present uncertain state in the elementary and secondary fields will be satisfactorily adjusted through the action and influence of this National Committee.

Meanwhile, the Association, through the MONTHLY, through its national meetings and the meetings of its ten sections, through the widening circle of undergraduate mathematical clubs, through the expanding influence of its members as they become more and more conscious of the opportunity for improvement in the collegiate field and more and more enthused with zeal for promoting the interests of mathematics both pedagogically and scientifically, will go forward vicariously fulfilling the prophetic vision which Professor Moore had in mind for the Society in 1902 when he said, in his presidential address: "Do you not feel with me that the American Mathematical Society, as the organic representative of the highest interests of mathematics in this country, should be directly related with the movement for reform? And, to this end, that the Society enlarging its membership by the introduction of a large body of the strongest teachers of mathematics in the secondary schools, should give continuous attention to the question of improvement of education in mathematics, in institutions of all grades?" Thus, what Professor Moore and many others had hoped would be the rôle of the Society, has, by mutual and satisfactory arrangements, fallen to the lot of the Association.

In conclusion, let me speak briefly of the serious economic conditions which confront the Association in common with the Society and most other scientific organizations. The great increase in the cost of printing and supplies will doubtless make necessary an advance in the annual dues of the Association by at least thirty-three and one third per cent. [an advance since authorized]. This seems a modest increase in comparison with most other kinds of expenses, but it will be a real burden for a large number of members in the colleges and universities where salary advances have not kept pace with the increased cost of living. Moreover, this increase will scarcely meet the added expense for printing without cutting down the number of pages in the MONTHLY and this at a time when the amount of high-grade material on hand for publication is much greater than ever before, and when plans were already developing to increase the number of issues to eleven and eventually to twelve by adding one or two numbers devoted mainly to expository and historical articles of an elementary character.

But even all this is by no means the most serious phase of the situation. It is clear that the time has come when we should no longer allow one or two members of the Association, merely as a labor of love, to perform the very arduous services which devolve upon them. This was doubtless unavoidable in the early years of our development before the full significance of the movement which the MONTHLY inaugurated and the Association perpetuated, was widely comprehended. But the sacrifices and whole-hearted devotion given to a cause by its pioneers should not be presumed upon as indicating the normal method of procedure when the cause has become well established and its supporters are large in number.

To be more explicit, the responsibilities carried by our Secretary-Treasurer and by our Editor-in-Chief are burdensome beyond the belief of anyone who has not had actual experience in such matters, and we have no justification as a body in allowing such time and strength-consuming service to be given without compensation. We would all declare with enthusiasm that such service is invaluable and that its influence is of far-reaching importance, but we should also realize that the time given to it—amazing in amount, if we only knew all the facts—is taken at the expense of necessary recreation or of professional growth and advancement or of both. All of us expect to render a reasonable amount of altruistic service, but few of us are in a position to afford to devote a very large fraction of our time to such service for more than a brief period during some stress of circumstances, such, for instance, as happened during the war. In a word, I am advocating that the secretaryship and the chief editorship in such organizations as the Association and the Society should be offices with at least a moderate stipend attached. This, of course, is impossible at present in the Association, and always will be impossible so long as the dues are kept within the present range of reasonable reach of the average member.

One thing, however, the Council has decided must be done, namely, adequate paid assistance must be provided for these offices and the funds therefor must be found. Some paid assistance has, of course, been provided, but quite inadequate, either for the Secretary's office or for the editorial office. The Council

has voted to pay \$100 per month, beginning at once, for expert editorial and clerical assistance in Professor Archibald's office. This will cancel all our reserve before 1922, unless we secure other income than that derived from the dues. This I believe can be done. A temporary subsidy fund can be raised by the voluntary contributions of members or of other friends of mathematics. When it becomes known that a definite service may be rendered to the cause of mathematics by such contributions, interested persons will come forward and offer assistance. Such contributions will tide over a present difficult situation, but what is needed is a permanent endowment fund. This also can be secured, but it will take more time. The reason that such endowments have not been made before is that the need and the opportunity have not been known. Steps have already been taken to incorporate the Association so that we may be legally authorized to handle such trust funds.[1] An endowment can be built up from many sources as the years advance. The Association will thus become able to pay for important service rendered without advancing dues beyond the reach of the average college teacher, and it will be able to magnify its work and influence in a multitude of ways, some of which have been described above, but more of which will develop as those in hand are worked out. Opportunities will be found for service not only in the current and routine affairs of the Association, but also in the greater undertakings such as the proposed Dictionary of Mathematical Terms, which is already definitely projected and awaiting funds, or a biographical dictionary or a history of mathematics, which may later become definite projects, any one of which would constitute an enduring monument to the person who should provide the funds to make such a project attainable. It seems not too much to hope that within a reasonable time the Association, perhaps in combination with the Society, should have a central office where its business and editorial affairs of whatever magnitude may be conducted efficiently and with just compensation to those who render such important service.

I have thus endeavored to describe some of the mathematical activities in America during the past twenty years and to show the relation of the Association to these activities during its brief history, first in regard to research scholarship in its broader sense and secondly, in regard to the teaching of mathematics in all its phases. I have tried to show that the Association has put into effective operation a really national investigation of the mathematical situation in the schools, conducted by experts in both teaching and scholarship whose findings and recommendations must inspire confidence and promote reforms wherever they may be needed. And, finally, I have tried to emphasize the pressing need of financial support for such an organization as the Association through subsidies and endowments which shall make possible the conduct of its great altruistic service without, on the one hand, unjustly overburdening the volunteer workers, and without, on the other hand, raising the annual dues beyond the reasonable reach of large numbers to whom the service of the Association is especially helpful. It is the privilege and duty of every member of the Association to make known this need and opportunity as widely as circumstances permit.

[1] The legal steps have since been completed.

THE GRAPHICAL SOLUTION OF SPHERICAL TRIANGLES.

BY HARRY C. BRADLEY, Massachusetts Institute of Technology.

Corresponding to every spherical triangle, there exists at the center of the sphere a trihedral angle, whose face angles are equal to the sides of the spherical triangle, and whose dihedral angles are equal to the angles of the triangle. Graphical solutions of the trihedral are readily obtained by descriptive geometry.

Gaspard Monge, the French genius who may be called the father of descriptive geometry, does not appear to have given us any solutions for the trihedral. At any rate, an edition of his descriptive geometry, dated 1820, the earliest edition to which I have had access, omits the subject of trihedrals entirely. The excellent work published by Prof. Albert E. Church of the U. S. Military Academy, West Point, in 1864, contains graphical solutions of the trihedral corresponding to all six cases which arise in the solution of spherical triangles, and is the earliest dated work which I have as yet discovered. A number of modern texts give these solutions. A list (far from complete) appears at the end of this article.[1]

[1] An early reference to the graphical solution of spherical triangles, similar to that of Professor Bradley's paper, is Chapter 17 ("Résolution des Triangles sphériques par la Regle et le Compas") in A. Cagnoli's *Traité de Trigonométrie rectiligne et sphérique . . .* traduit de l'Italien par M. Chompré. Paris, 1786; chapter 19 of the second edition, Paris 1808. (*Trigonometria piana e sferica.* Edizione seconda notabilmente ampliate. Bologna, 1804: "Risoluzioni de'triangoli sferici con la riga e col compasso," pp. 346–349). For similar and allied discussions the following sources may be consulted.

C. Gudermann, *Lehrbuch der niederen Sphärik,* Münster, 1836.

G. K. L. von Littrow, "Ueber Herrn M. Eble's graphische Methoden der Auflösung sphärischen Dreiecke mit besonderer Rücksicht auf sein neuestes 'Stundenzeiger' oder 'Horoskop' genanntes Instrument," *Sitzungsberichte der math.-naturwiss. Klasse d. k. Akademie der Wissen.,* Vienna, vol. 42, 1860, pp. 203–212.

F. C. Penrose, "Description of an improved diagram for the graphical solution of spherical triangles, applicable to the questions arising out of the spheroidal figure of the earth," *Monthly Notices of the Royal Astronomical Society,* vol. 37, May, 1877, pp. 403–409.

L. Janse, "Over het graphisch oplossen van bolvornigé driehoeken en van daarop gegronde zeevaart en sterrekundige vraagstukken," *Nieuw Archief voor Wiskunde,* vol. 11, 1884, pp. 1–27; vol. 12, 1886, pp. 113–148.

C. H. Smith, "A graphic method of solving spherical triangles," *Amer. Jl. Math.,* vol. 6, 1884, pp. 175–176.

C. Wiener, *Lehrbuch der darstellenden Geometrie,* vol. 1, Leipzig, 1884, pp. 104–113.

P. Braun, "Das Trigonometer von C. Braun," *Math. naturwissensch. Ber. aus Ungarn,* reprinted in W. Dyck, *Katalog mathematischer und mathematischphysikalischer Modelle, Apparate und Instrumente,* München, 1892, pp. 160–161.

M. D'Ocagne, (*a*) *Journal de l'Ecole Polytechnique,* second series, vol. 4, 1898, p. 224; (*b*) *Traité de Nomographie,* Paris, 1899, pp. 326–330.

S. Haller, "Beitrag zur Geschichte der konstruktiven Auflösung sphärischer Dreiecke durch stereographische Projektion," *Biblioteca Mathematica,* n.s., vol. 13, 1899, pp. 71–80.

F. N. Willson, *Theoretical and practical Graphics,* New York, 1902, pp. 206–210.

M. D'Ocagne, "Sur la résolution nomographique des triangles sphériques," (*a*) *Bull. Soc. Math. de France,* vol. 32, 1904, pp. 196–203; (*b*) *Comptes rendus de l'Académie des Sciences,* vol. 138, 1904, pp. 70–72.

G. Pesci, "Resolução nomographica do triangulo de Posição" [translated from the Italian into Portuguese by Radler de Aquino], *Revista Maritima Brazileira,* Nov.–Dec., 1907; Feb., 1908.

These graphical solutions, however, ordinarily appear merely as exercises in descriptive geometry. Without some modification or adaptation, they are not suited for general use as an aid in checking numerical computations of spherical triangles. Usually all parts of the triangle are taken as less than 90°. When several of the parts lie between 90° and 180°, the resulting construction often becomes very difficult, even with an expert knowledge of descriptive geometry, to execute and to interpret correctly. Yet, as a check to numerical computations, the graphical solutions are not without value. Especially is this true in those cases where ambiguity exists in the numerical solution. The graphical construction shows clearly one solution, two solutions, or none; and in the case of two solutions, the correspondence of the parts.

To test the accuracy which may be expected from a graphical solution, I drew a hundred or so figures of moderate size, say six or seven inches across. The angles were measured with a semi-circular protractor, five inches in diameter, and were laid out only to the nearest whole degree. Disregarding some extraordinary agreements, probably more or less accidental, an average accuracy of 1° or 2° was readily obtained. This is quite sufficient to detect any gross error of calculation.

Now, as an aid to checking numerical calculations, the fewer graphical solutions which can be made to serve, the better. Fortunately, direct graphical solutions of all possible cases of spherical triangles, with all possible combinations of acute and obtuse angles for given parts, are unnecessary, provided we are willing to combine a little trigonometry with our descriptive geometry. The trigonometry required is of the simplest sort, namely:

1. The principle of polar triangles, by which sides are replaced by the supplements of angles, and angles by the supplements of sides. For instance, the graphical solutions for three given sides are simple and direct, while those for three given angles are not. Hence, by applying the principle of polar triangles,

R. de Aquino, "Nomograms for deducing altitude and azimuth and for star identification and finding course and distance in great circle sailing," *U. S. Naval Institute Proceedings*, vol. 34, 1908, pp. 633–646. See also W. C. P. Muir, *Treatise on Navigation*, fourth edition, Annapolis, Md., 1918, Appendix D, pp. 773–777: "Solution of the astronomical triangle by nomography."

G. Pesci, "Cenni sulla risoluzione del triangolo di posizione senza calcoli trigonometrici," *Rivista Marittima*, Roma, vol. 42, Sept., 1909, pp. 317–328 + 1 table. [Discusses the "compasso trigonometrico" of G. F. Richer described by F. Callet in the supplement to Bézout's Spherical Trigonometry and Navigation, Paris, 1798].

G. Pesci, "Sulla risoluzioni dei triangoli sferici senza calcoli trigonometrici" *Supplemento al Periodico de Matematica*, Livorno, vol. 13, 1910, pp. 65–73.

C. Schoy, *Beiträge zur konstruktiven Lösung sphärischastronomischer Aufgaben*. Leipzig, 1910. 7 + 40 pp. + 8 plates.

G. Loria, *Vorlesungen über darstellende Geometrie*, translated by F. Schütte, part 2, Leipzig, 1913, pp. 3–15.

M. D'Ocagne, *Cours de Géométrie*, tome 2, Paris, 1918, pp. 312–314.

H. G. G., "A new graphic method in nautical astronomy," *Nature*, vol. 102, Oct. 24, 1918, pp. 155–6.

A. Hutchinson and H. B. Goodwin, "Graphic methods in astronomy," *Nature*, vol. 103, March 13, 20, 1919, pp. 25, 44. Graphic methods here employed are "recommended to crystallographers."

EDITOR.

both cases can be solved graphically by the construction for three given sides.

2. The principle of co-lunar triangles. If the given parts are of such magnitudes that the direct graphical solution is difficult of execution and interpretation, one, at least, of the three possible co-lunar triangles will give a simpler solution.

The problem which I set myself, then, was to find: 1. Which of the graphical solutions were simplest to construct and easiest to interpret; 2. The minimum number of such solutions necessary, after making any trigonometrical transformations in the data which seemed desirable. I submit the following constructions as my result. There is nothing especially new in them. Under the circumstances, there could not be. But one or two of them differ more or less in detail from anything I have ever happened to see published.

These constructions were all made by descriptive geometry. Those familiar with that subject will readily recognize the horizontal and vertical coördinate planes, the ground line, traces of oblique planes, and revolution about those traces. The constructions, however, may readily be made without knowledge of descriptive geometry, and to render them more generally useful will be described merely as constructions.[1] In each example, the angles of the given triangle are A, B, C, the opposite sides a, b, c. To make the examples more specific, actual numerical values (in degrees) for the parts of the triangle will be used throughout.

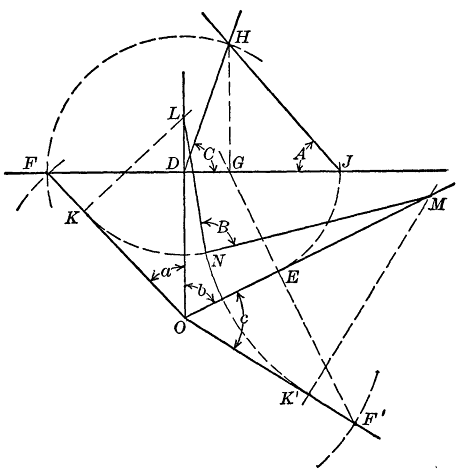

FIG. 1.

Case I. Given the three sides.

Example 1. All given sides acute. Given $a = 43°$, $b = 64°$, $c = 58°$.

This is solved in Fig. 1. Select any point O, and draw a vertical line OD. To the left of this line, lay off the smallest of the three given sides, in this case $a = 43°$. To the right of OD, lay off the largest side, $b = 64°$. Beyond this, lay off the remaining side, $c = 58°$. Draw a horizontal line intersecting the sides of a at F and D. Make $OF' = OF$. From F', draw $F'E$ perpendicular to OE, intersecting FD at G. At G, draw a vertical line. With D as center, radius DF, draw an arc intersecting this vertical at H. Draw HD. Then HDG is the angle $C = 70°$. With G as center, radius GE, draw an arc to intersect FD at J. Draw HJ. Then HJG is the angle $A = 49°$.

To find the remaining angle, opposite the middle side b, take any point K on OF. Draw from K a perpendicular to OF to meet OD at L. Make $OK' = OK$. From K' draw $K'M$ perpendicular to OF' to meet OE at M. With L as center,

[1] We have in this article a good illustration of the employment of the methods of descriptive geometry in pure mathematics as recommended very strongly by Professor Gino Loria in letters to Professor Roever published last year in this MONTHLY (*1919*, 45–53).—EDITOR.

radius LK, draw an arc. With M as center, radius MK', draw an arc, meeting the first arc at N. Then angle LNM is the angle $B = 85°$.

Answer. $A = 49°$, $B = 85°$, $C = 70°$.

Example 2. Two given sides acute, one obtuse. Given $a = 60°$, $b = 126°$, $c = 76°$.

The construction is shown in Fig. 2. It differs from Fig. 1 only in the size of the given parts. The figure is lettered the same as Fig. 1, and the same directions will apply.

Answer. $A = 35°$, $B = 148°$, $C = 40°$.

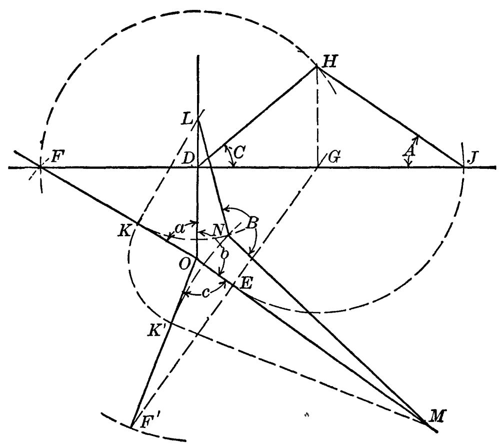

FIG. 2.

Example 3. Two given sides obtuse, one acute. Given $a = 60°$, $b = 100°$, $c = 140°$.

Form the co-lunar triangle $a = 60°$, $b' = 180° - b = 80°$, $c' = 180° - c = 40°$. Solve this by the construction of Fig. 1. This gives $A = 55°$, $B' = 112°$, $C' = 37°$. Whence, taking the co-lunar triangle of this result, we have the

Answer. $A = 55°$, $B = 68°$, $C = 143°$.

Example 4. All given sides obtuse. Given $a = 100°$, $b = 110°$, $c = 135°$.

Form the co-lunar triangle $a = 100°$, $b' = 70°$, $c' = 45°$. Solve this by the construction of Fig. 2, giving $A = 129°$, $B' = 48°$, $C' = 34°$. Whence,

Answer. $A = 129°$, $B = 132°$, $C = 146°$.

Example 5. An impossible case. Given $a = 30°$, $b = 80°$, $c = 40°$.

If we overlook the fact that $a + c < b$, the construction will bring us to our senses. Proceeding as in Fig. 1, we find point G, and draw a vertical line at this point. But with the parts as given, DG will be greater than DF. Hence, when we take D as center, DF as radius, and draw an arc, the arc will not intersect the vertical line at G. Therefore there is no solution.

Case II. Given the three angles.

As already stated, no direct constructions for this case are needed. Take the polar triangle corresponding to the given triangle, and solve by Case I.

Case III. Given two sides and the included angle.

Example 6. All given parts acute. Given $a = 43°$, $b = 64°$, $C = 70°$.

This is the same triangle solved from different given parts in Fig. 1, and the completed solution will be the same figure. To construct the figure from the above given parts, draw a vertical line OD. To the left of OD lay off the smaller of the given sides, $a = 43°$. To the right, lay off $b = 64°$. Draw the horizontal line FD. At D lay off the given angle C as shown. With D as center, radius

DF, draw an arc to intersect *DH* at *H*. Draw a vertical line *HG*, to meet *FD* produced at *G*. Draw *GE* perpendicular to *OM*. With *O* as center, radius *OF*, draw an arc to intersect *GE* produced at *F'*. Draw *OF'*. Then *EOF'* is the third side, $c = 58°$. The three sides now being known, the remaining angles, $A = 49°$ and $B = 85°$, are found as previously explained.

Answer. $c = 58°$, $A = 49°$, $B = 85°$.

Example 7. The two given sides acute, the given angle obtuse. Given $a = 61°$, $b = 53°$, $C = 135°$.

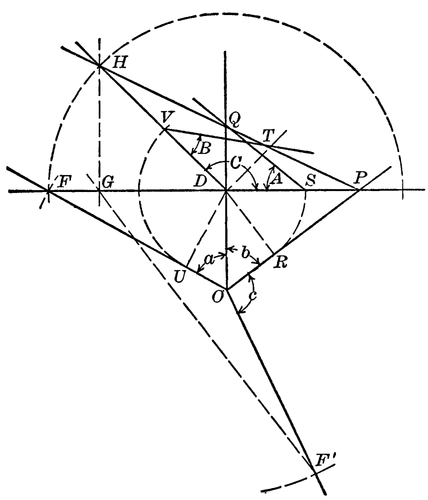

FIG. 3.

This example is solved in Fig. 3. Draw the vertical line *OD*. To the left of *OD* lay off the *larger* of the given sides, $a = 61°$. To the right of *OD* lay off the smaller given side, $b = 53°$. Draw a horizontal line *FD*, intersecting the sides of the angle *a* at *F* and *D*. Produce *FD* to intersect the remaining side of angle *b* at *P*; this can always be done, since the smaller side has been placed at the right of *OD*. At *D*, lay off the given angle $C = 135°$, as shown. With center *D*, radius *DF*, draw an arc intersecting *DH* at *H*. From *H* draw a vertical line to intersect *FD* at *G*. From *G* draw a line perpendicular to *OP*. With *O* as center, radius *OF*, draw an arc to intersect this line at *F'*. Then *POF'* is the third side, $c = 102°$.

Connect *H* with *P*, intersecting *OD* produced at *Q*. Draw *DR* perpendicular to *OP*. On *DP*, make distance $DS = DR$. Then *DSQ* is the angle opposite the side *DOF*, $A = 39°$.

From *D*, draw *DT* perpendicular to *DH*, intersecting *HP* at *T*. Also draw *DU* perpendicular to *OF*. Lay off on *DH* the distance $DV = DU$. Draw *VT*. Then *DVT* is the angle opposite the middle side, $B = 35°$.

Answer. $c = 102°$, $A = 39°$, $B = 35°$.

No other figures are needed for this Case. Any other conditions can be reduced to one of the two preceding solutions by the use of a co-lunar triangle. (See Examples 3 and 4.)

Case IV. Given two angles and the included side.

Take the polar triangle corresponding to the given triangle, and solve by Case III.

There are no impossible cases under Cases III and IV, and the construction can always be made.

Case V. Given two sides and the angle opposite to one of them.

Example 8. All three given parts acute. Given $a = 45°$, $b = 58°$, $A = 39°$.

The construction is shown in Fig. 4. Draw the vertical line *OD*. To the

left of OD lay off the side whose opposite angle is given, $a = 45°$. To the right of OD, lay off the other given side, $b = 58°$. Draw a horizontal line to give intersections F, D, and P. Draw DR perpendicular to OP. On DP, make $DS = DR$. At S, lay off the given angle A as shown, obtaining the intersection Q on OD produced. Draw PQ. With D as center, radius DF, draw an arc to intersect PQ. Since the side a is less than the side b, $DF < DP$, and the arc will intersect PQ in two points to the left of P, namely, H and H', each of which will give a solution to the problem.

For the first answer, from H draw a vertical line, intersecting DP at G. From G draw GE perpendicular to OP. With center O, radius OF, intersect this line at F'. Then EOF' is the third side, $c_1 = 18°$. Draw DH. Then GDH is the angle opposite this side, $C_1 = 16°$. The angle opposite the middle side, b, is found at N by the construction explained in Fig. 1; $B_1 = 131°$.

First Answer. $c_1 = 18°, C_1 = 16°, B_1 = 131°$.

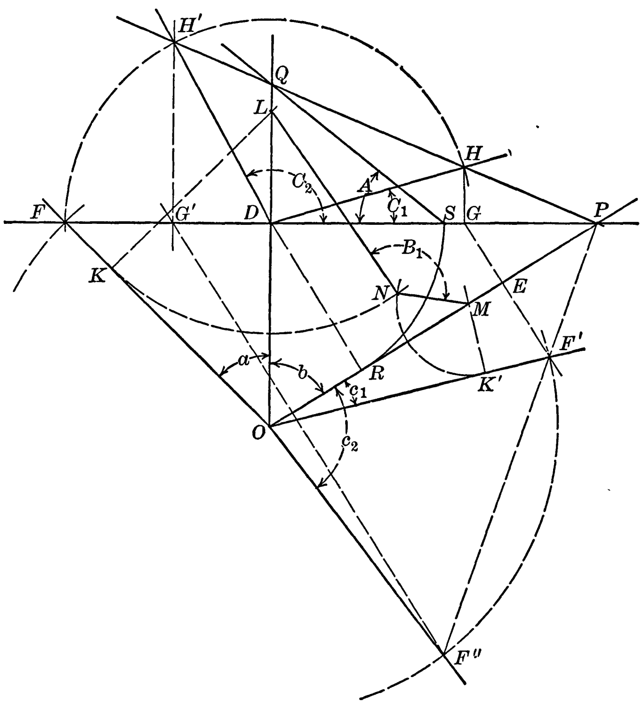

FIG. 4.

Proceeding similarly with the point H', we find EOF'' is $c_2 = 84°$, and GDH' is $C_2 = 118°$. No construction is needed for the remaining angle, B_2, since this angle is known by trigonometry to be the supplement of B_1. Hence $B_2 = 49°$.

Second Answer. $c_2 = 84°$, $C_2 = 118°$, $B_2 = 49°$.

Note. Should the intersection at F' prove rather flat, the point F' may be located more accurately from the fact that it lies on the line PF'', as shown in the figure.

Example 9. Given $a = 57°$, $b = 63°$, $A = 70°$.

If we proceed as explained for Fig. 4, the circle with D as center, radius DF, will be found to be tangent to PQ. There is then but one point H, and one solution. Going on with the construction from the point H, the third side is EOF', $c = 34°$. The angle opposite this side is GDH, $C = 38°$. The angle opposite the middle side needs no construction; by trigonometry, $B = 90°$.

Answer. $c = 34°$, $B = 90°$, $C = 38°$.

Example 10. Given $a = 38°$, $b = 59°$, $A = 53°$.

If we proceed as before, we shall find in this example that the radius DF is so short that the arc from F will not intersect PQ. There is consequently no solution.

Example 11. Given $a = 61°$, $b = 53°$, $A = 39°$.

This is the same triangle that was solved from different given parts in Fig. 3, and the completed construction will be the same figure.

The construction is started as explained for Fig. 4, Ex. 8, and proceeds as there explained until we draw the arc with D as center, radius DF. In this example, the side a, opposite the given angle A, is greater than the other given side b. Hence $DF < DP$, and the arc drawn from F will intersect the line PQ at but one point to the left of P. This intersection, H, gives the only solution to this example. Draw DH. Then PDH is the angle included between the given sides a and b, $C = 135°$.

Two sides and the included angle being known, the construction can now be completed as explained in case III. See Example 7.

Answer. $c = 102°$, $B = 35°$, $C = 135°$.

Example 12. Given $a = 63°$, $b = 58°$, $A = 75°$.

This is apparently the same case as that of the preceding example, since the side opposite the given angle is larger than the other. However, on attempting the construction, the given parts will be found to be of such sizes that the intersection H lies between P and Q. The angle included between the given sides is therefore acute, and the construction is best completed by the method used in Fig. 4 for the point H. See Example 8.

Answer. $c = 60°$, $B = 67°$, $C = 70°$.

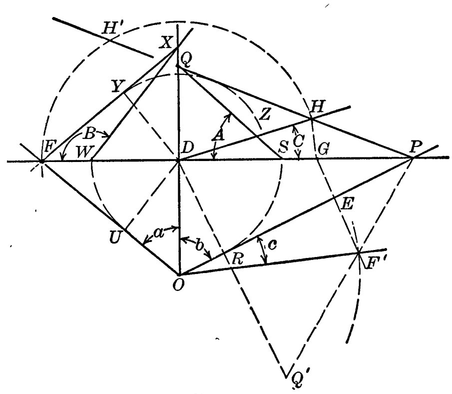

FIG. 5.

Example 13. The two given sides acute, the given angle obtuse. Given $a = 51°$, $b = 64°$, $B = 129°$.

The construction is shown in Fig. 5. Draw the vertical line OD. At the *right* of this line, lay off the side whose opposite angle is given, $b = 64°$. Lay off the other side, $a = 51°$, to the left of OD. Draw a horizontal line to give the intersections F, D, and P. From D draw DU perpendicular to OF. Make the distance $DW = DU$. At W lay off the given angle, $B = 129°$, as shown, giving the intersection X on OD produced. Draw FX. From D draw DY perpendicular to FX. With D as center, radius DY, draw the arc YZ. From P draw the line PQ tangent to this arc, intersecting OD produced at Q.

With D as center, radius DF, draw an arc to intersect PQ. The intersection H', nearer F, does not give a solution to the problem. A solution exists only if a second intersection, H, can be found on PQ to the left of P. It is evident that this

W. S. Hall, *Descriptive Geometry*, New York, 1903.
S. E. Warren, *Elements of Descriptive Geometry*, New York, 1905.
J. B. Millar, *Elements of Descriptive Geometry*, New York, 1905.

QUESTIONS AND DISCUSSIONS.

Edited by W. A. Hurwitz, Cornell University, Ithaca, N. Y.

REPLIES.

34 [*1917*, 134, 341; *1920*, 114, 301, 405]. Given the mixed integral and functional equation

$$\int_{x=0}^{x=h} f(x)dx = \frac{h}{6}\left[f(0) + 4f\left(\frac{h}{2}\right) + f(h)\right],$$

to determine the function $f(x)$. This equation is of rather fundamental practical value as it has to do with the most general solid whose volume is given by the prismatoid formula.

I. Remarks by A. A. Bennett, Baltimore, Md.

For the functional equation of Simpson's Rule, the origin is a point specially characterized. If smoothness at the origin be not required, solutions which are continuous and contain an infinite number of parameters may be obtained which present irregularities at this point. In particular the following one-parameter family of analytic functions with an essential singularity at the origin satisfy the equation:

$$x^a \sin (b \log x - c),$$

where a and b are constants satisfying certain transcendental equations and where c is an arbitrary parameter. This solution may furthermore be added to any other solution to yield a solution.

For the equation

$$\int_0^h f(x)dx = \frac{h}{6}\left[f(0) + 4f\left(\frac{h}{2}\right) + f(h)\right],$$

if f is differentiable,[1] and $f(0) = 0$, becomes, after differentiation

$$6f(x) = [4f(x/2) + f(x)] + x[2f'(x/2) + f'(x)].$$

Testing $f(x) \equiv x^a \sin (b \log x - c)$ we have

$$f'(x) = x^{a-1}[a \sin (b \log x - c) + b \cos (b \log x - c)].$$

Substituting throughout this gives

$$\begin{aligned} 6x^a \sin (b \log x - c) = (x^a/2^{a-2}) \sin (b \log x/2 - c) + x^a \sin (b \log x - c) \\ + (x^a/2^{a-2})[a \sin (b \log x/2 - c) + b \cos (b \log x/2 - c)] \\ + x^a[a \sin (b \log x - c) + b \cos (b \log x - c)]. \end{aligned}$$

Collecting terms, and canceling x^a, this yields

$$\begin{aligned} (5 - a) \sin (b \log x - c) - b \cos (b \log x - c) \\ = (1/2^{a-2})[(a + 1) \sin (b \log x/2 - c) + b \cos (b \log x/2 - c)]. \end{aligned}$$

Writing $b \log x/2 - c$ as $(b \log x - c) - b \log 2$ and expanding the terms in which this occurs, then equating coefficients of $\sin (b \log x - c)$ and of $\cos (b \log x - c)$, one obtains

$$\begin{cases} (5 - a)\cdot 2^{a-2} = (a + 1) \cos (b \log 2) + b \sin (b \log 2) \\ b\cdot 2^{a-2} = - b \cos (b \log 2) + (a + 1) \sin (b \log 2). \end{cases}$$

These equations are satisfied by an infinite set of values of a, b.[2]

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[1] The assumption of differentiability is not essential. Direct substitution of the function $x^a \sin (b \log x - c)$ in the given equation leads to a pair of conditions on a, b equivalent to those derived in the text.

[2] See the remarks by the Editor, immediately following the present article.

W. S. Hall, *Descriptive Geometry,* New York, 1903.
S. E. Warren, *Elements of Descriptive Geometry,* New York, 1905.
J. B. Millar, *Elements of Descriptive Geometry,* New York, 1905.

QUESTIONS AND DISCUSSIONS.

EDITED BY W. A. HURWITZ, Cornell University, Ithaca, N. Y.

REPLIES.

34 [*1917,* 134, 341; *1920,* 114, 301, 405]. Given the mixed integral and functional equation

$$\int_{x=0}^{x=h} f(x)dx = \frac{h}{6}\left[f(0) + 4f\left(\frac{h}{2}\right) + f(h)\right],$$

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I. REMARKS BY A. A. BENNETT, Baltimore, Md.

For the functional equation of Simpson's Rule, the origin is a point specially characterized. If smoothness at the origin be not required, solutions which are continuous and contain an infinite number of parameters may be obtained which present irregularities at this point. In particular the following one-parameter family of analytic functions with an essential singularity at the origin satisfy the equation:

$$x^a \sin (b \log x - c),$$

where a and b are constants satisfying certain transcendental equations and where c is an arbitrary parameter. This solution may furthermore be added to any other solution to yield a solution.

For the equation

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if f is differentiable,[1] and $f(0) = 0$, becomes, after differentiation

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Substituting throughout this gives

$$\begin{aligned} 6x^a \sin (b \log x - c) = {} & (x^a/2^{a-2}) \sin (b \log x/2 - c) + x^a \sin (b \log x - c) \\ & + (x^a/2^{a-2})[a \sin (b \log x/2 - c) + b \cos (b \log x/2 - c)] \\ & + x^a[a \sin (b \log x - c) + b \cos (b \log x - c)]. \end{aligned}$$

Collecting terms, and canceling x^a, this yields

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Writing $b \log x/2 - c$ as $(b \log x - c) - b \log 2$ and expanding the terms in which this occurs, then equating coefficients of $\sin (b \log x - c)$ and of $\cos (b \log x - c)$, one obtains

$$\begin{cases} (5 - a)\cdot 2^{a-2} = (a + 1) \cos (b \log 2) + b \sin (b \log 2) \\ \qquad b\cdot 2^{a-2} = - b \cos (b \log 2) + (a + 1) \sin (b \log 2). \end{cases}$$

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[1] The assumption of differentiability is not essential. Direct substitution of the function $x^a \sin (b \log x - c)$ in the given equation leads to a pair of conditions on a, b equivalent to those derived in the text.

[2] See the remarks by the Editor, immediately following the present article.

In order to exhibit the special rôle of the point $x = 0$, we shall obtain from the given equation another not involving this point; and then examine the nature of solutions of the new equation and under what conditions the old equation can be derived from the new. We assume first that $f(x)$ is continuous, and start with the given equation.

$$\int_0^x f(t)dt = \frac{x}{6}[f(0) + 4f(x/2) + f(x)]. \tag{1}$$

This is a special case of

$$\int_0^x f(t)dt = \frac{x}{6}[C + 4f(x/2) + f(x)]. \tag{2}$$

In (2), give x the constant value $a \neq 0$; thus

$$\int_0^a f(t)dt = \frac{a}{6}[C + 4f(a/2) + f(a)]; \tag{3}$$

and solve for C, thus

$$C = \frac{6}{a}\int_0^a f(t)dt - 4f(a/2) - f(a). \tag{4}$$

Substitute in (2),

$$\int_0^x f(t)dt = \frac{x}{6}\left[\frac{6}{a}\int_0^a f(t)dt - 4f(a/2) - f(a) + 4f(x/2) + f(x)\right]. \tag{5}$$

In (5), replace x by the constant $2a$,

$$\int_0^{2a} f(t)dt = \frac{a}{3}\left[\frac{6}{a}\int_0^a f(t)dt - 4f(a/2) + 3f(a) + f(2a)\right]. \tag{6}$$

Subtract (5),

$$\begin{aligned}\int_x^{2a} f(t)dt = \left(2 - \frac{x}{a}\right)\int_0^a f(t)dt - \tfrac{2}{3}(2a - x)f(a/2) + \tfrac{1}{6}(6a + x)f(a)\\ + \tfrac{1}{3}af(2a) - \tfrac{2}{3}xf(x/2) - \tfrac{1}{6}xf(x).\end{aligned} \tag{7}$$

From (6), subtract $\int_0^a f(t)dt$, giving

$$\int_a^{2a} f(t)dt = \int_0^a f(t)dt - \tfrac{4}{3}af(a/2) + af(a) + \tfrac{1}{3}af(2a) \tag{8}$$

or

$$\int_0^a f(t)dt = \int_a^{2a} f(t)dt + \tfrac{4}{3}af(a/2) - af(a) - \tfrac{1}{3}af(2a). \tag{9}$$

Substitute in (7), giving the equation

$$\begin{aligned}\int_x^{2a} f(t)dt = \left(2 - \frac{x}{a}\right)\int_a^{2a} f(t)dt + \tfrac{2}{3}(2a - x)f(a/2) - \tfrac{1}{6}(6a - 7x)f(a)\\ - \tfrac{1}{3}(a - x)f(2a) - \tfrac{2}{3}xf(x/2) - \tfrac{1}{6}xf(x).\end{aligned} \tag{10}$$

Solving for $4xf(x/2)$, we have

$$\begin{aligned}4xf(x/2) = -xf(x) - 6\int_x^{2a} f(t)dt + 6\left(2 - \frac{x}{a}\right)\int_a^{2a} f(t)dt\\ + (8a - 4x)f(a/2) - (6a - 7x)f(a) - (2a - 2x)f(2a).\end{aligned} \tag{1'}$$

For the new equation (1′) we have the following

THEOREM: *Given any positive number, a, and any real one-valued function continuous in the interval $I : a \leqq x \leqq 2a$, the end points being included, and further given an arbitrary value, A,—there exists a unique solution of* (1) *coinciding with the given function in I, taking on the value A at $a/2$ and continuous for all real positive values of x.*

It is at once obvious that the value of $f(1')$ in $I : a \leqq x \leqq 2a$ and its value at $a/2$ determine from (1′) its value in $a/2 \leqq x \leqq a$; by substitution of $x = 2a$ in (1′) it appears that $f(x)$ is continuous in both directions at $x = a$. By the methods of integral equations the value of $f(x)$ is determined by (1′) in $2a \leqq x \leqq 4a$. By repetition of this process the definition of the function can be extended repeatedly to intervals on the left, each half as long as the preceding, and to intervals on the right, each twice as long as the preceding. It need not be true, however, that the values so obtained on the left approach any limit at all as x approaches zero.

In order to return from (1′) to (1) we make the following assumptions:

(I) $f(x)$ is continuous, $x > 0$, and satisfies (1′) for some value of the constant a.

(II) $f(x)$ is bounded in a neighborhood of $x = 0$.

(III) If $\int_0^x f(t)dt$ can be written in the form $\frac{x}{6}\left[C + 4f\left(\frac{x}{2}\right) + f(x)\right]$, then $f(0)$ exists and $C = f(0)$.

The condition (II) may be replaced by the two weaker conditions:

(IIa) $\lim\limits_{x \to 0} xf(x) = 0$,

(IIb) $\int_0^x f(t)dt$ converges for $x > 0$.

Condition (III) may be replaced by

(IIIa) $f(0)$ is defined and $f(x)$ is continuous at $x = 0$. If (IIIa) is assumed, (II) is superfluous.

Starting with

$$4xf(x/2) = -xf(x) - 6\int_x^{2a} f(t)dt + 6\left(2 - \frac{x}{a}\right)\int_a^{2a} f(t)dt + (8a - 4x)f(a/2) - (6a - 7x)f(a) - (2a - 2x)f(2a) \qquad (1')$$

we have, upon letting x approach 0,—thus utilizing condition (II), or (IIa) and (II$_b$)—the following.

$$0 = -6\int_0^{2a} f(t)dt + 12\int_a^{2a} f(t)dt + 8af(a/2) - 6af(a) - 2af(2a),$$

whence we reobtain,

$$\int_0^a f(t)dt = \int_a^{2a} f(t)dt + \tfrac{4}{3}af(a/2) - af(a) - \tfrac{1}{3}af(2a). \qquad (9)$$

Eliminating $\int_a^{2a} f(t)dt$ between (9) and (1′) we have again

$$\int_x^{2a} f(t)dt = \left(2 - \frac{x}{a}\right)\int_0^a f(t)dt - \tfrac{2}{3}(2a - x)f(a/2) + \tfrac{1}{6}(6a + x)f(a) + \tfrac{1}{3}af(2a) - \tfrac{2}{3}xf(x/2) - \tfrac{1}{6}xf(x). \qquad (7)$$

In (7), letting x approach 0, we obtain

$$\int_0^{2a} f(t)dt = \frac{a}{3}\left[\frac{6}{a}\int_0^a f(t)dt - 4f(a/2) + 3f(a) + f(2a)\right]. \qquad (6)$$

Subtracting (7) from (6), we have,

$$\int_0^x f(t)dt = \frac{x}{6}\left[\frac{6}{a}\int_0^a f(t)dt - 4f(a/2) - f(a) + 4f(x/2) + f(x)\right], \qquad (5)$$

which is of the form

$$\int_0^x f(t)dt = \frac{x}{6}[C + 4f(x/2) + f(x)]. \qquad (2)$$

By employing condition (III), we return to

$$\int_0^x f(t)dt = \frac{x}{6}[f(0) + 4f(x/2) + f(x)], \qquad (1)$$

as desired.

Thus solutions of (1′) which satisfy (I), (II), (III) or (I), (IIa), (IIb), (III) or (I), (IIIa) will be solutions of (1). Solutions of (1′) include, however, functions not satisfying these further conditions bearing on the point $x = 0$ and therefore not serving as solutions of the original equation.

II. Remarks by the Editor.

It will be of value to indicate the relationship of the new solutions found by Professor Bennett to the work which has previously appeared in connection with this question. A convenient method of approach is found in the fact, used in earlier replies, that $f(x) = x^n$ is a solution of the

vanishes for $u \neq 0$. Thus (7) has at most one positive solution, and similarly at most one negative solution.

Finally let $\alpha = 0$. Equations (4), (3) reduce to

$$3 \sin \beta l = \beta(1 + \cos \beta l),$$

$$\beta \sin \beta l = 3(1 - \cos \beta l);$$

and this pair of equations is equivalent to the single equation

$$3 \sin \tfrac{1}{2}\beta l = \beta \cos \tfrac{1}{2}\beta l.$$

Since it is clear that no value which makes $\cos \frac{1}{2}\beta l$ vanish can be a solution, the only admissible values of β are those for which

$$\tan \tfrac{1}{2}\beta l = \tfrac{1}{3}\beta, \tag{8}$$

which may be written

$$\tan u - ku = 0, \tag{9}$$

where

$$u = \tfrac{1}{2}\beta l, \qquad k = \frac{2}{3l}.$$

This is a type of equation which arises in many problems of analysis. Inspection of the graphs of $\tan u$ and ku leads at once to the well-known fact, which can readily be proved by rigorous analytic methods, that (8) has an infinite number of solutions, approximating more and more closely, as they increase numerically, to odd multiples of $\pi/2$.

We condense all these results into the statement: The function x^n, where n is a constant whose real part is positive, satisfies the proposed functional equation for the following values of n and no others:

(*a*) The real values $n = 1, 2, 3$.

(*b*) An infinite set of values of the form $n = 2 + \beta i$, where β is a solution[1] of (8).

Any of the second set of solutions leads to real solutions of the type considered by Professor Bennett. In fact, since

$$x^n = x^{m+2} = x^{\alpha+2+\beta i} = x^{\alpha+2} \cos (\beta \log x) + i x^{\alpha+2} \sin (\beta \log x),$$

for any n yielding a complex solution we have the two real solutions $x^{\alpha+2} \cos (\beta \log x)$ and $x^{\alpha+2} \sin (\beta \log x)$, whence also the more general solution $x^{\alpha+2} \sin (\beta \log x - c)$, which is a linear combination of them. Equations (3), (4) are equivalent to Professor Bennett's pair of equations for a, b with $a = \alpha + 2$, $b = \beta$. We have thus shown that Professor Bennett's formula gives a solution only (except for the trivial cases $b = 0$, $a = 1, 2, 3$) when $a = 2$, and that in this instance an infinite number of values of b will be possible. It is obvious that much more general solutions can be constructed, of the form $x^2F(x)$, where

$$F(x) = \Sigma[A_p \cos (\beta_p \log x) + B_p \sin (\beta_p \log x)],$$

proper precautions being taken to insure the validity of the series.

The importance of these new solutions lies in the following facts. The function $x^2 \sin (\beta \log x - c)$ possesses a continuous first derivative even at the origin. In previous editorial comment, it was stated that "it seems reasonable to adopt as a goal with reference to this question the proof that if the equation holds for some range of the variable h, the function $F(x)$ can only be a polynomial of degree $\leqq 3$, under restrictions as light as possible—*e.g.*, that $F(x)$ should be continuous and possess a stated number (as small as it can be made) of derivatives." The statement has now been proved, with the number of derivatives as great as *six*, by Professor Gillespie. Professor Bennett's example, together with the present comments, shows that the theorem is not true with the number of derivatives as small as *one*. Can the gap between these two results be bridged?

DISCUSSIONS.

In a previous number of the MONTHLY (*1920*, 53) Mr. W. F. Cheney gave an instance of a geometric proof of the law of tangents in trigonometry. In the first discussion below Professor Lovitt shows how a number of such proofs may be obtained. The two papers contain references to a number of other similar proofs.

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[1] The smallest positive value of β is approximately 12.94.

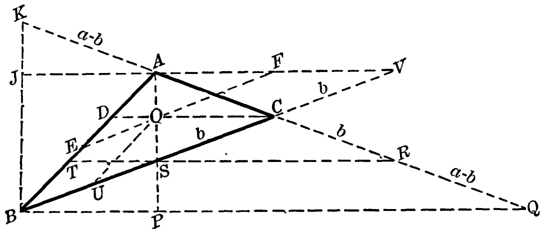
In the usual form of the law of the mean for derivatives, $f(x + h) - f(x) = hf'(x + \theta h)$, the quantity θ depends on both x and h. It is natural to inquire into the character of the functional dependence of θ on x and h. A general investigation of this dependence has been made by Hedrick, *Annals of Mathematics*, series 2, p. 177. Professor Downing in the second discussion undertakes the more special task of studying the nature of θ for simple forms of the function $f(x)$. He takes up the cases of linear, quadratic, and cubic functions.

I. Geometrical Proofs of the Law of Tangents.

By W. V. Lovitt, Colorado College.

The geometrical proof depends in many cases upon finding pairs of lines the ratio of whose lengths is $(a - b)/(a + b)$, where a and b are sides of a given triangle. We proceed then to find such pairs of lines.

Make



$$AC = SC = CV = CR = b;$$

make

$$CK = CQ = CB = a.$$

Then

$$AK = BS = RQ = a - b,$$

and

$$BV = RK = a + b.$$

Draw CD bisecting the angle ACB. The lines VJ, TR, CD, and BQ are parallel. Draw AP and $BK \perp BQ$. Draw $EOF \parallel BC$ and $OU \parallel AB$.

Then

$$BU = \tfrac{1}{2}(a - b), \qquad UC = \tfrac{1}{2}(a + b),$$

and

$$\frac{a-b}{a+b} = \frac{BS}{BV} = \frac{EO}{EF} = \frac{BU}{UC} = \frac{DO}{OC} = \frac{TS}{AV} = \frac{TS}{SR} = \frac{BP}{PQ} = \frac{JA}{JV} = \frac{BS}{AQ} = \frac{AK}{BV}. \quad (1)$$

We also find

$$\tfrac{1}{2}(A + B) = \angle OSC = \angle OAC = \angle JBV = \angle JKA,$$

$$\tfrac{1}{2}(A - B) = \angle BAP = \angle JBA.$$

In many geometrical proofs of the law of tangents the only points of difference are in the precise figure and method used in establishing some one or more of the equalities (1). If equalities (1) were a well known part of the course in elementary geometry, then the several different proofs of the law of tangents would be simpler.

We give below some proofs which are believed to be new. With one excep-

In the usual form of the law of the mean for derivatives, $f(x+h) - f(x) = hf'(x + \theta h)$, the quantity θ depends on both x and h. It is natural to inquire into the character of the functional dependence of θ on x and h. A general investigation of this dependence has been made by Hedrick, *Annals of Mathematics*, series 2, p. 177. Professor Downing in the second discussion undertakes the more special task of studying the nature of θ for simple forms of the function $f(x)$. He takes up the cases of linear, quadratic, and cubic functions.

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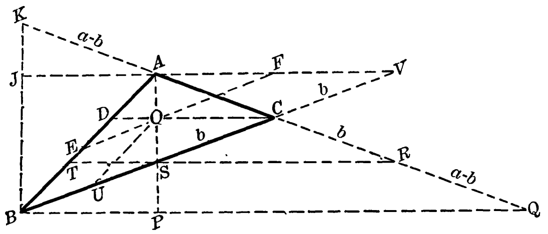

and

$$BV = RK = a + b.$$

Draw CD bisecting the angle ACB. The lines VJ, TR, CD, and BQ are parallel. Draw AP and $BK \perp BQ$. Draw $EOF \parallel BC$ and $OU \parallel AB$.

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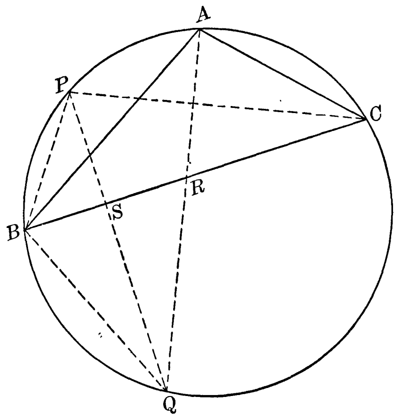
$$\tan \tfrac{1}{2}(A - B) = \frac{AJ}{BJ}; \qquad \tan \tfrac{1}{2}(A + B) = \frac{VJ}{BJ};$$

$$\frac{\tan \frac{1}{2}(A - B)}{\tan \frac{1}{2}(A + B)} = \frac{AJ}{VJ} = \frac{AK}{BV} = \frac{a - b}{a + b};$$

or

$$\tan \tfrac{1}{2}(A - B) = \frac{AJ}{BJ} = \frac{AK \sin \frac{1}{2}(A + B)}{BV \cos \frac{1}{2}(A + B)} = \frac{a - b}{a + b} \tan \tfrac{1}{2}(A + B),$$

(6) Circumscribe a circle about ABC. Draw CP bisecting the angle C. Make $CR = CA = b$. Then $BR = a - b$.



$$\angle\, CPS = \angle\, CAR = \tfrac{1}{2}(A + B),$$

$$\angle\, BPS = \angle\, BAR = \tfrac{1}{2}(A - B),$$

$$\angle\, BQS = \angle\, RQS = \angle\, BCP = \frac{C}{2},$$

or arc BP = arc PA. Hence

$$\angle\, RSQ = 90°$$

and

$$BS = SR = \tfrac{1}{2}(a - b),$$

$$CS = \tfrac{1}{2}(a + b).$$

Then we have

$$\tan \tfrac{1}{2}(A - B) = \frac{BS}{PS}; \qquad \tan \tfrac{1}{2}(A + B) = \frac{SC}{PS};$$

$$\frac{\tan \frac{1}{2}(A - B)}{\tan \frac{1}{2}(A + B)} = \frac{BS}{CS} = \frac{a - b}{a + b}.$$

In addition to the proofs here given I find distinct proofs in R. E. Moritz, *Elements of Plane Trigonometry*, p. 131; Hall and Frink, *Trigonometry*, p. 54; F. Durell, *Plane Trigonometry*, p. 108; in W. F. Cheney, *1920*, 53–54; and in works by the following authors listed *1920*, 53: E. J. Wilczynski, Young and Morgan, and E. W. Hobson.[1]

II. Note on the Law of the Mean.

By H. H. Downing, University of Kentucky.

With certain well-known restrictions on $f(x)$ and its derivative, $f'(x)$, the law of the mean states that

$$\frac{f(x + h) - f(x)}{h} = f'(x + \theta h), \tag{1}$$

where θ is a positive proper fraction.

[1] Yet another proof is given by G. Lindborg in *Tidskrift för Elementär Matematik, Fysik och Kemi*, vol. 3, May, 1920, p. 201.—Editor.

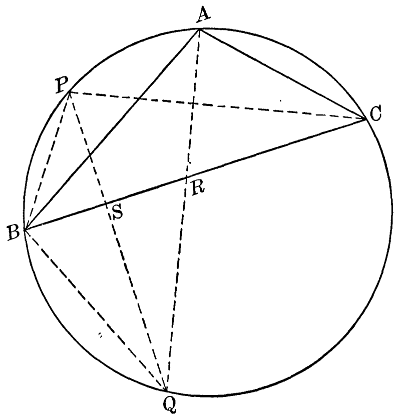
$$\tan \tfrac{1}{2}(A - B) = \frac{AJ}{BJ}; \qquad \tan \tfrac{1}{2}(A + B) = \frac{VJ}{BJ};$$

$$\frac{\tan \frac{1}{2}(A - B)}{\tan \frac{1}{2}(A + B)} = \frac{AJ}{VJ} = \frac{AK}{BV} = \frac{a-b}{a+b};$$

or

$$\tan \tfrac{1}{2}(A - B) = \frac{AJ}{BJ} = \frac{AK \sin \frac{1}{2}(A + B)}{BV \cos \frac{1}{2}(A + B)} = \frac{a-b}{a+b} \tan \tfrac{1}{2}(A + B),$$

(6) Circumscribe a circle about ABC. Draw CP bisecting the angle C. Make $CR = CA = b$. Then $BR = a - b$.



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$$CS = \tfrac{1}{2}(a + b).$$

Then we have

$$\tan \tfrac{1}{2}(A - B) = \frac{BS}{PS}; \qquad \tan \tfrac{1}{2}(A + B) = \frac{SC}{PS};$$

$$\frac{\tan \frac{1}{2}(A - B)}{\tan \frac{1}{2}(A + B)} = \frac{BS}{CS} = \frac{a-b}{a+b}.$$

In addition to the proofs here given I find distinct proofs in R. E. Moritz, *Elements of Plane Trigonometry*, p. 131; Hall and Frink, *Trigonometry*, p. 54; F. Durell, *Plane Trigonometry*, p. 108; in W. F. Cheney, *1920*, 53–54; and in works by the following authors listed *1920*, 53: E. J. Wilczynski, Young and Morgan, and E. W. Hobson.[1]

II. Note on the Law of the Mean.

By H. H. Downing, University of Kentucky.

With certain well-known restrictions on $f(x)$ and its derivative, $f'(x)$, the law of the mean states that

$$\frac{f(x + h) - f(x)}{h} = f'(x + \theta h), \tag{1}$$

where θ is a positive proper fraction.

[1] Yet another proof is given by G. Lindborg in *Tidskrift för Elementär Matematik, Fysik och Kemi*, vol. 3, May, 1920, p. 201.—Editor.

If x increases these points move towards the left along straight lines through the origin, and the hyperbolas corresponding to two values of x are homothetic with respect to the origin.

The interesting point about the equations is the nature of the coefficients of h.

It is further interesting to notice those points on the hyperbola which may be used when the conditions of the law of the mean are satisfied: that is, when θ is positive and less than one, or when ϕ is numerically less than h, and has the same sign. Also, notice may be made of these relations when the point on the cubic whose abscissa is x, is to the left of the inflection point, coincides with the inflection point, or is to the right of it.

RECENT PUBLICATIONS.

REVIEWS.

The Early Mathematical Manuscripts of Leibniz translated from the Latin texts published by Carl Immanuel Gerhardt with critical and historical notes. By J. M. CHILD, Chicago-London, The Open Court Publishing Co., 1920. Price $2.00.

American readers will be glad to possess in English translation the manuscripts of Leibniz containing his early work in the creation of the calculus that were preserved in the Royal Library in Hanover and were first published by C. I. Gerhardt in the years 1846–1855. We have here seventeen manuscripts bearing dates which fall within the time 1673–1677 and also two manuscripts of later date which give two accounts, one brief, the other more extended, of the origin of the calculus—both from the pen of Leibniz himself. Child breaks the chronological order by printing the last two documents first, thus laying stress on the parts which after all are really less important; for it is the papers of 1673–1677 that contain the cardinal points of the record. The book begins with the controversy on the priority of discovery which, by rights, should have been postponed to the end. Child translates also Gerhardt's essay on "Leibniz in London," which includes three manuscripts of Leibniz, and a second essay by Gerhardt, on "Leibniz and Pascal." Child adds profuse historical notes and arguments.

As a rule, the style is clear, but we mention a few exceptions. On page 8, line 23, a sentence is quoted, but it does not clearly appear which of three men mentioned is the author of it; only by inference from what follows later can the reader place the author with certainty. On the same page, line 26, Child quotes from "the memoir referred to," when as yet, no memoir has been mentioned; the authorship of the quotation at the bottom of page 8 is not revealed in the text.

In some instances Child is guilty of inconsistencies and errors. On page 9 he gives 1708, instead of 1704, as the date of the first complete publication of Newton's *Tractatus de quadratura curvarum.* On page 13, line 38, the date of the second edition of Barrow's *Lectiones Opticæ et Geometricæ* is given as 1874; of course, it should be 1674. On page 6 Child declares that Fatio concludes that

Leibniz, as the second discoverer of the calculus, "has borrowed from Newton," while on page 9 Child says, "Fatio does not dare to make a direct assertion, only an insinuation."

Some of the footnotes contain conjectures relating to historical events and the motives which prompted the movements of the actors. These conjectures are of value, provided the rapid reader does not lose sight of the fact that they are only conjectures. Indeed, some of Child's contentions are far from convincing. Take the matter of the "Characteristic triangle." Leibniz used it but does not claim it as his own. He says he got it from Pascal. Child argues that Leibniz must have gotten it from Barrow, on the ground that Leibniz obtained a copy of Barrow's *Lectiones Opticæ et Geometricæ* a month or so before he got a copy of Pascal, and that Leibniz's geometrical figure more closely resembles Barrow's than that of Pascal. Child may be correct in his view that for geometrical knowledge, Leibniz was indebted to Barrow more deeply than is ordinarily believed.

As regards the Newton-Leibniz controversy, Child agrees with A. De Morgan in characterizing the publication of the *Commercium epistolicum* by the Royal Society of London as an act of disgraceful unfairness; Child discards the hypothesis that Leibniz took his calculus from Newton, and says: (p. 5) "I hold quite other views as to the possible source of Leibniz's inspiration, if indeed he is not to be credited with perfectly independent discovery."

There are a few American reviewers who unconsciously have acquired the mannerism of belittling American contributions to the history of science and extolling foreign ones. While in the past, the present reviewer may have exhibited an exactly opposite tendency, he has no hesitation in this case to express the opinion that students of the history of mathematics will feel greatly indebted to Child for his translations and for his conscientious, fair-minded and thorough study of the documents involved.

FLORIAN CAJORI.

UNIVERSITY OF CALIFORNIA.

Infinitesimal Calculus. By F. S. CAREY. (Longmans Mathematical Series.) London, Longmans, 1919. 8vo. 20 + 352 + 9 pages. Price 14 shillings.

How far should one go in introducing new ideas at the beginning of a first course in the calculus? And how much time can one afford to spend in explaining those ideas? It is in answering these two questions that the text before us differs most radically from our standard texts. Most writers seem to agree that while in an introductory course the notions of variable, function and limit must be discussed, a very brief discussion is best, introducing as little notation and as few new terms as possible, with the thought presumably that those notions will become most rapidly and truly understood through use.

The attitude of the author is given in the preface. "Believing that there is no royal road which leads smoothly and directly to the Infinitesimal Calculus, the author has made no attempt to evade all the difficulties which at the outset face the student in this subject. The road has, however, been laid in the first

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I think, than to use Duhamel's theorem, and as logical. It should be added, however, that in the treatment of multiple integrals the author is not so successful, and is forced to disregard small quantities of higher order without explanation.

The American reader is impressed with the amount of geometry that is given. The chapters on curve tracing, envelopes, evolutes, roulettes, and graphics and nomography have much in them that is not included in our introductory texts. As there are no counterbalancing omissions, we would find it necessary to lengthen our course to include it all. The book furnishes an excellent reference work for this material. The last chapter, on "Graphics, nomography," gives in ten pages a good idea of the possibilities of the subject, and should, as the author states, "enable the reader to follow the complicated nomograms which are largely used in France and other countries."

The book as a whole is very carefully written, with surprisingly few errors. An occasional slip is however noted. For example, on page 49, we read that "if $f(x)$ is increasing as x passes through $x = a$, $f'(a)$ is positive," overlooking the possibility of $f'(a) = 0$. Other errors were noted, but are of no great importance.

E. J. MOULTON.

Periodic Orbits. By F. R. MOULTON in collaboration with DANIEL BUCHANAN, THOMAS BUCK, F. L. GRIFFIN, W. R. LONGLEY, W. D. MACMILLAN. (Publication no. 161) Carnegie Institution of Washington, 1920. 4to. 14 + 524 pp.

Quotation from the Introduction: "The problem of three bodies received a great impetus in 1878, when Hill published his celebrated researches upon the lunar theory. His investigations were carried out with practical objects in mind, and comparatively little attention was given to the underlying logic of the processes which he invented. For example, the legitimacy of the use of infinite determinants was assumed, the validity of the solution of infinite systems of non-linear equations was not questioned, and the conditions for the convergence of the infinite series which he used were stated to be quite unknown. These deficiencies in the logic of his work do not detract from the brilliancy and value of his ideas, and his skill in carrying them out excites only the highest admiration.

"The work of Hill was followed in the early nineties by the epoch-making researches of Poincaré, which were published in detail in his *Les Méthodes Nouvelles de la Mécanique Céleste.* Poincaré brought to bear on the problem all the resources of modern analysis. The new methods of treating the difficult problem of three bodies which he invented (*sic*) were so numerous and powerful as to be positively bewildering. They opened so many new fields that a generation will be required for their complete exploration. On the one hand, the results were in the direction of purely theoretical considerations, in which Birkhoff has recently made noteworthy extensions; on the other hand, they foreshadowed somewhat dimly methods which will doubtless be of great importance in practical applications in celestial mechanics. The researches of Poincaré are scarcely less revolutionary in character than were those of Newton when he discovered the law of gravitation and laid the foundations of celestial mechanics.

"In 1896 Sir George Darwin published an extensive paper on the problem of three bodies in *Acta Mathematica.* In mathematical spirit it was similar to the work of Hill; indeed, the methods used were essentially those of Hill, but the problem treated was considerably different. For a ratio of the finite masses of ten to one, Darwin undertook to discover by numerical processes all the periodic orbits of certain types and to follow their changes with varying values of the Jacobian constant of integration. This program was excellently carried out at the cost of a great amount of labor. It gave specific numerical results for many orbits in a particular example.

"The investigations contained in this volume were begun in 1900 and, with the exception of

the last chapter, they were completed by 1912. Those not made by myself were carried out by students who made their doctorates under my direction.

"The following chapters have been heretofore published in substance:"

I. Sections III and IV. *American Journal of Mathematics*, vol. 33 (1911).
II. *Astronomical Journal*, vol. 25 (1907).
III. *Rendiconti Matematico di Palermo*, vol. 32 (1911).
IV. *Transactions of the American Mathematical Society*, vol. 11 (1910).
VII. *Mathematische Annalen*, vol. 73 (1913).
VIII. *Annals of Mathematics*, second series, vol. 12 (1910).
XI–XIV. *Transactions of the American Mathematical Society*, vols. 7 (1906), 13 (1912), 8 (1907), and 9 (1908) respectively.
XV. *Proceedings of the London Mathematical Society*, series 2, vol. 2 (1912).

"The investigations and computations contained in the last chapter were completed in 1917. . . . Most of the computations on which many of the results of the last chapter are based were made by Dr. W. L. Hart and Dr. I. A. Barnett. Without assistance of such a high order the laborious computations could not have been carried out."

Contents: I: "Certain theorems on implicit functions and differential equations" (Moulton and Mac Millan), 1–54; II: "Elliptic motion," 55–66; III: "The spherical pendulum," 67–98; IV: "Periodic orbits about an oblate spheroid" (MacMillan), 99–150; V: "Oscillating satellites about the straight-line equilibrium points (first method)," 151–198; VI: "Oscillating satellites (second method)," 199–216; VII: "Oscillating satellites when the finite masses describe elliptic orbits," 217–284; VIII: "The straight-line solutions of the problem of n bodies," 285–298; IX: "Oscillating satellites near the Lagrangian equilateral triangle points" (Buck), 299–324; X: "Isosceles-triangle solutions of the problem of three bodies" (Buchanan), 325–356; XI: "Periodic orbits of infinitesimal satellites and inferior planets," 357–378; XII: "Periodic orbits of superior planets," 379–388; XIII: "A class of periodic orbits of a particle subject to the attraction of n spheres having prescribed motion" (Longley), 389–424; XIV: "Certain periodic orbits of k finite bodies revolving about a relatively large central mass" (Griffin), 425–456; XV: "Closed orbits of ejection and related periodic orbits," 457–484; XVI: "Synthesis of periodic orbits in the restricted problem of three bodies," 485–524.

Elementary Functions and Applications. By A. S. GALE and C. W. WATKEYS. New York, Henry Holt and Company, 1920. 12mo. 20 + 436 pages. Price \$2.60.

From the preface: "This book presents a coherent year's work in mathematics for college freshmen, consisting of a study of the elementary functions, algebraic and transcendental, and their applications to problems arising in various fields of knowledge. The treatment is confined to functions of one variable, with incidental exceptions, and complex values of the independent and dependent variables are excluded. The subject matter includes the essentials of plane trigonometry and topics from advanced algebra, analytic geometry, and calculus.

"The text is the result of experiments beginning in 1907–8. It has been used in the classroom since 1913–14, and each year extensive revisions have been made. Hence the content of the course, the order of topics, and the manner of presentation are based upon the experience of several years.

"The unity of the course is gained by an explicit analysis of the functions studied, which enables the student to comprehend the purpose of the course as a whole and the nature of the investigation of properties of functions of a given type. This analysis consists of three parts:

"First. Relations between a given function and its graph. . . . Most of these relations are considered in the first chapter so that at the start the student is made aware of a number of questions which will be investigated in studying a particular type of functions.

"Second. Relations between pairs of functions and their graphs. . . . These geometric transformations are introduced in connection with simple algebraic functions so that they are familiar tools by the time they are needed for the study of transcendental functions.

"Third. Analogous properties of functions which have no immediate graphical interpretation. Several properties of x^n are grouped together . . . in order to indicate further questions which should be investigated in studying transcendental functions.

"Emphasis is also placed on characteristic properties which distinguish one class of functions from another."

Contents—Chapter I: Functions, equations, and graphs, 1–45; II: Linear functions, 46–86; III: Algebraic functions, 87–157; IV: Trigonometric functions, 158–213; V: Exponential and logarithmic functions, 214–263; VI: Differentiation of algebraic functions, 264–300; VII: Integration, 301–331; VIII: Properties of trigonometric functions (Logarithmic solution of triangles, Cases III and IV), 332–364; IX: Theory of measurement [Statistical methods, permutations, combinations, the binomial expansion, probability, compound events, mortality tables, frequency distributions, averages, measures of variability, equation of the frequency curve representing a symmetrical distribution, the probable error, least squares, correlations], 365–432; Index, 433–436.

Análisis Matemático. Volumen 1: Las nociones fundamentales. By UGO BROGGI. (Publicaciones de la Facultad de Ciencias Fisicas, Matemáticas y Astronomicas, Textos y conferencias no. 44.) La Plata, Imprenta y casa editora 'Coni,' Buenos Aires, 1919. Royal 8vo. 152 p. Price 6 pesos.

Part 1: Coördinates and tangents, 1–46; part 2: The notions of derivative and integral 47–90; part 3: Applications (Elementary methods of integration, definite integrals of geometry indeterminate expressions, integrals between infinite and improper limits, exercises), 91–148; alphabetical index, 149–152.

NOTES.

The first article (pages 33–37) of the second number of the second volume (1920) of *Norsk Mathematisk Tidsskrift* is a paper on "The main lines of H. G. Zeuthen's scientific productivity," read by Professor Heegard before the Norwegian Mathematical Society. It is accompanied by the reproduction of a recent portrait of Zeuthen.

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Mathematics Teacher, in reorganized form, is to be the official organ of the National Council of Teachers of Mathematics (cf. *1920*, 241). Mr. J. R. CLARK of the Lincoln School, New York, is to be the editor-in-chief and the first number is to be issued in January, 1921. There are to be eight numbers a year and the subscription price is $2.00.

In *Archivio di Storia della Scienza*, volume 1, no. 3, published in May, 1920 (cf. *1919*, 404; *1920*, 217) Aldo Mieli continues his methodical bibliography (350 titles) of works on the history of science, 332–356. There is also a review by A. Mieli of two works published by the John Crerar Library, Chicago: *A List of Books on the History of Science, January, 1911* (Chicago, 1911); *Supplement, December, 1916* (Chicago, 1917), 361–363.

A new edition of Professor H. S. Carslaw's book *Introduction to the Theory of Fourier's Series and Integrals and the Mathematical Theory of the Conduction of Heat*, published in 1906 by Macmillan and Co., London, and now for some time out of print, is at present in the press. The book has been completely rewritten and will consist of two volumes. The first deals with "Infinite series and in-

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tegrals," with special reference to Fourier's series and integrals; the second with the "Conduction of heat." It is expected that the first volume will appear early next year and the second in the course of that year.

We have received the first fifteen numbers (May, 1919–July, 1920, 12 numbers to a volume) of the Spanish monthly periodical *Revista de Matematicas y Fisicas Elementales*, "designed for secondary, normal, special and higher schools," and published at Buenos Aires. Each number contains short articles (for example on "The indefinite integral," "An artifice in successive approximations," "Trial in classification of cubics," "Logarithmic calculation"), Notes, Miscellanea (a collection of statements of theorems and historical notes), Bibliography (brief reviews, contents of periodicals, titles of new books), Questions and discussions, and Problems proposed and solved—The publication at Buenos Aires of the bimonthly *Revista de Matematicas* ceased in February, 1918, with the concluding number of the second volume.

ARTICLES IN CURRENT PERIODICALS.

American Journal of Mathematics, volume 42, no. 2, April (published in July), 1920: "On the convergence of certain classes of series of functions" by R. D. Carmichael, 77–90; "On the solution of linear equations in infinitely many variables by successive approximations" by J. L. Walsh, 91–96; "Self-dual plane curves of the fourth order" by L. E. Wear, 97–118; "On the groups of isomorphisms of a system of Abelian groups of order p^m and type $(n, 1, 1, \cdots, 2)$" by L. C. Mathewson, 119–128; "On the satellite line of the cubic" by R. M. Winger, 129–135.

Bolletino di Bibliografia e Storia delle Scienze Matematiche, second series, volume 2, 1919, no. 1: [Authorized translation into Italian of L. C. Karpinski's article "The History of Science" in *School and Society*, December 21, 1918 [*1919*, 163]], 15–27.—No. 2: Review by G. L[oria] of D. E. Smith's edition of De Morgan's *Budget of Paradoxes* (Chicago, 1915), L. C. Karpinski's *Robert of Chester's Latin Translation of the Algebra of Al-Khowarizmi* (New York, 1915), and of R. C. Archibald's *Euclid's Book on Divisions of Figures* (Cambridge, 1915), 38–41 and 43–44; Review by G. Vivanti of M. Bôcher's *Leçons sur les méthodes de Sturm dans la théorie des équations différentielles* (Paris, 1917) and of H. Hancock's *Theory of Maxima and Minima* (Boston, 1917), 46–49; Brief notes on the following articles by scholars in America: "Pierre Laurent Wantzel," "Plans for a history of mathematics in the nineteenth century," and "Origin of the name 'mathematical induction'" by F. Cajori; "The rôle of the concept of infinity in the work of Lucretius" by C. J. Keyser; "The teaching of the history of science" by G. Sarton; and "Medicine and mathematics in the sixteenth century" by D. E. Smith, 58–64.

Bulletin of the American Mathematical Society, volume 26, no. 10, July, 1920: "The April meeting of the American Mathematical Society in New York" by F. N. Cole, 433–444; "Stieltjes derivatives" by P. J. Daniell, 444–448; "Sheffer's set of five postulates for Boolean algebras in terms of the operation 'rejection' made completely independent" by J. S. Taylor, 449–454; "Rotation surfaces of constant curvature in space of four dimensions" by C. L. E. Moore, 454–460.

Bulletin of the Calcutta Mathematical Society, volume 10, no. 4, March, 1920: "On the diffraction of light by a transparent wedge" by S. Banerji, 199–206; "On a geometrical treatment of the scattering of light by a perfectly reflecting cone" by A. Datta, 207–212; "A note on the deformation of surface" by B. Sen, 213–218; "On the stability of two co-axial rectilinear vortices of compressible fluid" by B. Datta, 219–228; "On some properties of natural numbers" by H. Datta, 229–238; "A note on Whittaker's formula for the solution of algebraic or transcendental equations" by B. Pal, 239–242; Review by A. C. Bose of C. E. Cullis's *Matrices and Determinoids* (2 vols., Cambridge, 1913–1918), 243–256.

Enseignement Mathématique, volume 21, no. 1, June 1920: "Comment un conservateur pourait-il arriver au seuil de la mécanique nouvelle?" by T. Levi-Civita, 5–28 (conference at the mathematical seminary of the University of Rome, March 8, 1919); "Remarques sur la théorie

des ensembles et les antinomies cantoriennes, II" by D. Miriamanoff, 29–52; Les deux suites Fibonacciennes fondamentales (u_n) (v_n); tables de leurs termes jusqu'à $n = 120$" by C. A. Laisant, 52–56; "C. A. Laisant (1841–1920)" by H. Fehr, 57; Mélanges, Chronique, 58–63; Review of D. E. Smith's *Number Stories of Long Ago* (Boston, 1919), 67–68.

HERMATHENA, no. 42, 1920: "Heron's formula for cube root" by J. G. Smyly, 64–67; "Some examples of Greek arithmetic" by J. G. Smyly, 105–114.

ISIS, volume 3, January, 1920: "The purpose of Zeno's arguments on motion" by F. Cajori, 7–20.

JOURNAL OF THE INSTITUTE OF ACTUARIES, volume 52, April, 1920: "An elementary demonstration of Stirling's approximate formula for the value of factorial n" by G. J. L., 102–106.

MATHEMATICS TEACHER, volume 12, no. 4, June, 1920: "Some helps and hindrances in teaching mathematics in the secondary schools" by L. E. Lynde, 139–153; "The effect of post-armistice conditions on mathematical courses and methods" by H. English, 154–166; "Teaching practical mathematics efficiently" by C. H. Sampson, 167–171. "New Books," 172–174.

NATURE, volume 105, June 10, 1920: "The approximate evaluation of definite integrals between finite limits" by A. F. Dufton, 455–456—June 17: "A new method for approximate evaluation of definite integrals between finite limits" by T. Y. Baker, 486—July 1: "Use of Sumner lines in navigation" by T. H. Tizard, 552–554; "Dr. F. A. Tarleton" by R. A. P. Rogers, 554; "British aëronautics," 561–562.

PHILOSOPHICAL MAGAZINE, sixth series, volume 40, July, 1920: "A new reading of relativity" by F. Slate, 31–49; "The bearing of rotation on relativity" by R. A. Sampson, 67–72; "On the measurement of time and other magnitudes" by N. R. Campbell, 161–162.

POPULAR ASTRONOMY, volume 28, no. 6, June–July, 1920: "The theory of relativity" by W. H. Pickering, 334–344,—No. 7, August–September: "John Alfred Brashear, 1840–1920" (portrait frontispiece) by F. Schlesinger, 373–379; "Aintoff's equal area projection of the sphere" by R. L. Faris, 385–386.

PROCEEDINGS OF THE AMERICAN ACADEMY OF ARTS AND SCIENCES, volume 55, no. 4, May, 1920: "Orbits from assumed laws of motion" by Arthur Searle, 189–207.—No. 7, June: "Some geometric investigations on the general problem of dynamics" by J. Lipka, 283–322.

PROCEEDINGS OF THE NATIONAL ACADEMY OF SCIENCES, volume 6, no. 5, May, 1920: "On Kummer's memoir of 1857 concerning Fermat's last theorem" by H. S. Vandiver, 266–269.

REVUE DU MOIS, 15th year, April 10, 1920: "Les mathématiques à l'Université de Strasbourg" by M. Fréchet, 337–362.

REVUE GÉNÉRALE DES SCIENCES, volume 31, June 15, 1920: "Un nouvel appareil pour le tracé des coniques," 329–330 (an account of the conigraph as described in the *Proc. Royal Society*, see this MONTHLY, *1920*, 317); "La géométrie differentielle projective des résaux" by G. Tzitzéica, 349–352 [". . . Cependant, c'est à M. Wilczynski que revient le mérite d'avoir créé une théorie projective, aussi complète que possible, des propriétés infinitésimales des courbes planes, des courbes gauches et des surfaces réglées. Il a employé à cet effet certains invariants et covariants des équations différentielles linéaires du troisiéme et du quatriéme ordre et des systèmes d'équations du second ordre. Il a appliqué aussi une méthode analogue à l'étude des surfaces et des congruences de droites de notre espace.

"Cette méthode a donné de beaux résultats. Toutefois les calculs, un peu longs et compliqués, laissent au second plan, les propriétés géométriques. Aussi, les études synthétiques de quelques géomètres italiens out rendu à cette branche de la géomètrie différentielle un aspect plus intuitif, en y ajoutant de plus la considération des figures dans un espace projectif à plus de trois dimensions."]

REVUE SCIENTIFIQUE, volume 58, June 12, 1920: "L'espace et le temps dans la physique moderne" by L. Bloch, 333–341.—June 26: "La théorie d'Einstein et la nouvelle loi de la gravitation" by J. Bosler, 353–359.

SCHOOL AND SOCIETY, volume 12, August 7, 1920: "An odd method for determining the year of birth" by G. A. Miller, 106–107 [De Morgan's conundrum, "I was x years old in the year x^2."]

SCIENCE PROGRESS, volume 15, July, 1920: "Mathematical philosophy" by Bertrand Russell, 101 [Letter dated April 20; "Sir,—The review of my *Introduction to Mathematical Philosophy* by the late P. E. B. Jourdain, in the April number of *Science Progress*, contains some statements which I cannot pass by in silence.

its middle point M is the centre of gravity of the A_λ, and if P is any point in space at a distance a from M, then $\sum_\lambda PA_\lambda = n(R^2 + a^2)$"]; "Muss die Sonne ihren höchsten Stand immer im Meridian erreichen"? by P. Kiesling, 34–35; "Die Kugel als regelmässiger Vielflächner" by F. Elsas, 36–37; "Zur Maclaurins Trisektrix, zur Kissoide und zur Versiera" by A. Peschke, 37; "Zur Lehre von den harmonischen Strahlen" by J. Mahrenholz, 37; "Ueber eine Beziehung zwischen Normale und Brennstrahlen in einem Punkte einer Ellipse" by J. Mahrenholz, 37–38; Various other brief notes, 38–40.

Zeitschrift für mathematischen und naturwissenschaftlichen Unterricht, volume 51, no. 4–5, published May 20, 1920: "Das d'Hondtsche Wahlsystem." "Eine Anwendung der graphischen Methoden" by E. Sós, 89–93; "Mathematische Betrachtungen über das geltende politische Wahlverfahren" by H. Franke, 94–106; "Arbeit und Boden in der Volkswirtschaftslehre und Mechanik" by J. Braun, 106–109; "Optische Geometrie" by R. Böger, 110–118; "Eine neue Verallgemeinerung des pythagoräischen Lehrsatzes" by A. Voigt, 118–122; "Das Ecktangentendreieck" by J. Mahrenholz, 123; "Der Saccheri-Legendresche Satz lautet: Ist in einem einzigen Dreieck die Winkelsumme $\lesseqqgtr 180°$, so ist sie es in jedem" by K. Fladt, 124–125 [a very simple proof for the Euclidean case, which, however, the author states he has not yet succeeded in extending to the non-Euclidean cases;] "Aufgabenrepertorium," 126–132; "Bücher, besprechungen," 140–141.

UNDERGRADUATE MATHEMATICS CLUBS.

Edited by U. G. Mitchell, University of Kansas, Lawrence.

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"'Comparing a Quaternion investigation, no matter in what department, with the equivalent Cartesian one, even when the latter has availed itself to the utmost of the improvements suggested by Higher Algebra, one can hardly

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help making the remark that they contrast even more strongly than the decimal notation with the binary scale, or with the old Greek arithmetic—or than the well-ordered subdivisions of the metrical system with the preposterous no-systems of Great Britain, a mere fragment of which (in the form of Table of Weights and Measures) form, perhaps the most effective, if not the most ingenious, of the many instruments of torture employed in our elementary teaching.'—P. G. TAIT. (Quoted by MORITZ, *Memorabilia Mathematica*, p. 280.)

'Quaternions came from Hamilton after his really good work had been done; and though beautifully ingenious have been an unmixed evil to those who have touched them in any way, including Clerk Maxwell.'—WILLIAM THOMSON. (Quoted by MORITZ, *Memorabilia Mathematica*, p. 279.)

So T and T′ disagree
On parts of their philosophy."

It may be possible that other clubs are keeping similarly complete records of their proceedings. If so, the editor would be glad to have the fact reported.

CLUB ACTIVITIES.

THE CORNELL PARABOLA, Cornell University, Ithaca, N. Y.

The Cornell Parabola was organized by the students of the upper classes in November, 1919. It was modelled along the lines of the Oliver Mathematical Club, which has existed among the faculty members for many years, and the organization was very informal. There is no constitution, there are no dues and there are only two officers—a chairman, popularly known as the "Focus" and a secretary, better known as the "Directrix." A program committee consisting of four students and one faculty member planned the work.

The attendance for all of the meetings averaged over fifteen and there were always spirited discussions and incidental problems at the close of each program.

The only social event of the year was the Christmas party which was held at the home of one of the professors. The chief game of the evening was a contest in supplying mathematical terms for the missing words in a "Romance of Polly Hedron and Ray Show." There was a Christmas tree loaded with candy boxes in all the regular polyhedron shapes.

The officers of the club were: Chairman, Theodore L. Bennett '21; secretary, Elfrida Heath '20; chairman program committee, Nellie G. Tallman '20; faculty adviser, Dr. Helen B. Owens.

Programs for the year 1919–20 are given below.

"Reliability of averages" by Herbert Sturges, Gr; "Properties of determinants" by Theodore L. Bennett '21; "The ancients' knowledge of mathematics" by Elfrida G. Heath '20; "Some applications of mathematics to physics" by Roy Kennedy '21; "College students' errors in mathematics" by Alan D. Campbell, Instructor in Mathematics; "Fallacious proofs in geometry" by Professor Frederick W. Owens; Parabola Christmas Party; "Possible constructions with straight edge and compass" by Professor Owens; "Topics from

mechanics and hydraulics" by Salvador Quinones '21; "Famous problems of mathematics" by L. Kalles '19; "Consequences of a triangular unit of area" by Professor Walter B. Carver; "Properties of the parabola" by Allen D. Campbell, Instructor in Mathematics; "The theory of the pendulum" by Joseph A. Beeker '19; "Research problem in electricity" by Ruth Yeaton, Gr; "Value of mathematics to the engineer" by Burdette K. Northrup '19; "Value of mathematics to the physicist and chemist" by Austin Bailey, Gr.

THE MATHEMATICAL CLUB, Harvard University, Cambridge, Mass.
[*1918*, 186–7, 449–50; *1919*, 262–3.]

The officers for 1919–20 were: President, Joseph L. Walsh, Gr.; secretary-treasurer, Harold T. Davis, Gr.; faculty adviser, Professor William C. Graustein.

Programs for 1919–20 are given below.

October 1, 1919: "The elementary functions" by Professor William F. Osgood.

October 15: "Problems connected with submarine acoustics" by Professor Oliver D. Kellogg.

October 29: "Elementary notions of general analysis" by Dr. Israel A. Barnett, Instructor.

November 12: "Correlation" by Professor Edward V. Huntington.

December 3: "Linkages" by Bancroft H. Brown, Instructor.

December 17: "Nomography" by Rexford S. Tucker, Gr.

January 7, 1920: "Numerical integration" by Charles A. Rupp, Instructor.

February 11: "The application of the method of least squares to some geometrical problems" by Professor Julian L. Coolidge.

February 25: "Operations generating algebra" by Dr. Norbert Wiener, Instructor in Mathematics, Massachusetts Institute of Technology.

March 10: "Approximation to continuous functions" by Lewis E. Ward, Gr.

March 24: "Rectifiable and other curves" by Professor Edward B. Van Vleck, Professor of Mathematics, University of Wisconsin.

April 7: "Polygons whose vertices lie on a convex curve" by Forrest H. Murray, Gr.

April 28: "Surfaces of negative curvature" by Dr. Harold C. M. Morse, Instructor.

May 12: "A solution of the biquadratic" by Professor William C. Graustein.

May 24: "The location of the roots of the derivative of a polynomial" by Joseph L. Walsh, Gr.

Officers-elect for the year 1920–21: President, Rudolph E. Langer, Gr.; secretary-treasurer, Bancroft H. Brown, Instructor; faculty adviser, Professor Oliver D. Kellogg.

THE MATHEMATICAL CLUB OF SMITH COLLEGE, Northampton, Mass.
[*1918*, 91, 455; *1920*, 184.]

Below are given the programs for the year 1919–20.

October 20, 1919: Discussion of plans for the year. Reports of the September

meetings of the American Mathematical Society and the Mathematical Association of America.

November 10: "Map drawing, with special reference to the use of maps in the Great War" by Professor Suzan R. Benedict.

December 1: "Leibnitz" by Ellen Hastings '20; "Descartes" by Marjorie Lord '20; "Newton" by Helen Frank '20.

December 15: Christmas Party.

February 19, 1920: "Applications of mathematics" by Margaret Doran '20.

March 8: "Introductory paper concerning the three famous problems of antiquity" by Agnes Grant '20; "Trisection of the angle" by Isabel Painter '20; "Squaring the circle" by Miriam Courtney '20.

April 12: "Duplication of the cube" by Carolyn Boudo '20; "The Pythagorean theorem" by Ruth Colsten '20.

May 3: "A study of transformations" by Dorothy Graves '21.

May 24: Spring Party.

THE PASCAL CIRCLE, Trinity College, Washington, D. C.
[*1920*, 425.]

An account of this club's activities for the year 1919–20 was given last month. No previous account of the club having appeared in this department, an inquiry was addressed to Professor Marie Cecelia Mangold concerning the previous history of the club. The information given below was received in reply.

The Pascal Circle was founded in October 1916 with the following officers: President, Anna Marie Lawler '17; secretary, Zita Louise Donahoe '17.

Two meetings per month were held and some of the topics discussed were the following: The mathematical basis of magic number cards and tricks with cards—Geometrical fallacies—Popular "catch" problems—Magic squares and magic pencils—The arcascope and the slide rule—Geometric paper folding—The Pascal triangle and the Pascal theorem—Nine-point circle.

Officers for the year 1917–18 were: President, Virginia Alma Keller '18; secretary, Frances Dix Wyman '19. Some meetings were held in the fall of 1917, but on account of sickness the Circle was disbanded until late in the year 1918–19 when it was reorganized with Katherine Mary Martin '19 as president and Beatrice Agnes Convey '20 as secretary. Three of the subjects discussed that year were: Locating machine guns by means of sound—The mathematical principle underlying the Russian peasant method of multiplication—Ciphers and cryptograms.

SOLUTIONS OF PROBLEMS

2790 [1919, 414]. Proposed by J. W. LASLEY, JR., University of North Carolina.

How shall we buy twelve eggs for eighty cents, if hen eggs sell at five cents each, duck eggs at seven cents each, and turkey eggs at eight cents each, and if we buy some of each?

SOLUTION BY C. A. ISAACS, State College of Washington.

Let x, y, z, be the number of eggs of each kind, respectively. Then

$$x + y + z = 12,$$

$$5x + 7y + 8z = 80.$$

Eliminating x,

$$2y + 3z = 20, \quad \text{or} \quad y = 10 - \frac{3z}{2}.$$

Since y is an integer, this value of y, though fractional in form, must be an integer. Therefore, z is even. Assigning even values to z, beginning with 2, we get three sets of answers satisfying the conditions of the problem: $z = 2, y = 7, x = 3$; $z = 4, y = 4, x = 4$; $z = 6, y = 1, x = 5$.

Also solved by H. N. CARLETON, H. L. OLSON, ARTHUR PELLETIER, and E. E. WHITFORD.

NOTES AND NEWS

It is hoped that readers of the MONTHLY will cooperate in contributing to the general interest of this department by sending items to the Editor-in-Chief.

Miss MARY C. BALL has been appointed instructor in mathematics at Northwestern University.

Assistant Professor J. N. MICHIE, of the Agricultural and Mechanical College of Texas, has been appointed adjunct professor of applied mathematics at the University of Texas.

Assistant Professor A. C. MADDOX, of the Oklahoma Agricultural and Mechanical College, has been appointed professor of mathematics at the Louisiana State Normal School at Natchitoches.

Professor J. B. FAUGHT of the Kent (Ohio) State Normal College has been made professor of mathematics in Yankton College.

Dr. G. W. SMITH, of the University of Kentucky, has been appointed assistant professor of mathematics in the University of Kansas. He taught during the past summer in the summer session of the University of Colorado.

Professor R. A. WELLS, of Park College, has been made associate professor of mathematics at Michigan State Normal College.

Mr. CORNELIUS GOUWENS, of the University of Kansas, has been appointed assistant professor of mathematics at Iowa State College.

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At the University of Iowa, Assistant Professor E. W. CHITTENDEN has been promoted to an associate professorship; Dr. W. H. WILSON has been promoted from an instructorship to an associateship; Dr. ROSCOE WOODS, of the University of Illinois, has been appointed instructor in mathematics; Mr. H. M. JEFFERS of the Lick Observatory has been appointed instructor in mathematics and astron-

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Dr. A. S. Hathaway, since 1891 professor of mathematics at the Rose Polytechnic Institute, has retired from active service. He is succeeded by Associate Professor C. P. Sousley of Pennsylvania State College.

Professor H. E. Buchanan, of the University of Tennessee, has resigned to become professor and head of the department of mathematics at Tulane University.

Professor E. W. Brown, of Yale, is on leave of absence for the first semester of this year, a part of the time being spent at Christ's College, Cambridge.

Mr. H. A. Simmons has been appointed instructor in mathematics at the University of Michigan.

Dr. Henry W. Stager, for many years head of the department of mathematics in Fresno Junior College, Fresno, California, and more recently with the United States Railroad Administration, has been appointed instructor of mathematics in the University of Washington.

At the United States Naval Academy, Assistant Professors J. A. Bullard, J. N. Galloway, A. Dillingham, and G. R. Clements have been promoted to associate professorships; Mr. H. M. Robert, Jr., Mr. M. A. Eason, Dr. L. S. Dederick, Dr. L. T. Wilson, Mr. H. H. Gaver, and Dr. W. F. Shenton have been promoted to assistant professorships, and Mr. E. R. C. Miles, Mr. A. J. Barrett, Mr. A. A. Robinson, and Mr. E. A. Bailey have been appointed to instructorships. There are now forty men in the mathematics department at the Academy. All grades have received increases in salary during the past year, instructors now being paid $2,800 per annum.

Sir Joseph Larmor, of the University of Cambridge, has been elected "correspondant" in place of Professor Liapounoff (cf. *1920*, 179, 384) in the section of geometry of the Academy of Sciences of the Institute of France.

Dr. Pierre Boutroux, professor of differential and integral calculus at the University of Poitiers, and recently of Princeton University, has been appointed professor of the history of science at the Collège de France.

Dr. Ludwik Silberstein, well known on this side of the Atlantic for his books on the *Theory of Relativity* (1914) and *Elements of Vector Algebra* (1919), and for his mathematical papers dealing with electromagnetism, optics, projective geometry, spectrum theory, etc., has left England and is now associated with the research laboratory staff of the Eastman Kodak Company (see *1920*, 217).

At the thirteenth regular meeting of the Association of Mathematics Teachers of New Jersey, at Rutgers College on October 30, the following papers were read by members of the Association: "The report of the National Committee from the viewpoint of college entrance requirements" by Dean H. E. Hawkes of Columbia University; "The law of exponents" by Professor Richard Morris of Rutgers College.

CONTENTS

EDITORIAL CORRESPONDENCE AND BOOKS FOR REVIEW should be addressed to the EDITOR-IN-CHIEF, R C. ARCHIBALD, Brown University, Providence, R. I.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS Oberlin, Ohio.

Fifth Summer Meeting of the Association, Chicago, September 6, 1920;
Fifth Annual Meeting, Chicago, December 28–29, 1920

The following are dates of Section meetings of the Association in 1920, unless otherwise specified:

ILLINOIS, Univ. of Chicago, December 29

IOWA, Univ. of Iowa, Iowa City, May 1; West High School Bldg., Des Moines, Nov. 5

KANSAS, State Agricultural College, Manhattan, April 3; Topeka, January 22, 1921

KENTUCKY, Centre College, Danville, April 17

MARYLAND–DISTRICT OF COLUMBIA–VIRGINIA, Goucher College, Baltimore, Md., May 15; Naval Academy, Annapolis, Md., December 11

MINNESOTA, St. Catherine's College, St. Paul, June 5

MISSOURI, Kansas City Junior College, November 13; Univ. of Missouri, November 25–26, 1921

OHIO, Ohio State Univ., Columbus, April 2

ROCKY MOUNTAIN, Colorado College, Colorado Springs, April 2

Important Notice

The Mathematical Association of America, like all other organizations of an educational character, gives manifold more than it receives from its constituents. This discrepancy is accounted for by the gratuitous and arduous work given to the Association by its devoted servants.

Since it is impossible to raise the dues above a certain maximum without going beyond the reach of very many of those to whom the Association means most, it seems clear that an endowment fund is the best solution of the difficulty. Now that the Association is incorporated it is legally qualified to administer such a fund.

An endowment is needed not only to prevent a reduction of the number of pages in the MONTHLY, but also to enable the Association to make just compensation to its servants, and to go forward with its important projects such, for example, as the preparation and publication of a Mathematical Dictionary which is so greatly needed in the English language.

It is believed that, when these conditions are widely known among the friends of mathematics, financial support of this kind will be forthcoming.

Legal Form for Gifts and Bequests

I hereby give[1] to the Board of Trustees of the Mathematical Association of America the sum of .... Dollars,

to be known as the ....

.... Fund, and to be used[2]

for { Endowment—the income only of which may be expended.
Special Projects—for which both principal and income may be expended. }

Witness: Signature ....

....

[1] In case of a bequest, the first line should read "I hereby give and bequeath," etc.
[2] Indicate which one of the two purposes is desired, and omit the other.